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Intuitionistic fuzzy differential equation with nonlocal condition

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Abstract: In this paper, we shall prove the existence and uniqueness theorem of a solution to the nonlocal intuitionistic fuzzy differential equation using the concept of intuitionistic fuzzy semigroup and the contraction mapping principle.
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1 Introduction

A large class of physically important problems is described by fuzzy differential equations. Kaleva [4] discussed the properties of differentiable fuzzy set valued mappings and give the existence and uniqueness theorem for a solution of the fuzzy differential equations $x_0(t) = f(t, x(t))$ when f satisfies the Lipschitz condition. And Feng [5] studied the existence and uniqueness of a solution, the continuity of the solution with respect to the initial value and the stability of fuzzy stochastic differential equations.

On the other hand, Jae Ug Jeong [1] gave the the existence and uniqueness theorem of a solution to the nonlocal fuzzy differential equation using the contraction mapping principle.

In this work, we study the existence and uniqueness of solutions for intuitionistic fuzzy differential equations with nonlocal conditions of the following form:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) & t \in I = [0, a] \\ x(0) = x_0 + g\Big(t_1, t_2, \ldots, t_p, x(.)\Big) \end{cases} \tag{1}$$

where A generates an intuitionistic fuzzy strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathbb{F}_1$ see [3], $x_0 \in \mathbb{F}_1$ f and g are given functions to be specified later.

The symbol $g\Big(t_1, t_2, \ldots, t_p, x(.)\Big)$ is used in the sense that in the place of . we can substitute only elements of the $\{t_1, t_2, \ldots, t_p\}$. For example,$g\Big(t_1, t_2, \ldots, t_p, x(.)\Big)$ can be defined by the formula

$$g\Big(t_1, t_2, \ldots, t_p, x(.)\Big) = c_1 x(t_1) + c_2 x(t_2) + \cdots + c_p x(t_p)$$

where c_i $(i = 1, 2, \ldots, p)$ are given constants.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us denote by $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.

Definition 1. *We denote*

$$I\!F_1 = \Big\{(u, v) : \mathbb{R} \to [0, 1]^2 \;|\forall x \in \mathbb{R} \;/ 0 \leq u(x) + v(x) \leq 1\Big\}$$

where

1. (u, v) *is normal i.e there exists* $x_0, x_1 \in \mathbb{R}$ *such that* $u(x_0) = 1$ *and* $v(x_1) = 1$.
2. u *is fuzzy convex and* v *is fuzzy concave.*
3. u *is upper semi-continuous and* v *is lower semi-continuous*
4. *supp*$(u, v) = cl\Big(\{x \in \mathbb{R} : v(x) < 1\}\Big)$ *is bounded.*

For $\alpha \in [0, 1]$ *and* $(u, v) \in I\!F_1$*, we define*

$$\Big[(u, v)\Big]^{\alpha} = \{x \in \mathbb{R} \mid v(x) \leq 1 - \alpha\} \text{ and } \Big[(u, v)\Big]_{\alpha} = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$$

Remark 1. *We can consider* $\Big[(u, v)\Big]_{\alpha}$ *as* $[u]^{\alpha}$ *and* $\Big[(u, v)\Big]^{\alpha}$ *as* $[1 - v]^{\alpha}$ *in the fuzzy case.*

Definition 2. *The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$0_{(1,0)}(x) = \begin{cases} (1, 0) & x = 0 \\ (0, 1) & x \neq 0 \end{cases}$$

Definition 3. *Let* (u, v) ,$(u', v') \in I\!F_1$ *and* $\lambda \in \mathbb{R}$*, we define the addition by :*

$$\Big((u, v) \oplus (u', v')\Big)(z) = \Bigg(\sup_{z=x+y} \min\Big(u(x), u'(y)\Big); \inf_{z=x+y} \max\Big(v(x), v'(y)\Big)\Bigg)$$

$$\lambda(u, v) = \begin{cases} (\lambda u, \lambda v) & \textit{if } \lambda \neq 0 \\ 0_{(0,1)} & \textit{if } \lambda = 0 \end{cases}$$

According to Zadeh's extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space $\mathbb{F}_1$ as follows:

$$\Big[(u,v)\oplus(z,w)\Big]^{\alpha} = \Big[(u,v)\Big]^{\alpha} + \Big[(z,w)\Big]^{\alpha}, \qquad \Big[\lambda(u,v)\Big]^{\alpha} = \lambda\Big[(u,v)\Big]^{\alpha}$$
$$\Big[(u,v)\oplus(z,w)\Big]_{\alpha} = \Big[(u,v)\Big]_{\alpha} + \Big[(z,w)\Big]_{\alpha}, \qquad \Big[\lambda(u,v)\Big]_{\alpha} = \lambda\Big[(u,v)\Big]_{\alpha}$$

where $(u,v),(z,w)\in\mathbb{F}_1$ and $\lambda\in\mathbb{R}$.
We denote

$$[(u,v)]_l^{+}(\alpha) = \inf\{x\in\mathbb{R} \mid u(x)\geq\alpha\}, \quad [(u,v)]_r^{+}(\alpha) = \sup\{x\in\mathbb{R} \mid u(x)\geq\alpha\}$$
$$[(u,v)]_l^{-}(\alpha) = \inf\{x\in\mathbb{R} \mid v(x)\leq 1-\alpha\}, \quad [(u,v)]_r^{-}(\alpha) = \sup\{x\in\mathbb{R} \mid v(x)\leq 1-\alpha\}$$

Remark 2.

$$\Big[(u,v)\Big]_{\alpha} = \Big[[(u,v)]_l^{+}(\alpha), [(u,v)]_r^{+}(\alpha)\Big]$$
$$\Big[(u,v)\Big]^{\alpha} = \Big[[(u,v)]_l^{-}(\alpha), [(u,v)]_r^{-}(\alpha)\Big]$$

Theorem 1. *let $\mathcal{M} = \{M_\alpha,\ M^\alpha\ :\ \alpha\in[0,1]\}$ be a family of subsets in $\mathbb{R}$ satisfying Conditions $(i)-(iv)$*

i) $\alpha\leq\beta \Rightarrow M_\beta\subset M_\alpha$ *and* $M^\beta\subset M^\alpha$

ii) M_α *and* M^α *are nonempty compact convex sets in* $\mathbb{R}$ *for each* $\alpha\in[0,1]$.

iii) *for any non-decreasing sequence* $\alpha_i\to\alpha$ *on* $[0,1]$, *we have* $M_\alpha = \bigcap_i M_{\alpha_i}$ *and* $M^\alpha = \bigcap_i M^{\alpha_i}$.

iv) *For each* $\alpha\in[0,1]$, $M_\alpha\subset M^\alpha$ *and define* u *and* v, *by*

$$u(x) = \begin{cases} 0 & \text{if } x\notin M_0 \\ \sup\{\alpha\in[0,1]\ :\ x\in M_\alpha\} & \text{if } x\in M_0 \end{cases}$$
$$v(x) = \begin{cases} 1 & \text{if } x\notin M^0 \\ 1-\sup\{\alpha\in[0,1]\ :\ x\in M^\alpha\} & \text{if} x\in M^0 \end{cases}$$

Then $(u,v)\in\mathbb{F}_1$.

Proof. See [2]. □

The space $\mathbb{F}_1$ is metrizable by the distance of the following form :

$$d_\infty\Big((u,v),(z,w)\Big) = \frac{1}{4}\sup_{0<\alpha\leq1}\left\|\Big[(u,v)\Big]_r^{+}(\alpha) - \Big[(z,w)\Big]_r^{+}(\alpha)\right\|$$
$$+\frac{1}{4}\sup_{0<\alpha\leq1}\left\|\Big[(u,v)\Big]_l^{+}(\alpha) - \Big[(z,w)\Big]_l^{+}(\alpha)\right\| + \frac{1}{4}\sup_{0<\alpha\leq1}\left\|\Big[(u,v)\Big]_r^{-}(\alpha) - \Big[(z,w)\Big]_r^{-}(\alpha)\right\|$$
$$+\frac{1}{4}\sup_{0<\alpha\leq1}\left\|\Big[(u,v)\Big]_l^{-}(\alpha) - \Big[(z,w)\Big]_l^{-}(\alpha)\right\|$$

where $\|\ \|$ denotes the usual Euclidean norm in $\mathbb{R}$.

Theorem 2. $\left(I\!\!F_1, d_\infty\right)$ *is a complete metric space.*

Proof. See [2]. $\square$

3 The L_2-space

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space.

Definition 4. *An intuitionistic fuzzy random variable (i.f.r.v, for short) is a Borel measurable function* $X : (\Omega, \mathcal{A}) \to \left(I\!\!F_1, d_\infty\right)$

The norm $\| \ \|$ of an intuitionistic fuzzy number $(u, v) \in I\!\!F_1$ is defined by

$$\|(u,v)\| = d_\infty\Big((u,v), 0_{(1,0)}\Big) = \Big\| \Big[(u,v)\Big]_0 \Big\| = \frac{1}{2} \sup_{a \in [(u,v)]_0} |a| + \frac{1}{2} \inf_{b \in [(u,v)]^0} |b|$$

If $E\|X\| < \infty$, then the expected value EX exists. X is called a second-order i.f.r.v, provided $E\|X\|^2 < \infty$. Let

$$L\Big(\Omega, \mathcal{A}, P\Big) = \{X | X \text{ is a i.f.r.v with } \int_\Omega d_\infty(X, 0_{(1,0)})^2 dP(\omega) < \infty\}$$

The family of all second-order i.f.r.v.'s also denoted by $L_2(I\!\!F_1)$ (L_2, for short).
Any two i.f.r.v.'s X and Y are called equivalent if $P(X \neq \mathrm{Y}) = 0$. The all equivalent elements in L_2 are identified. Define

$$\varphi(X, \mathrm{Y}) = \left(\int_\Omega d_\infty(X, \mathrm{Y})^2 dP\right)^{\frac{1}{2}}, \ X, \mathrm{Y} \in L_2$$

The norm $\|X\|_2$ of an element $X \in L_2$ is defined by

$$\|X\|_2 = \varphi(X, 0_{(1,0)}) = \left(\int_\Omega d_\infty(X, 0_{(1,0)})^2 dP\right)^{\frac{1}{2}}$$

Proposition 1. (L_2, φ) *is a complete metric space.*

In addition φ satisfies that

$$\varphi(X + Z, \mathrm{Y} + Z) = \varphi(X, \mathrm{Y}) \ , \ \varphi(\lambda X, \lambda \mathrm{Y}) = |\lambda| \varphi(X, \mathrm{Y}) \tag{2}$$

$$\varphi(\lambda X, kX) \leq |\lambda - k| \|X\|_2 \tag{3}$$

for any $X, \mathrm{Y}, Z \in L_2$ and $\lambda, k \in \mathbb{R}$

Definition 5. *Let $(X_n)_{n \geq 1}$ be a sequence in L_2, we call that X_n converges in mean square or m.s. converges to X as $n \to \infty$, if $\varphi(X_n, X) \to 0$, write $X_n \xrightarrow{m.s} X$ or $\lim\limits_{n \to \infty} X_n = X$*

Definition 6. *Let T be a finite or an infinite interval in $\mathbb{R}$. A mapping $X : T \to L_2$ is called a second-order intuitionistic fuzzy stochastic process (i.f.s.p. for short). If X is continuous at a $t \in T$ with respect to the metric φ then we call X continuous in mean square or m.s. continuous at t. If X is m.s. continuous at every $t \in T$ then we call X m.s. continuous.*

4 Nonlocal intuitionistic fuzzy differential equation

Throughout this work, we suppose that

(H_1) $A : D(A) \subseteq \mathbb{F}_1 \to \mathbb{F}_1$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and there exist constants M and $\omega \in \mathbb{R}_+^*$ such that $\varphi\left(T(t)x, T(t)y\right) \leq Me^{\omega t}\varphi\left(x, y\right),\ for\ t \geq 0\ ,\ x, y \in L_2 \cap D(A)$.

(H_2) $f : I \times L_2 \to L_2$ is m.s. continuous intuitionistic fuzzy mapping with respect to t, which satisfies a generalized Lipschitz condition, i.e., there exists constant K_1 such that $\varphi(f(t, x), f(t, y)) \leq K_1\varphi(x, y)$.

(H_3) $g : I^p \times L_2 \to L_2$ satisfies a generalized Lipschitz condition, i.e., there exists constant K_2 such that, $\forall t \in I, x, y \in L_2$ and $x_0 \in L_2$ $\varphi\Big(g\Big(t_1, t_2, \dots, t_p, x(.)\Big), g\Big(t_1, t_2, \dots, t_p, y(.)\Big)\Big) \leq K_2\varphi(x, y)$.

We consider the nonlocal intuitionistic fuzzy differential equation:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) & t \in I = [0, a] \\ x(0) = x_0 + g\Big(t_1, t_2, \dots, t_p, x(.)\Big) \end{cases} \tag{4}$$

where $0 < t_1 < t_2 < \dots < t_p \leq a$

Definition 7. *A function $x(.) : I \to L_2$ is said a mild solution of* (4)*, if*

$$x(t) = T(t)\Big[x_0 + g\Big(t_1, t_2, ..., t_p, x(.)\Big)\Big] + \int_0^t T(t - s)f(s, x(s))ds$$

for $0 \leq t \leq a$.

Theorem 3. *Assume that assumptions $(H_1) - (H_3)$ hold. Then* (4) *has a unique mild solution on the interval $[0, \xi]$ where*

$$\xi = \min\left\{a\ ,\ \frac{1}{\omega}\log\Big(\frac{b - \varepsilon + \frac{N_1 M}{\omega}}{MN_2 + \frac{N_1 M}{\omega}}\Big)\ ,\ \frac{1}{\omega}\log\Big(\frac{1 + \frac{K_1 M}{\omega}}{K_2 M + \frac{K_1 M}{\omega}}\Big)\right\}$$

and

$$\varphi\Big(f(t, x), 0_{(1,0)}\Big) \leq N_1\ ,\ \varphi\Big(g\Big(t_1, t_2, \dots, t_p, x(.)\Big), 0_{(1,0)}\Big) \leq N_2$$

Proof. Let $B = \{x \in L_2 \mid H(x, x_0) \leq b\}$ be the space of m.s. continuous intuitionistic fuzzy mappings with

$$H(x, y) = \sup_{0\leq t\leq \xi} \varphi\Big(x(t), y(t)\Big)$$

and b a positive number. Define a mapping $P : B \to B$ by

$$Px(t) = T(t)\Big[x_0 + g\Big(t_1, t_2, \dots, t_p, x(.)\Big)\Big] + \int_0^t T(t - s)f(s, x(s))ds$$

First of all, we show that P is m.s. continuous and $H(Px, x_0) \leq b$. Since f is m.s. continuous, we have

$$\varphi(Px(t+h), Px(t)) = \varphi\Big(T(t+h)\Big[x_0 + g\Big(t_1, t_2, \ldots, t_p, x(.)\Big)\Big]$$
$$+ \int_0^{t+h} T(t+h-s)f(s, x(s))ds, T(t)\big[x_0 + g(t_1, t_2, \ldots, t_p, x(.))\big]$$
$$+ \int_0^t T(t-s)f(s, x(s))ds\Big)$$
$$\leq \varphi\Big(T(t+h)x_0, T(t)x_0\Big) + \varphi\Big(T(t+h)g\Big(t_1, t_2, \ldots, t_p, x(.)\Big), T(t)g\Big(t_1, t_2, \ldots, t_p, x(.)\Big)\Big)$$
$$+ \varphi\Big(\int_0^{t+h} T(t+h-s)f(s, x(s))ds, \int_0^t T(t-s)f(s, x(s))ds\Big)$$
$$\leq \varphi\Big(T(t+h)x_0, T(t)x_0\Big) + \varphi\Big(T(t+h)g\Big(t_1, t_2, \ldots, t_p, x(.)\Big), T(t)g\Big(t_1, t_2, \ldots, t_p, x(.)\Big)\Big)$$
$$+ \varphi\Big(\int_0^{h} T(t+h-s)f(s, x(s))ds, 0_{(1,0)}\Big)$$
$$+ \varphi\Big(\int_h^{t+h} T(t+h-s)f(s, x(s))ds, \int_0^t T(t-s)f(s, x(s))ds\Big)$$
$$\leq Me^{\omega t}\Big[\varphi\big(T(h)x_0, x_0\big) + \varphi\Big(T(h)g(t_1, t_2, \ldots, t_p, x(.)), g(t_1, t_2, \ldots, t_p, x(.))\Big)\Big]$$
$$+ \varphi\Big(\int_0^{h} T(t+h-s)f(s, x(s))ds, 0_{(1,0)}\Big)$$
$$+ \int_0^t Me^{\omega(t-s)}\varphi\Big(f(s+h, x(s+h)), f(s, x(s))\Big)ds$$

It is clear that $\varphi\big(T(h)x_0, x_0\big) \to 0$, $\varphi\Big(T(h)g(t_1, t_2, \ldots, t_p, x(.)), g(t_1, t_2, \ldots, t_p, x(.))\Big) \to 0$ and

$$\varphi\Big(\int_0^h T(t+h-s)f(s, x(s))ds, 0_{(1,0)}\Big)) \to 0$$

as $h \to 0$.

And by the dominated convergence theorem:

$$\int_0^t Me^{\omega(t-s)}\varphi\Big(f(s+h, x(s+h)), f(s, x(s))\Big)ds \to 0$$

That is, the map P is m.s. continuous on I. Furthermore,

$$\begin{aligned}
\varphi(Px(t), x_0) &= \varphi\Big(T(t)\big[x_0 + g(t_1, t_2, \ldots, t_p, x(.))\big] + \int_0^t T(t-s)f(s, x(s))ds, x_0\Big) \\
&\leq \varphi\big(T(t)x_0, x_0\big) + \varphi\big(T(t)g(t_1, t_2, \ldots, t_p, x(.)), 0_{(1,0)}\big) \\
&+ \varphi\Big(\int_0^t T(t-s)f(s, x(s))ds, 0_{(1,0)}\Big) \\
&\leq \varepsilon + Me^{\omega t}\varphi\big(g(t_1, t_2, \ldots, t_p, x(.)), 0_{(1,0)}\big)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t Me^{\omega(t-s)}\varphi\Big(f(s,x(s)),0_{(1,0)}\Big)ds \\
&\leq \varepsilon+N_2Me^{\omega t}+\frac{N_1M}{\omega}[e^{\omega t}-1]
\end{aligned}
$$

and so

$$
\begin{aligned}
H(Px,x_0) &= \sup_{0\leq t\leq \xi}\varphi(Px(t),x_0) \\
&\leq \varepsilon+N_2Me^{\omega\xi}+\frac{N_1M}{\omega}[e^{\omega\xi}-1] \\
&\leq b.
\end{aligned}
$$

Since (L_2,φ) is a complete metric space, a standard proof applies to show that

$$C([0,\xi],L_2)=\{x:[0,\xi]\to L_2|\ x(t) \text{ is m.s. continuous}\}$$

is complete. Now we show that B is a closed subset of $C([0,\xi],L_2)$. Let $\{x_n\}$ be a sequence in B such that $x_n\to x\ \in C([0,\xi],L_2)$ as $n\to\infty$. Then

$$
\begin{aligned}
\varphi(x(t),x_0) &\leq \varphi(x(t),x_n(t))+\varphi(x_n(t),x_0) \\
H(x,x_0) &= \sup_{0\leq t\leq\xi}\varphi(x(t),x_0) \\
&\leq H(x,x_n)+H(x_n,x_0) \\
&\leq \varepsilon+b
\end{aligned}
$$

for sufficiently large n and arbitrary $\varepsilon>0$. So $x\in B$. This implies that B is a closed subset of $C([0,\xi],L_2)$. Therefore B is a complete metric space. Next, we will show that P is a contraction mapping. For $x,y\ \in B$

$$
\begin{aligned}
\varphi(Px(t),Py(t)) &\leq \varphi\big(T(t)g(t_1,t_2,\ldots,t_p,x(.)),T(t)g(t_1,t_2,...,t_p,y(.))\big)+ \\
&\quad\ \varphi\Big(\int_0^t T(t-s)f(s,x(s))ds,\int_0^t T(t-s)f(s,y(s))ds\Big) \\
&\leq K_2Me^{\omega t}\varphi(x,y)+K_1M\int_0^t e^{\omega(t-s)}\varphi\Big(f(s,x(s)),f(s,y(s))\Big)ds
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
H(Px,Py) &\leq \sup_{0\leq t\leq\xi}\Big\{K_2Me^{\omega t}\varphi(x,y)+K_1M\int_0^t e^{\omega(t-s)}\varphi\Big(f(s,x(s)),f(s,y(s))\Big)ds\Big\} \\
&\leq \Big(K_2Me^{\omega\xi}+\frac{K_1M}{\omega}[e^{\omega\xi}-1]\Big)H(x,y)
\end{aligned}
$$

Since $K_2Me^{\omega\xi}+\frac{K_1M}{\omega}[e^{\omega\xi}-1]<1$, P is a contraction map. Therefore P has a unique fixed point
$Px=x\ \in C([0,\xi],L_2)$, that is

$$x(t)=T(t)\big[x_0+g(t_1,t_2,...,t_p,x(.))\big]+\int_0^t T(t-s)f(s,x(s))ds.$$

This completes the proof. $\square$

Theorem 4. *Suppose that f, g are the same as in Theorem 3. Let $x(t,x_0)$,$y(t,y_0)$ be solutions of Eq.(4) to x_0, y_0, respectively. Then there exist constants c_1 and c_2 such that*

1. $H\big(x(.,x_0),y(.,y_0)\big) \le c_1\varphi(x_0,y_0)$ *for any* $x_0,y_0 \in L_2$
2. $H\big(x(.,x_0),0_{(1,0)}\big) \le c_2\big(\varphi(x_0,0_{(1,0)}) + N_1 + N_3\big)$ *where*

$$\varphi\big(g(t_1,t_2,...,t_p,x(.)),0_{(1,0)}\big) \le N_1 \text{ and } \int_0^t e^{-\omega s}\varphi\Big(f(s,0_{(1,0)}),0_{(1,0)}\Big) \le N_3$$

Proof. **1.** For any $t \in [0,\xi]$ we have

$$\begin{aligned}
&\varphi\big(x(t,x_0),y(t,y_0)\big) \le \varphi\big(T(t)x_0,T(t)y_0\big)\\
&+\varphi\big(T(t)g(t_1,t_2,\dots,t_p,x(.,x_0)),T(t)g(t_1,t_2,\dots,t_p,y(.,y_0))\big)\\
&+\varphi\Big(\int_0^t T(t-s)f(s,x(s,x_0))ds,\int_0^t T(t-s)f(s,y(s,y_0))ds\Big)\\
&\le Me^{\omega t}\Big[\varphi\big(x_0,y_0\big)+\varphi\big(g(t_1,t_2,\dots,t_p,x(.,x_0)),g(t_1,t_2,\dots,t_p,y(.,y_0))\big)\Big]\\
&+Me^{\omega t}\int_0^t e^{-\omega s}\varphi\Big(f(s,x(s,x_0)),f(s,y(s,y_0))\Big)ds\\
&\le Me^{\omega\xi}\Big[\varphi\big(x_0,y_0\big)+K_2\varphi\big(x(.,x_0),y(.,y_0)\big)\Big]\\
&+K_1Me^{\omega\xi}\int_0^t e^{-\omega s}\varphi\Big(x(s,x_0),y(s,y_0)\Big)ds
\end{aligned}$$

From Gronwall's inequality, we get

$$\begin{aligned}
\varphi\big(x(t,x_0),y(t,y_0)\big) &\le Me^{\omega\xi}\Big[\varphi\big(x_0,y_0\big)+K_2\varphi\big(x(.,x_0),y(.,y_0)\big)\Big]\exp\Big(K_1Me^{\omega\xi}\int_0^t e^{-\omega s}ds\Big)\\
&\le Me^{\omega\xi}\Big[\varphi\big(x_0,y_0\big)+K_2\varphi\big(x(.,x_0),y(.,y_0)\big)\Big]\exp\Big(K_1Me^{\omega\xi}\frac{1-e^{-\omega t}}{\omega}\Big)
\end{aligned}$$

Thus we have

$$H\big(x(.,x_0),y(.,y_0)\big) \le Me^{\omega\xi}\Big[\varphi\big(x_0,y_0\big)+K_2H\big(x(.,x_0),y(.,y_0)\big)\Big]\exp\Big(K_1M\frac{e^{\omega\xi}-1}{\omega}\Big)$$

i.e,

$$\Big(1-K_2Me^{\omega\xi}\exp\Big(K_1M\frac{e^{\omega\xi}-1}{\omega}\Big)\Big)H\big(x(.,x_0),y(.,y_0)\big) \le Me^{\omega\xi}\exp\Big(K_1M\frac{e^{\omega\xi}-1}{\omega}\Big)\varphi\big(x_0,y_0\big)$$

Consequently, we obtain

$$H\big(x(.,x_0),y(.,y_0)\big) \le \frac{Me^{\omega\xi}\exp\Big(K_1M\frac{e^{\omega\xi}-1}{\omega}\Big)}{\Big(1-K_2Me^{\omega\xi}\exp\Big(K_1M\frac{e^{\omega\xi}-1}{\omega}\Big)\Big)}\varphi\big(x_0,y_0\big)$$

Taking $c_1 = \frac{Me^{\omega\xi}\exp\left(K_1M\frac{e^{\omega\xi}-1}{\omega}\right)}{\left(1-K_2Me^{\omega\xi}\exp\left(K_1M\frac{e^{\omega\xi}-1}{\omega}\right)\right)}$, we obtain $H\big(x(.,x_0), y(.,y_0)\big) \leq c_1\varphi\big(x_0, y_0\big)$.

2. For any $t \in [0,\xi]$ we have

$$\begin{aligned}
&\varphi\big(x(t,x_0), 0_{(1,0)}\big) \leq \varphi\big(T(t)x_0, 0_{(1,0)}\big) + \varphi\big(T(t)g(t_1,t_2,\ldots,t_p,x(.,x_0)), 0_{(1,0)}\big)\\
&+\varphi\bigg(\int_0^t T(t-s)f(s,x(s,x_0))ds, 0_{(1,0)}\bigg)\\
&\leq Me^{\omega t}\Big[\varphi\big(x_0,0_{(1,0)}\big) + \varphi\big(g(t_1,t_2,\ldots,t_p,x(.,x_0)), 0_{(1,0)}\big)\Big]\\
&+Me^{\omega t}\int_0^t e^{-\omega s}\varphi\bigg(f(s,x(s,x_0)), 0_{(1,0)}\bigg)ds\\
&\leq Me^{\omega\xi}\Big[\varphi\big(x_0,0_{(1,0)}\big) + \varphi\big(g(t_1,t_2,\ldots,t_p,x(.,x_0)), 0_{(1,0)}\big)\Big]\\
&+Me^{\omega\xi}\bigg[\int_0^t e^{-\omega s}\varphi\bigg(f(s,x(s,x_0)), f(s,0_{(1,0)})\bigg) + \int_0^t e^{-\omega s}\varphi\bigg(f(s,0_{(1,0)}), 0_{(1,0)}\bigg)\bigg]ds\\
&\leq Me^{\omega\xi}\Big[\varphi\big(x_0,0_{(1,0)}\big) + \varphi\big(g(t_1,t_2,\ldots,t_p,x(.,x_0)), 0_{(1,0)}\big)\Big]\\
&+Me^{\omega\xi}\bigg[K_1\int_0^t e^{-\omega s}\varphi\bigg(x(s,x_0), 0_{(1,0)}\bigg) + \int_0^t e^{-\omega s}\varphi\bigg(f(s,0_{(1,0)}), 0_{(1,0)}\bigg)\bigg]ds
\end{aligned}$$

From Gronwall's inequality, we get

$$\begin{aligned}
\varphi\big(x(t,x_0),0_{(1,0)}\big) &\leq Me^{\omega\xi}\bigg[\varphi\big(x_0,0_{(1,0)}\big) + \varphi\big(g(t_1,t_2,\ldots,t_p,x(.,x_0)),0_{(1,0)}\big)\\
&\quad+\int_0^t e^{-\omega s}\varphi\bigg(f(s,0_{(1,0)}),0_{(1,0)}\bigg)\bigg]\exp\bigg(K_1Me^{\omega\xi}\int_0^t e^{-\omega s}ds\bigg)\\
&\leq Me^{\omega\xi}\Big[\varphi\big(x_0,0_{(1,0)}\big) + N_1 + N_3\Big]\exp\bigg(K_1M\frac{e^{\omega\xi}-1}{\omega}\bigg)
\end{aligned}$$

Taking $c_2 = Me^{\omega\xi}\exp\left(K_1M\frac{e^{\omega\xi}-1}{\omega}\right)$, we get

$$\begin{aligned}
H\big(x(.,x_0),0_{(1,0)}\big) &= \sup_{0\leq t\leq\xi}\varphi\big(x(t,x_0),0_{(1,0)}\big)\\
&\leq c_2\Big[\varphi\big(x_0,0_{(1,0)}\big) + N_1 + N_3\Big]
\end{aligned}$$

This completes the proof. □

We consider the following intuitionistic fuzzy differential equations with nonlocal conditions

$$x(t) = T(t)\big[x_0 + g(t_1,t_2,\ldots,t_p,x(.))\big] + \int_0^t T(t-s)f(s,x(s))ds \tag{5}$$

$$x_n(t) = T(t)\big[x_{n,0} + g_n(t_1,t_2,\ldots,t_p,x_n(.))\big] + \int_0^t T(t-s)f_n(s,x_n(s))ds, \;\; n \geq 1. \tag{6}$$

If Eqs. (5) and (6) satisfy the conditions of Theorem 3, then they have unique solutions $x(t)$, and $x_n(t)$, $t \in [0,\xi]$, respectively.

Theorem 5. *Suppose that f, g are the same as in Theorem 3. If*

$$\varphi\big(x_{n,0}, x_0\big) \to 0,$$

$$\varphi\Big(g_n(t_1, t_2, \ldots, t_p, x(.)), g(t_1, t_2, \ldots, t_p, x(.))\Big) \to 0$$

and

$$\sup_{0\le t\le \xi} \varphi\Big(f_n(t, y), f(t, y)\Big) \to 0 \textit{ as } n \to \infty \textit{ for each } y \in L_2,$$

then

$$\sup_{0\le t\le \xi} \varphi\Big(x_n(t), x(t)\Big) \to 0 \textit{ as } n \to \infty.$$

Proof. For any $t \in [0, \xi]$ we have

$$\begin{aligned}
&\varphi\big(x_n(t), x(t)\big) \le \varphi\big(T(t)x_{n,0}, T(t)x_0\big) \\
&+ \varphi\Big(T(t)g_n(t_1, t_2, \ldots, t_p, x_n(.)), T(t)g(t_1, t_2, \ldots, t_p, x(.))\Big) \\
&+ \varphi\Big(\int_0^t T(t-s)f_n(s, x_n(s))ds, \int_0^t T(t-s)f(s, x(s))ds\Big) \\
&\le \varphi\big(T(t)x_{n,0}, T(t)x_0\big) + \varphi\Big(T(t)g_n(t_1, t_2, \ldots, t_p, x_n(.)), T(t)g_n(t_1, t_2, \ldots, t_p, x(.))\Big) \\
&+ \varphi\Big(T(t)g_n(t_1, t_2, \ldots, t_p, x(.)), T(t)g(t_1, t_2, \ldots, t_p, x(.))\Big) \\
&+ \varphi\Big(\int_0^t T(t-s)f_n(s, x_n(s))ds, \int_0^t T(t-s)f_n(s, x(s))ds\Big) \\
&+ \varphi\Big(\int_0^t T(t-s)f_n(s, x(s))ds, \int_0^t T(t-s)f(s, x(s))ds\Big).
\end{aligned}$$

$$\begin{aligned}
\varphi\big(x_n(t), x(t)\big) \le{}& Me^{\omega\xi}\Big[\varphi\big(x_{n,0}, x_0\big) + \varphi\Big(g_n(t_1, t_2, \ldots, t_p, x_n(.)), g_n(t_1, t_2, \ldots, t_p, x(.))\Big) \\
&+ \varphi\Big(g_n(t_1, t_2, \ldots, t_p, x(.)), g(t_1, t_2, \ldots, t_p, x(.))\Big) \\
&+ \int_0^t e^{-\omega s}\varphi\Big(f_n(s, x(s))ds, f(s, x(s))\Big)ds\Big] \\
&+ Me^{\omega\xi}\int_0^t e^{-\omega s}\varphi\Big(f_n(s, x_n(s)), f_n(s, x(s))\Big)ds \\
&\le Me^{\omega\xi}\Big[\varphi\big(x_{n,0}, x_0\big) + K_2\varphi\Big(x_n(.), x(.)\Big) \\
&+ \varphi\Big(g_n(t_1, t_2, \ldots, t_p, x(.)), g(t_1, t_2, \ldots, t_p, x(.))\Big) \\
&+ \int_0^t e^{-\omega s}\varphi\Big(f_n(s, x(s))ds, f(s, x(s))\Big)ds\Big] \\
&+ K_1Me^{\omega\xi}\int_0^t e^{-\omega s}\varphi\Big(x_n(s), x(s)\Big)ds.
\end{aligned}$$

From Gronwall's inequality, we get

$$\varphi\big(x_n(t),x(t)\big) \leq Me^{\omega\xi}\bigg[\varphi\big(x_{n,0},x_0\big)+K_2\varphi\bigg(x_n(.),x(.)\bigg) \\ +\varphi\bigg(g_n(t_1,t_2,...,t_p,x(.)),g(t_1,t_2,...,t_p,x(.))\bigg) \\ +\int_0^t e^{-\omega s}\varphi\bigg(f_n(s,x(s))ds,f(s,x(s))\bigg)ds\bigg]\exp\bigg(K_1Me^{\omega\xi}\frac{1-e^{\omega t}}{\omega}\bigg).$$

That is,

$$\bigg(1-K_2Me^{\omega\xi}\exp\bigg(K_1M\frac{e^{\omega\xi}-1}{\omega}\bigg)\bigg)\sup_{0\leq t\leq\xi}\varphi\big(x_n(t),x(t)\big) \\ \leq Me^{\omega\xi}\bigg[\varphi\big(x_{n,0},x_0\big)+\varphi\bigg(g_n(t_1,t_2,\ldots,t_p,x(.)),g(t_1,t_2,\ldots,t_p,x(.))\bigg) \\ +\sup_{0\leq t\leq\xi}\int_0^t e^{-\omega s}\varphi\bigg(f_n(s,x(s)),f(s,x(s))\bigg)ds\bigg]\exp\bigg(K_1M\frac{e^{\omega\xi}-1}{\omega}\bigg) \quad (7)$$

And

$$\begin{aligned}\varphi\bigg(f_n(s,x(s)),f(s,x(s))\bigg) &\leq \varphi\bigg(f_n(s,x(s)),f_n(s,0_{(1,0)})\bigg)+\varphi\bigg(f_n(s,0_{(1,0)}),f(s,0_{(1,0)})\bigg)\\ &\quad+\varphi\bigg(f(s,0_{(1,0)}),f(s,x(s))\bigg)\\ &\leq 2K_1\varphi\bigg(x(s),0_{(1,0)}\bigg)+\sup_{0\leq t\leq\xi}\varphi\bigg(f_n(s,0_{(1,0)}),f(s,0_{(1,0)})\bigg)\\ &\leq 2K_1c_2\bigg(\varphi(x_0,0_{(1,0)})+N_1+N_3\bigg)+1\end{aligned}$$

as soon as n is large enough, where we used **2.** of Theorem 4.

Hence, using the dominated convergence theorem in (7), we obtain the conclusion of the theorem. □

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