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On intuitionistic fuzzy chained modules

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Abstract: In this paper, we introduce and explore some novel concepts within the frame work of intuitionistic fuzzy module theory. First, we define the notion of an intuitionistic fuzzy chained module as a generalisation of chained modules, establishing it foundational properties. We then characterise intuitionistic fuzzy chained modules, in terms of its level-cut submodules. In addition to this, we describe an intuitionistic fuzzy chained module in terms of its intuitionistic fuzzy cyclic submodules. Finally, we demonstrate that under specific conditions, the intuitionistic fuzzy multiplication module is an intuitionistic fuzzy chained module.
Keywords: Intuitionistic fuzzy chained module, Intuitionistic fuzzy (cyclic) submodule, Intuitionistic fuzzy multiplication module, Intuitionistic fuzzy (prime) ideal.
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1 Introduction

Uniserial modules, also known as chained modules, are modules over a ring in which every submodule is contained in a chain of submodules. A module M over a ring R is uniserial if every submodule of M is linearly ordered by inclusion. This means that for any two submodules N and K of M, either $N \subseteq K$ or $K \subseteq N$. Uniserial modules play a significant role in module theory and provide insights into the structure of modules over rings. Some of the important properties of uniserial modules are: (1) All submodules except M and $\{0\}$ are simultaneously essential and superfluous; (2) The cyclic submodules of M are linearly ordered; (3) M is a uniform module; (4) M is indecomposable module; (5) If M is a multiplication module over a uniserial ring R, then M is a uniserial module; (6) If M has a maximal submodule, then M is a local module; (7) Image and pre-image of a uniserial module is a uniserial module. The detail study about uniserial modules and their properties can be found in [1, 13, 21] and [22].

The concept of intuitionistic fuzzy sets was introduced by K.T. Atanassov [2, 3] as a generalization of that of fuzzy sets developed by L. A. Zadeh [23], and it is a very effective tool to study the case of vagueness. Further, many researchers [7, 8, 10, 11, 15, 18], applied this notion in various branches of mathematics, especially in algebra, and defined intuitionistic fuzzy subgroups, intuitionistic fuzzy subrings (ideals), and intuitionistic fuzzy submodules, intuitionistic fuzzy essential (superfluous) submodules, and so forth. For detailed study of these concepts, we recommend the reader to follow [5]. In the present paper, we develop the notion of intuitionistic fuzzy chained modules as a generalization of the notion of chained module (or uniserial module) in crisp module theory. The intuitionistic fuzzy chained modules are then described in terms of their level-cut submodules. Further, we use the intuitionistic fuzzy cyclic submodules of an intuitionistic fuzzy chained module to characterise it. Lastly, we show that the intuitionistic fuzzy multiplication module is an intuitionistic fuzzy chained module when the underling ring is a chained ring.

2 Preliminaries

Given a non-empty set M, an intuitionistic fuzzy subset (IFS) A is an ordered function $(\mu_A, \nu_A) : M \to [0,1] \times [0,1]$ with $\mu_A(x) + \nu_A(x) \leq 1, \forall x \in M$. We denote by $IFS(M)$, the set of all IFSs of M. For $A, B \in IFS(M)$ we write $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in M$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$. Further, if $f : X \to Y$ is a mapping and A, B be respectively IFS of X and Y. Then $f(A)$ is an IFS of Y is defined as $\mu_{f(A)}(y) = \sup\{\mu_A(x) : f(x) = y\}$, $\nu_{f(A)}(y) = \inf\{\nu_A(x) : f(x) = y\}$, for all $y \in Y$ and $f^{-1}(B)$ is an IFS of X is defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$, $\forall x \in X$. An IFS A of X is called f-invariant if $A(x) = A(y)$ whenever $f(x) = f(y)$, where $x, y \in X$. By an intuitionistic fuzzy point (IFP) $x_{(p,q)}$ of M, $x \in M$, $p, q \in [0,1]$ such that $p + q \leq 1$ we mean $x_{(p,q)} \in IFS(M)$ is defined by

$$x_{(p,q)}(y) = \begin{cases} (p,q), & \text{if } y = x \\ (0,1), & \text{if otherwise.} \end{cases}$$

If $A \in IFS(M)$ and $x \in M$ such that $\mu_A(x) \geq p$ and $\nu_A(x) \leq q$, then $x_{(p,q)} \subseteq A$. We write it as $x_{(p,q)} \in A$. The intuitionistic fuzzy characteristic function of M with respect to a subset N is denoted by χ_N and is defined as:

$$\mu_{\chi_N}(y) = \begin{cases} 1, & \text{if } y \in N \\ 0, & \text{if otherwise} \end{cases}; \quad \nu_{\chi_N}(y) = \begin{cases} 0, & \text{if } y \in N \\ 1, & \text{otherwise.} \end{cases}$$

If $x = \theta$ and $p = 1, q = 0$, then $x_{(p,q)} = \theta_{(1,0)}$ (or $N = \{\theta\}$) is called the intuitionistic fuzzy zero point of M and is denoted by $\chi_{\{\theta\}}$.

Lemma 2.1. [9] *If f is a function defined on a set M, A_1 and A_2 are IFSs of M, B_1 and B_2 are IFSs of $f(M)$. Then the following are true:*

1. $A_1 = f^{-1}(f(A_1))$, *whenever A_1 is f-invariant.*
2. $f(f^{-1}(B_1)) = B_1$.
3. *If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.*
4. *If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.*

The following are three very basic definitions given in [8, 10] and [11].

Definition 2.2. [10] Let $X \in IFS(R)$. Then X is called an intuitionistic fuzzy subring of R if for all $x, y \in R$, the followings are satisfied

(i) $\mu_X(x - y) \geq \mu_X(x) \wedge \mu_X(y)$; (ii) $\nu_X(x - y) \leq \nu_X(x) \vee \nu_X(y)$;

(iii) $\mu_X(xy) \geq \mu_X(x) \wedge \mu_X(y)$; (iv) $\nu_X(xy) \leq \nu_X(x) \vee \nu_X(y)$.

Definition 2.3. [10] Let $X \in IFS(R)$. Then X is called an intuitionistic fuzzy ideal (IFI) of R if for all $x, y \in R$, the followings are satisfied

(i) $\mu_X(x - y) \geq \mu_X(x) \wedge \mu_X(y)$; (ii) $\nu_X(x - y) \leq \nu_X(x) \vee \nu_X(y)$;

(iii) $\mu_X(xy) \geq \mu_X(x) \vee \mu_X(y)$; (iv) $\nu_X(xy) \leq \nu_X(x) \wedge \nu_X(y)$.

Definition 2.4. [8, 11] Let $X \in IFS(M)$. Then X is called an intuitionistic fuzzy module (IFM) of M if for all $x, y \in M, r \in R$, the followings are satisfied

(i) $\mu_X(x - y) \geq \mu_X(x) \wedge \mu_X(y)$; (ii) $\nu_X(x - y) \leq \nu_X(x) \vee \nu_X(y)$;

(iii) $\mu_X(rx) \geq \mu_X(x)$; (iv) $\nu_X(rx) \leq \nu_X(x)$.

(v) $\mu_X(\theta) = 1$; (vi) $\nu_X(\theta) = 0$.

Clearly, $\chi_{\{\theta\}}$, χ_M are IFMs of M and these are called trivial IFMs of M. Let $IFM(M)$ denote the set of all intuitionistic fuzzy R-modules of M and $IFI(R)$ denote the set of all intuitionistic fuzzy ideals of R. We note that when $R = M$, then $X \in IFM(M)$ if and only if $\mu_X(\theta) = 1, \nu_X(\theta) = 0$ and $X \in IFI(R)$.

Let $X \in IFS(M)$ and $p, q \in [0, 1]$ with $p + q \leq 1$. Then the set $X_{(p,q)} = \{x \in M : \mu_X(x) \geq p$ and $\nu_X(x) \leq q\}$ is called the (p, q)-cut subset of M with respect to X. If $A, B \in IFS(M)$, then for each $p, q \in [0, 1]$ with $p + q \leq 1$ we have $(A \cap B)_{(p,q)} = A_{(p,q)} \cap B_{(p,q)}$ and $A = B$ if and only if $A_{(p,q)} = B_{(p,q)}$.

Definition 2.5. [14] Let X and Y be two intuitionistic fuzzy modules of R-modules M_1 and M_2 respectively, $f : X \to Y$ is called an intuitionistic fuzzy homomorphism if $f : M_1 \to M_2$ is an R-homomorphism and $Y(f(x)) = X(x)$ for each $x \in M_1$

Remark 2.6. [14] (1) Let M_1 and M_2 be two R-modules, $f : M_1 \to M_2$ is an R-epimorphism. If A is an intuitionistic fuzzy submodule of M_1, then $f(A)$ is an intuitionistic fuzzy submodule of M_2.
(2) Let M_1 and M_2 be two R-modules, $f : M_1 \to M_2$ is an R-homomorphism. If B is an intuitionistic fuzzy submodule of M_2, then $f^{-1}(B)$ is an intuitionistic fuzzy submodule of M_1.

Definition 2.7. [4,16] An IFI X of a ring R is called an intuitionistic fuzzy prime ideal if for any IFIs A, B of R such that $AB \subseteq X$ implies $A \subseteq X$ or $B \subseteq X$.

Definition 2.8. [18] Let X and A be two intuitionistic fuzzy modules of an R-module M. Then A is called an intuitionistic fuzzy submodule of X if $A \subseteq X$.

Definition 2.9. [20] An intuitionistic fuzzy module X of an R-module M is called intuitionistic fuzzy simple module if and only if X has no proper intuitionistic fuzzy submodules.

Definition 2.10. [12] An intuitionistic fuzzy module X of an R-module M is called intuitionistic fuzzy cyclic module, if there exists $x_{(s,t)} \in X$ such that each $y_{(p,q)} \in X$ written as $y_{(p,q)} = r_{(j,k)}x_{(s,t)}$ for some intuitionistic fuzzy point $r_{(j,k)} \in IFP(R)$. In this case, we shall write $X = \langle x_{(s,t)} \rangle$ to denote the intuitionistic fuzzy cyclic module generated by $x_{(s,t)}$.

Definition 2.11. [17] Suppose A and B be two intuitionistic fuzzy modules of an R-module M. We define $(A : B)$ by

$$(A : B) = \bigcup \{r_{(j,k)} : r \in R, j, k \in (0, 1] \text{ with } j + k \leq 1, r_{(j,k)}B \subseteq A\}$$

is called the residual quotient of A and B. Clearly, $(A : B) \in IFS(R)$. By ([17], Theorem (3.6)) if $A, B \in IFM(M)$, then $(A : B) \in IFI(R)$.

Definition 2.12. [15] An IFSM A of a R-module M is said to be an intuitionistic fuzzy superfluous (or small) submodule (IFSSM) of M (or of χ_M), if for any IFSM B of M, $A+B = \chi_M \Rightarrow B = \chi_M$, or equivalently, $B \neq \chi_M \Rightarrow A + B \neq \chi_M$.
We denote it by $A \ll_{IF} \chi_M$ or $A \ll_{IF} M$.

Definition 2.13. [5] An IFSM $A(\neq \chi_{\{\theta\}})$ of a R-module M is said to be an intuitionistic fuzzy essential submodule (IFESM) of M (or of χ_M) , if for any IFSM B of M, $A \cap B = \chi_{\{\theta\}} \Rightarrow B = \chi_{\{\theta\}}$, or equivalently, $B \neq \chi_{\{\theta\}} \Rightarrow A \cap B \neq \chi_{\{\theta\}}$.
We denote it by $A \sqsubseteq_{IF} \chi_M$ or $A \sqsubseteq_{IF} M$.

3 Intuitionistic fuzzy chained modules

In this section, we introduce the concepts of an intuitionistic fuzzy chained modules and examine some of their properties.

Definition 3.1. Let X be an intuitionistic fuzzy module of an R-module M. If, for every intuitionistic fuzzy submodule A, B of X, either $A \subseteq B$ or $B \subseteq A$, then X is referred to as an intuitionistic fuzzy chained module (IFCM).

Theorem 3.2. *All intuitionistic fuzzy submodules of an intuitionistic fuzzy chained module X of an R-module M, other than $\chi_{\{\theta\}}$ and χ_M are simultaneously intuitionistic fuzzy superfluous submodule as well as intuitionistic fuzzy essential submodule.*

Proof. Let $A(\neq \chi_{\{\theta\}}, \chi_M)$ be an IFSM of an intuitionistic fuzzy chained module X of an R-module M. Let B be an IFSM of X. Then either, $A \subseteq B$ or $B \subseteq A$.
Suppose that B be an IFSM of X such that $A + B = \chi_M$, then we have
Case (i) When $B \subseteq A$, then $\chi_M = A + B \subseteq A + A = A$, which is not possible.
Case (ii) When $A \subseteq B$, then $\chi_M = A + B \subseteq B + B = B$ implies that A is an IFSSM.

Now, suppose that B be an IFSM of X such that $A \cap B = \chi_{\{\theta\}}$, then we have
Case (i) When $A \subseteq B$, then $\chi_{\{\theta\}} = A \cap B = A$, which is not possible.
Case (ii) When $B \subseteq A$, then we have $\chi_{\{\theta\}} = A \cap B = B$ implies that A is an IFESM.
Hence A is simultaneously an IFSSM as well as an IFESM. □

To prove our next theorem, first we prove the following lemma:

Lemma 3.3. *Let A and B be two IFSs of R. Then $A \subseteq B$ if and only if $A_{(s,t)} \subseteq B_{(s,t)}$, for each $s, t \in [0, 1]$ with $s + t \leq 1$.*

Proof. It is easy so it is omitted. □

The following theorem characterizes intuitionistic fuzzy chained module in terms of its level-cut submodules.

Theorem 3.4. *An intuitionistic fuzzy module X of an R-module M is an IFCM if and only if $X_{(s,t)}$ is a chained module, for each $s, t \in [0, 1]$ with $s + t \leq 1$.*

Proof. Firstly, let X be an IFCM. To show that $X_{(s,t)}$ is a chained module for each $s, t \in [0, 1]$ with $s + t \leq 1$. Let N, K be submodules of $X_{(s,t)}$. Define IFSs A and B of $X_{(s,t)}$ as

$$\mu_A(x) = \begin{cases} s, & \text{if } x \in N \\ 0, & \text{otherwise} \end{cases} ; \nu_A(x) = \begin{cases} t, & \text{if } x \in N \\ 1, & \text{otherwise} \end{cases}$$

$$\mu_B(x) = \begin{cases} s, & \text{if } x \in K \\ 0, & \text{otherwise} \end{cases} ; \nu_B(x) = \begin{cases} t, & \text{if } x \in K \\ 1, & \text{otherwise} \end{cases}$$

It is easy to see that A, B are IFSMs of X with $A_{(s,t)} = N$, $B_{(s,t)} = K$. Since X is an IFCM, so we have $A \subseteq B$ or $B \subseteq A$. Hence $A_{(s,t)} \subseteq B_{(s,t)}$ or $B_{(s,t)} \subseteq A_{(s,t)}$ (by Lemma (3.3)). Thus $N \subseteq K$ or $K \subseteq N$.

Conversely, let $X_{(s,t)}$ be a chained module for each $s, t \in [0, 1]$ with $s + t \leq 1$. To show that X is an IFCM. Let A, B be two IFSMs in X. Then $A_{(s,t)}, B_{(s,t)}$ are submodules in $X_{(s,t)}$, for each $s, t \in (0, 1]$ with $s + t \leq 1$. Since $X_{(s,t)}$ is a chained R-module, therefore, $A_{(s,t)} \subseteq B_{(s,t)}$ or $B_{(s,t)} \subseteq A_{(s,t)}$ which implies $N \subseteq K$ or $K \subseteq N$ (by Lemma (3.3)). □

Example 3.5. (1) Let $M = \mathbb{Z}_8$ as a $\mathbb{Z}$-module. Let $X = \chi_M$, then for all $s, t \in (0, 1]$ with $s + t \leq 1$ we have $X_{(s,t)} = \mathbb{Z}_8$. But $\mathbb{Z}_8$ is a chained module. Hence X is an IFCM (by Theorem (3.4)).

(2) Every intuitionistic fuzzy simple module is an IFCM.

Remark 3.6. If $Y \subseteq X$ and X is an IFCM, then Y is also an IFCM.

Proof. Let A, B be two IFSMs of Y then A, B are also IFSMs of X, since X is an IFCM. Then $A \subseteq B$ or $B \subseteq A$ which imply Y is an IFCM. □

Definition 3.7. [5, 6] An intuitionistic fuzzy module X is called uniform if $A \cap B \neq \chi_{\{\theta\}}$ for any non-trivial IFSMs A and B of X. Equivalently, X is called an intuitionistic fuzzy uniform module if $A \neq \chi_{\{\theta\}}$ and for every IFSM $B \neq \chi_{\{\theta\}}$ of X is an intuitionistic fuzzy essential in X.

Proposition 3.8. *An intuitionistic fuzzy module X of an R-module M is an intuitionistic fuzzy uniform if and only if $X_{(s,t)}$ is a uniform module, for all $s, t \in (0, 1]$ with $s + t \leq 1$.*

Proof. Firstly, let X be an intuitionistic fuzzy uniform module. To prove that $X_{(s,t)}$ is a uniform module, for all $s, t \in (0, 1]$ with $s + t \leq 1$. Let N, K be submodules of $X_{(s,t)}$. Define IFSs A, B of $X_{(s,t)}$ as

$$\mu_A(x) = \begin{cases} s, & \text{if } x \in N \\ 0, & \text{otherwise} \end{cases} \quad ; \nu_A(x) = \begin{cases} t, & \text{if } x \in N \\ 1, & \text{otherwise} \end{cases}$$

$$\mu_B(x) = \begin{cases} s, & \text{if } x \in K \\ 0, & \text{otherwise} \end{cases} \quad ; \nu_B(x) = \begin{cases} t, & \text{if } x \in K \\ 1, & \text{otherwise} \end{cases}$$

Clearly A, B are intuitionistic fuzzy submodules of X. But $A_{(s,t)} = N$, $B_{(s,t)} = K$ since X is intuitionistic fuzzy uniform module, then $A \cap B \neq \chi_{\{\theta\}}$.
This imply $(A \cap B)_{(s,t)} \neq \{\theta\}$ and so $A_{(s,t)} \cap B_{(s,t)} \neq \{\theta\}$.
Thus $N \cap K \neq \{\theta\}$. Hence $X_{(s,t)}$ is a uniform module.

Conversely, let $X_{(s,t)}$ is a uniform module. To prove that X is an intuitionistic fuzzy uniform module. Suppose A, B be any two non-trivial intuitionistic fuzzy submodules of X. Then $A_{(s,t)}, B_{(s,t)}$ are submodules of $X_{(s,t)}$, for all $s, t \in (0, 1]$ with $s + t \leq 1$. Since $X_{(s,t)}$ is a uniform R-module then $A_{(s,t)} \cap B_{(s,t)} \neq \{\theta\}$ and so $(A \cap B)_{(s,t)} \neq \{\theta\}$. Thus $A \cap B \neq \chi_{\{\theta\}}$. Hence X is an intuitionistic fuzzy uniform module. □

Now, we establish a relationship between an intuitionistic fuzzy uniform module and an intuitionistic fuzzy chained module in the following proposition:

Proposition 3.9. *Every intuitionistic fuzzy chained module is an intuitionistic fuzzy uniform module.*

Proof. Let X be an IFCM of an R-module M. Let A, B be any two non-trivial IFSMs of X, then $A \subseteq B$ or $B \subseteq A$.
If $A \subseteq B$ then $A \cap B = A$ and if $B \subseteq A$ then $A \cap B = B$ which implies $A \cap B \neq \chi_{\{\theta\}}$. Hence X is an intuitionistic fuzzy uniform module. □

Remark 3.10. The converse of Proposition (3.9) is not true, see the following example.

Example 3.11. Let $M = \mathbb{Z}$ as a $\mathbb{Z}$-module and $X = \chi_M$ so that $X_{(s,t)} = \mathbb{Z}$ for all $s, t \in (0, 1]$ with $s + t \leq 1$. But $\mathbb{Z}$ is a uniform module. Therefore by Proposition (3.8) X is an intuitionistic fuzzy uniform module. But X is not an IFCM since there exists two IFSMs A, B of X defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in \langle 2 \rangle \\ 0.25, & \text{otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in \langle 2 \rangle \\ 0.5, & \text{otherwise} \end{cases}$$

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in \langle 5 \rangle \\ 0,25, & \text{otherwise} \end{cases} \quad ; \quad \nu_B(x) = \begin{cases} t, & \text{if } x \in \langle 5 \rangle \\ 0.5, & \text{otherwise} \end{cases}$$

such that $A \nsubseteq B$ and $B \nsubseteq A$.

Recall that if N and K are two submodules of an R-module M, then N and K are called comparable if $N \subseteq K$ or $K \subseteq N$.

We shall intuitionistic fuzzify this concept as follows:

Definition 3.12. Let A, B be two IFSMs of an intuitionistic fuzzy module X of an R-module M, then A and B are called comparable if $A \subseteq B$ or $B \subseteq A$.

Proposition 3.13. *An intuitionistic fuzzy module X of an R-module M is chained if and only if every two intuitionistic fuzzy cyclic submodules of X are comparable.*

Proof. Let A and B be two IFSMs of X. Suppose that $A \nsubseteq B$, we show that $B \subseteq A$. Since $A \nsubseteq B$, there exists $x_{(s,t)} \in A$ and $x_{(s,t)} \notin B$ and so $\langle x_{(s,t)} \rangle \subseteq A$ and $\langle x_{(s,t)} \rangle \nsubseteq B$. Let $y_{(p,q)} \in B$, then $\langle y_{(p,q)} \rangle \subseteq B$. Therefore $\langle x_{(s,t)} \rangle, \langle y_{(p,q)} \rangle$ are two intuitionistic fuzzy cyclic submodules of X, then either

$$\langle x_{(s,t)} \rangle \subseteq \langle y_{(p,q)} \rangle \text{ or } \langle y_{(p,q)} \rangle \subseteq \langle x_{(s,t)} \rangle.$$

If $\langle x_{(s,t)} \rangle \subseteq \langle y_{(p,q)} \rangle$, then $\langle x_{(s,t)} \rangle \subseteq B$ (since $\langle y_{(p,q)} \rangle \subseteq B$). Therefore $x_{(s,t)} \in B$, which is a contradiction.

If $\langle y_{(p,q)} \rangle \subseteq \langle x_{(s,t)} \rangle$, then $\langle y_{(p,q)} \rangle \subseteq A$ (since $\langle x_{(s,t)} \rangle \subseteq A$). Thus $B \subseteq A$ so X is an IFCM.

The converse is obvious. □

Remark 3.14. An IFCM is indecomposable.

Proof. If possible, let X be an IFCM and is decomposable. Then we have $X = A \oplus B$ for some non-trivial IFSMs A and B of X. So $A \cap B = \chi_{\{\theta\}}$ which is a contradiction to the fact that X is an intuitionistic fuzzy uniform module (by Proposition (3.9)). □

Next, we introduce the notion of intuitionistic fuzzy chained ring.

Definition 3.15. An intuitionistic fuzzy ring X of a ring R is called chained if and only if for each intuitionistic fuzzy ideals I, J of X either $I \subseteq J$ or $J \subseteq I$.

Remark 3.16. An intuitionistic fuzzy ring X is chained if and only if $X_{(s,t)}$ is chained ring for all $s, t \in (0, 1]$ with $s + t \leq 1$.

Proof. It is easy so it is omitted. □

Definition 3.17. [19] An intuitionistic fuzzy module X of an R-module M is called multiplication module if for each intuitionistic fuzzy submodule A of X, there exists a intuitionistic fuzzy ideal I of R such that $A = IX$.

Proposition 3.18. *Let X be an intuitionistic fuzzy multiplication module of an R-module M. If R is a chained ring, then X is an intuitionistic fuzzy chained module.*

Proof. Let A and B be two IFSMs of X. Then there exists IFIs I and J of R such that $A = IX$ and $B = JX$, since $I_{(s,t)}$ and $J_{(s,t)}$ are ideals of R and R is chained, therefore $I_{(s,t)} \subseteq J_{(s,t)}$ or $J_{(s,t)} \subseteq I_{(s,t)}$. Thus $I \subseteq J$ or $J \subseteq I$ (by Remark (3.3)) implies that $IX \subseteq JX$ or $JX \subseteq IX$. Thus $A \subseteq B$ or $B \subseteq A$. Hence X is an IFCM. □

Definition 3.19. [17] Let X be an IFCM of an R-module M and let

$$V(X) = \{(\chi_{\{\theta\}} : x_{(s,t)}) | x_{(s,t)} \in X\}$$

be the collection of IFIs of R.

Definition 3.20. [17] Let X be a non-constant intuitionistic fuzzy module of an R-module M. If A be any proper intuitionistic fuzzy submodule of X. Then the annihilator of A is denoted by $\operatorname{ann}(A)$ and is defined as:

$$\operatorname{ann}(A) = \bigcup\{r_{(j,k)} : r \in R, j, k \in (0, 1] \text{ with } j + k \leq 1, r_{(j,k)}A \subseteq \chi_{\{\theta\}}\}$$

Note that for any $r \in R$, we have $\mu_{\operatorname{ann}(A)}(r) = \sup\{j : j \in (0, 1], r_{(j,k)}A \subseteq \chi_{\{\theta\}}\}$ and $\nu_{\operatorname{ann}(A)}(r) = \inf\{k : k \in (0, 1], r_{(j,k)}A \subseteq \chi_{\{\theta\}}\}$. In other words, $\operatorname{ann}(A) = (\chi_{\{\theta\}} : A)$.

Definition 3.21. [17] An intuitionistic fuzzy module X is called faithful if

$$\operatorname{ann}(X) = \chi_{\{0\}}.$$

Remark 3.22. If X is an intuitionistic fuzzy faithful chained module, then

$$\bigcap\nolimits_{\chi_{\{\theta\}} \neq x_{(s,t)} \in X} (\chi_{\{\theta\}} : x_{(s,t)}) = \chi_{\{0\}}.$$

Proof. If possible, let $\bigcap_{\chi_{\{\theta\}} \neq x_{(s,t)} \in X} (\chi_{\{\theta\}} : x_{(s,t)}) \neq \chi_{\{0\}}$, then there exist $\chi_{\{0\}} \neq r_{(j,k)} \in IFP(R)$ such that $r_{(j,k)}x_{(s,t)} = \chi_{\{\theta\}}$, $\forall x_{(s,t)} \in IFP(X)$. Then $r_{(j,k)}X = \chi_{\{\theta\}}$, i.e., $r_{(j,k)} \in \operatorname{ann}(X)$, a contradiction (as $\operatorname{ann}(X) = \chi_{\{0\}}$) . □

Theorem 3.23. *If X is an IFCM of an R-module M, then*

(1) $V(X)$ is a linearly ordered set of IFIs of R.

(2) $P = \bigcup_{\chi_{\{\theta\}} \neq x_{(s,t)} \in X} (\chi_{\{\theta\}} : x_{(s,t)})$ is an intuitionistic fuzzy prime ideal of R.

Proof. **(i)**. Let $A = (\chi_{\{\theta\}} : x_{(s,t)})$ and $B = (\chi_{\{\theta\}} : y_{(p,q)})$ be any two members of $V(X)$, where $x_{(s,t)}, y_{(p,q)} \in X$ be such that $x_{(s,t)}, y_{(p,q)} \neq \chi_{\{\theta\}}$. Since X is an intuitionistic fuzzy chained module, therefore $X_{(s,t)}$ is chained module (by theorem (3.4)) implies that $V(X_{(s,t)})$ is a linearly ordered set of ideals of R (see [13], Remark (1.9)). Thus $V(X)$ is a linearly ordered set of intuitionistic fuzzy ideals of R.

(ii). $P = \bigcup_{\chi_{\{\theta\}} \neq x_{(s,t)} \in X} (\chi_{\{\theta\}} : x_{(s,t)})$ is an intuitionistic fuzzy ideal of R. To show that P is an intuitionistic fuzzy prime ideal, let $a_{(s,t)}, b_{(p,q)} \in IFP(R)$ such that $a_{(s,t)}b_{(p,q)} \in P$, then there is $\chi_{\{\theta\}} \neq x_{(u,v)} \in X$ such that $a_{(s,t)}b_{(p,q)} \in (\chi_{\{\theta\}} : x_{(u,v)})$ so $a_{(s,t)}b_{(p,q)}x_{(u,v)} = \chi_{\{\theta\}}$. This implies that $a_{(s,t)} \in (\chi_{\{\theta\}} : b_{(p,q)}x_{(u,v)})$.

Now, if $b_{(p,q)}x_{(u,v)} = \chi_{\{\theta\}}$, then $b_{(p,q)} \in (\chi_{\{\theta\}} : x_{(u,v)})$. Thus $b_{(p,q)} \in P$.
If $b_{(p,q)}x_{(u,v)} \neq \chi_{\{\theta\}}$, then $a_{(s,t)} \in (\chi_{\{\theta\}} : b_{(p,q)}x_{(u,v)})$, and so $a_{(s,t)} \in P$.
Hence P is an intuitionistic fuzzy prime ideal of R. □

4 Direct sum of intuitionistic fuzzy chained modules

We turn our attention to the direct sum of intuitionistic fuzzy chained modules.

Remark 4.1. If X and Y are two IFCMs of an R-module M_1 and M_2 respectively, then $X \oplus Y$ is not necessary an IFCM of $M_1 \oplus M_2$ as the following example shows:

Example 4.2. Let $M = \mathbb{Z}_6$ be a $\mathbb{Z}$-module. Suppose $X, Y \in IFS(M)$ defined by

$$\mu_X(a) = \begin{cases} 0.3, & \text{if } a \in \langle 2 \rangle \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_X(a) = \begin{cases} 0, & \text{if } a \in \langle 2 \rangle \\ 1, & \text{otherwise} \end{cases}$$

$$\mu_Y(a) = \begin{cases} 0.3, & \text{if } a \in \langle 3 \rangle \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_Y(a) = \begin{cases} 0, & \text{if } a \in \langle 3 \rangle \\ 1, & \text{otherwise} \end{cases}$$

It is clear that X and Y are IFCMs of Z_6. But $X \oplus Y$ is not an IFCM of $\mathbb{Z}_6 \oplus \mathbb{Z}_6$. Since there exist IFSMs A, B of $X \oplus Y$ defined by

$$\mu_A(a,b) = \begin{cases} 0.3, & \text{if } (a,b) \in \langle 2 \rangle \oplus \langle 0 \rangle \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_A(a,b) = \begin{cases} 0, & \text{if } (a,b) \in \langle 2 \rangle \oplus \langle 0 \rangle \\ 1, & \text{otherwise} \end{cases}$$

$$\mu_B(a,b) = \begin{cases} 0.3, & \text{if } (a,b) \in \langle 0 \rangle \oplus \langle 3 \rangle \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_B(a,b) = \begin{cases} 0, & \text{if } (a,b) \in \langle 0 \rangle \oplus \langle 3 \rangle \\ 1, & \text{otherwise} \end{cases}$$

Here we checked that $A(2,0) = 0.3$, $B(2,0) = 0$, that is $A \nsubseteq B$. Also $A(0,3) = 0$, $B(0,3) = 0.3$, that is $B \nsubseteq A$. Thus $X \oplus Y$ is not an IFCM of $\mathbb{Z}_6 \oplus \mathbb{Z}_6$.

Theorem 4.3. *Let X and Y be intuitionistic fuzzy modules of the R-modules M_1 and M_2, respectively. If $X \oplus Y$ is an IFCM of $M_1 \oplus M_2$, then X is an IFCM of M_1 and Y is an IFCM of M_2.*

Proof. Let A be an IFSM of an IFCM $X \oplus Y$ of an R-module $M_1 \oplus M_2$. Define IFSs $A_1 \in IFS(M_1)$ and $B_1 \in IFS(M_2)$ by

$$A_1(x) = A((x,0)) \text{ and } B_1(y) = A((0,y)), \text{ for all } x \in M_1 \text{ and } y \in M_2.$$

It is easy to verify that A_1 and B_1 are IFSMs of X and Y, respectively.
Also, for each $(x,y) \in M_1 \oplus M_2$, we have

$$\begin{aligned} \mu_A((x,y)) &= \mu_A[(x,0)+(0,y)]) \\ &= \mu_A((x,0)) \wedge \mu_A((0,y)) \\ &= \mu_{A_1}(x) \wedge \mu_{B_1}(y) \\ &= \mu_{A_1 \oplus B_1}((x,y)). \end{aligned}$$

Similarly, we can show that $\nu_A((x,y)) = \nu_{A_1 \oplus B_1}((x,y))$. Hence, $A = A_1 \oplus B_1$. Thus, we see that if A, B are two IFSMs of $X \oplus Y$, then there exists IFSMs A_1, A_2 of X and IFSMs B_1, B_2 of Y such that $A = A_1 \oplus B_1$ and $B = A_2 \oplus B_2$. As $X \oplus Y$ is an IFCM, so either $A \subseteq B$ or $B \subseteq A$. This imply that either $A_1 \oplus B_1 \subseteq A_2 \oplus B_2$ or $A_2 \oplus B_2 \subseteq A_1 \oplus B_1$ which further implies either $A_1 \subseteq A_2$, $B_1 \subseteq B_2$ or $A_2 \subseteq A_1$, $B_2 \subseteq B_1$. Thus X and Y are also IFCMs. □

Next, we shall identify the behaviors of IFCMs under homomorphism.

Theorem 4.4. *Let X and Y be intuitionistic fuzzy modules of an R-modules M_1 and M_2, respectively. Let $f : X \to Y$ be an intuitionistic fuzzy epimorphism. If X is an IFCM, then Y is also an IFCM.*

Proof. Let A, B are IFSMs of Y. Then $f^{-1}(A), f^{-1}(B)$ are IFSMs of X (Remark (2.6), (2)), since X is an IFCM, then either $f^{-1}(A) \subseteq f^{-1}(B)$ or $f^{-1}(B) \subseteq f^{-1}(A)$.

If $f^{-1}(A) \subseteq f^{-1}(B)$ then $f(f^{-1}(A)) \subseteq f(f^{-1}(B))$ (by Lemma (2.1), (2)) implies $A \subseteq B$. Similarly, if $f^{-1}(B) \subseteq f^{-1}(A)$, then $B \subseteq A$. Hence Y is an IFCM. □

Theorem 4.5. *Let X and Y be intuitionistic fuzzy modules of an R-modules M_1 and M_2, respectively. Let $f : X \to Y$ be an intuitionistic fuzzy homomorphism such that every IFSs of Y is f-invariant. If Y is an IFCM, then X is an IFCM.*

Proof. Let A, B are IFSMs of X. Hence $f(A), f(B)$ are IFSMs of Y (Remark (2.6), (1)), since Y is an IFCM then $f(A) \subseteq f(B)$ or $f(B) \subseteq f(A)$.

Now if $f(A) \subseteq f(B)$ then $f^{-1}(f(A)) \subseteq f^{-1}(f(B))$ (by Lemma (2.1), (4)) implies $A \subseteq B$. Similarly, if $f(B) \subseteq f(A)$, then $B \subseteq A$. Hence X is an IFCM. □

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