Notes on Intuitionistic Fuzzy Sets
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On intuitionistic fuzzy implications

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Abstract: In this paper we conduct a systematic algebraic study on the set $\mathbb{I}$ of all intuitionistic fuzzy implications. To this end, we propose a binary operation, denoted by $*$,which makes a $(\mathbb{I}, *)$ a monoid.we determine the largest subgroup $\mathfrak{K}$ of this monoid and using its representation define a group action of $\mathfrak{K}$ that partitions $\mathbb{I}$ into equivalence classes. Also we give novel way of generating newer fuzzy implications from given ones by a bijective transformations.
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1 Introduction

Study in intuitionistic fuzzy subsets and application of intuitionistic fuzzy control have been developed quickly since the definition of intuitionistic fuzzy sets was introduced by Atanassov in 1983. **IFSS** theory basically defies the claim that from the fact that an element x "belongs" to a given degree (say μ) to a fuzzy set A, naturally follows that x should "not belong" to A to the extent $1 - \mu$, an assertion implicit in the concept of a fuzzy set. On the contrary, **IFSS** assign to each element of the universe both a degree of membership μ and one of non-membership ν such that $\mu + \nu \leq 1$, thus relaxing the enforced duality $\nu = 1 - \mu$ from fuzzy set theory. Obviously, when $\mu + \nu = 1$ for all elements of the universe, the traditional fuzzy set concept is recovered.

Technology of intuitionistic fuzzy control has been applied to many fields including medical field [7, 8, 9]. But the basic theory of intuitionistic fuzzy control is inferior to its application, especially the theory of intuitionistic fuzzy reasoning. Since Zadeh [10] introduced the compositional rule of inference (**CRI**), many researchers have take advantage of fuzzy implication operators to represent the relation between two variables linked together by means of an $if - then$ rule. In intuitionistic fuzzy reasoning theory, intuitionistic fuzzy implication operators play the same important role.

This paper is organized as follows. In Section 2 we propose a binary operation $*$ on the set of all intuitionistic fuzzy implication $\mathbb{I}$ that makes $(\mathbb{I}, *)$ a monoid. This is the first work in which such a rich structure has been obtained on the entire set of intuitionistic fuzzy implications $\mathbb{I}$ In Section 3 We characterize the largest such subgroup $\mathcal{K}$ and, based on their representation, propose a group action of $\mathcal{K}$ on $\mathbb{I}$. Clearly, this group action partitions $\mathbb{I}$ into equivalence classes. And in Section 4 we propose a new method for the construction of new intuitionistic fuzzy implications. Finally we draw conclusions and indicate future lines of research.

2 Preliminaries

First we give the concept of intuitionistic fuzzy set defined by Atanassov and we recall some elementary definitions that we use in the sequel. Assume that X is the universe.

Definition 1 ([1, 2]). *The intuitionistic fuzzy subsets (in shorts **IFSS**) defined on a non-empty set X as objects having the form*

$$A = \{\langle x, \mu(x), \nu(x)\rangle : x \in X\}$$

where the functions $\mu : X \to [0,1]$ and $\nu : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively, and $0 \leq \mu(x) + \nu(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $\langle \mu, \nu\rangle$ for the intuitionistic fuzzy subset $A = \{\langle x, \mu(x), \nu(x)\rangle : x \in X\}$.

Definition 2 ([2]). *Let $A = \langle \mu_A, \nu_A\rangle$ and $B = \langle \mu_B, \nu_B\rangle$ IFSS of X. Then*
$A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$
$A = B$ iff $A \subset B$ and $B \subset A$
$A^c = \langle \nu_A, \mu_A\rangle$
$A \cap B = \langle \mu_A \wedge \mu_B, \nu_A \vee \nu_B\rangle$
$A \cup B = \langle \mu_A \vee \mu_B, \nu_A \wedge \nu_B\rangle$
$\Box A = \langle \mu_A, 1 - \mu_A\rangle$, $\Diamond A = \langle 1 - \nu_A, \nu_A\rangle$

We recall from [5] that $L^* = \{\tilde{x} = (x_1, x_2)/x_1 + x_2 \leq 1\}$ is a complete lattice with the order defined by

$$\tilde{x} \geq \tilde{y} \quad \textit{if and only if} \quad x_1 \geq y_1 \quad \textit{and} \quad x_2 \leq y_2$$

Now we recall the definition of intuitionistic fuzzy implication operator given by Atanassov and Gargov [3].

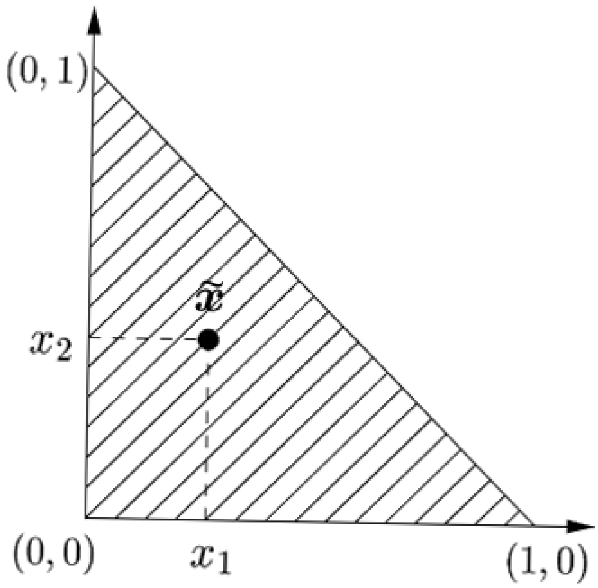

Figure 1: Graphical representation of the set L^*

Definition 3. *An intuitionistic fuzzy implication operator (IFIO) is any $I : L^{*2} \longrightarrow L^*$ mapping satisfying the border conditions:*
$I((0,1),(0,1)) = (1,0)$*;* $I((0,1),(1,0)) = (1,0)$
$I((1,0),(1,0)) = (1,0)$*;* $I((1,0),(0,1)) = (0,1)$
and the two following conditions:

1) *If $\tilde{x} \leq \tilde{y}$, then $\forall \tilde{z} \in L^* I(\tilde{x},\tilde{z}) \geq I(\tilde{y},\tilde{z})$*

2) *If $\tilde{y} \leq \tilde{z}$, then $\forall \tilde{x} \in L^* I(\tilde{x},\tilde{y}) \leq I(\tilde{x},\tilde{z})$*

Definition 4 ([6])**.** *If $(X,*)$ is a mathematical system such that $\forall$ a, b, c $\in$ X, $(a*b)*c = a*(b*c)$, then $*$ is called associative and $(X,*)$ is called a semigroup.*

3 Monoid structure on the set of all intuitionistic fuzzy implications

Let $\mathbb{I}$ be the set of all intuitionistic fuzzy implications. In this section, we begin by proposing a binary operation $*$ on the set $\mathbb{I}$ of all intuitionistic fuzzy implications and show that $(\mathbb{I},*)$ forms a monoid and discuss the properties preserved under this operation.

Definition 5. *For any two intuitionistic fuzzy implications I, J we define $I * J$: $L^{*2} \longrightarrow L^*$ as $(I * J)(\tilde{x},\tilde{y}) = I(\tilde{x}, J(\tilde{x},\tilde{y}))$,$\tilde{x}$,$\tilde{y} \in L^*$.*

The following result shows that $I * J$ is, indeed, an intuitionistic fuzzy implication.

Theorem 1. *$I * J$ is an intuitionistic fuzzy implication, i.e., $I * J \in \mathbb{I}$.*

Proof. **(i)** Let $\tilde{x_1}, \tilde{x_2}, \tilde{y} \in L^*$ be such that $\tilde{x_1} \geq \tilde{x_2}$. Then $J(\tilde{x_1}, \tilde{y}) \leq J(\tilde{x_2}, \tilde{y})$.
Then $(I * J)(\tilde{x_1}, \tilde{y}) = I(\tilde{x_1}, J(\tilde{x_1}, \tilde{y})) \leq I(\tilde{x_2}, J(\tilde{x_2}, \tilde{y})) = (I * J)(\tilde{x_2}, \tilde{y})$.
Then $I * J$ is decreasing for the first variable. Similarly one can show that $I * J$ is increasing in the second variable.

(ii) $(I * J)(\tilde{0}, \tilde{0}) = I(\tilde{0}, J(\tilde{0}, \tilde{0})) = I(\tilde{0}, \tilde{1}) = \tilde{1}$, $(I * J)(\tilde{1}, \tilde{1}) = I(\tilde{1}, J(\tilde{1}, \tilde{1})) = I(\tilde{1}, \tilde{1}) = \tilde{1}$, $(I * J)(\tilde{1}, \tilde{0}) = I(\tilde{1}, J(\tilde{1}, \tilde{0})) = I(\tilde{1}, \tilde{0}) = \tilde{0}$.

□

Theorem 2. $(\mathbb{I}, *)$ *forms a monoid, whose identity element is given by*

$$I_D(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \tilde{y} & if \quad \tilde{x} \neq \tilde{0} \end{cases}$$

Proof. From the previous theorem $*$ is a binary closed operation on the set $\mathbb{I}$. For associativity of $*$, let $I, J, K \in \mathbb{I}$ and $\tilde{x}, \tilde{y} \in L^*$. Then

$$\begin{aligned} (I * (J * K))(\tilde{x}, \tilde{y}) &= I(\tilde{x}, (J * K)(\tilde{x}, \tilde{y})) \\ &= I(\tilde{x}, (J(\tilde{x}, K(\tilde{x}, \tilde{y})))) \\ &= (I * J)(\tilde{x}, K(\tilde{x}, \tilde{y})) \\ &= ((I * J) * K)(\tilde{x}, \tilde{y}) \end{aligned}$$

Further,

$$\begin{aligned} (I * I_D)(\tilde{x}, \tilde{y}) &= I(\tilde{x}, I_D(\tilde{x}, \tilde{y})) \\ &= \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ I(\tilde{x}, \tilde{y}) & if \quad \tilde{x} \neq \tilde{0} \end{cases} \\ &= I(\tilde{x}, \tilde{y}) \end{aligned}$$

Similarly $I_D * I = I$ then I_D becomes the identity element in $\mathbb{I}$. □

Remark 1. $(\mathbb{I}, *)$ *is not a group. Indeed, take a*

$$I_1(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{y} & if \quad \tilde{x} = \tilde{1} \\ \tilde{1} & otherwise \end{cases}$$

and we have $I * I_1 = I_1$ for all $I \in \mathbb{I}$. Thus there does not exist any $J \in \mathbb{I}$ such that $J * I_1 = I_D$.

Lemma 1. *Let $I \in \mathbb{I}$; then I is invertible w.r.t $*$ if and only if there exists a unique $J \in \mathbb{I}$ such that for any $\tilde{x}, \tilde{y} \in L^*$ with $\tilde{x} \neq \tilde{0}$, $I(\tilde{x}, J(\tilde{x}, \tilde{y})) = \tilde{y} = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$*

Proof. Let I be an invertible element w.r.t $*$, i.e., there exists a unique $J \in \mathbb{I}$ such that $I * J = I_D = J * I$. In other words,
$I(\tilde{x}, J(\tilde{x}, \tilde{y})) = I_D(\tilde{x}, \tilde{y}) = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$, $\tilde{x}, \tilde{y} \in L^*$.
But for $\tilde{x} \neq \tilde{0}$ we have $I_D(\tilde{x}, \tilde{y}) = \tilde{y}$ thus for $\tilde{x} \neq \tilde{0}$, $I(\tilde{x}, J(\tilde{x}, \tilde{y})) = \tilde{y} = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$.

Conversely, assume that there exists a unique $J \in \mathbb{I}$ such that for $\tilde{x} \neq \tilde{0} \quad I(\tilde{x}, J(\tilde{x}, \tilde{y})) = I_D(\tilde{x}, \tilde{y}) = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$.
Since $I, J \in \mathbb{I}$ and $I * J, J * I \in \mathbb{I}$ we have $I(\tilde{x}, J(\tilde{x}, \tilde{y})) = I_D(\tilde{x}, \tilde{y}) = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$. Then I is invertible w.r.t $*$. □

Theorem 3 ([4]). *A function $\varphi : L^* \longrightarrow L^*$ is a continuous increasing bijection if, and only if, there exists a continuous increasing bijection $\lambda : [0,1] \longrightarrow [0,1]$ such that $\varphi(x) = (\lambda(x_1), 1 - \lambda(1 - x_2))$.*

Theorem 4. *The solutions of $I(\tilde{x}, J(\tilde{x}, \tilde{y})) = \tilde{y} = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$ are of the forms $I(\tilde{x}, , \tilde{y}) = \varphi(\tilde{y})$ and $J(\tilde{x}, \tilde{y}) = \varphi^{-1}(\tilde{y})$ for some continuous increasing bijection φ*

Proof. Let I and $J \in \mathbb{I}$ such that $I(\tilde{x}, J(\tilde{x}, \tilde{y})) = \tilde{y} = J(\tilde{x}, I(\tilde{x}, \tilde{y}))$ for all $\tilde{x} \neq \tilde{0}$ and $\tilde{y} \in L^*$. Let $\tilde{x} \neq \tilde{0}$ be fixed arbitrary and define two functions $\varphi_{\tilde{x_0}}, \psi_{\tilde{x_0}} : L^* \longrightarrow L^*$ as $\varphi_{\tilde{x_0}}(\tilde{y}) = I(\tilde{x_0}, \tilde{y})$ and $\psi_{\tilde{x_0}}(\tilde{y}) = J(\tilde{x_0}, \tilde{y})$. Clearly, both $\varphi_{\tilde{x_0}}, \psi_{\tilde{x_0}}$ are increasing function on L^*. Then $I(\tilde{x_0}, J(\tilde{x_0}, \tilde{y})) = \varphi_{\tilde{x_0}}(\psi_{\tilde{x_0}}(\tilde{y})) = (\varphi_{\tilde{x_0}} \circ \psi_{\tilde{x_0}})(\tilde{y}) = \tilde{y}$ for all $\tilde{y} \in L^*$. Similarly, $J(\tilde{x_0}, I(\tilde{x_0}, \tilde{y})) = \psi_{\tilde{x_0}}(\varphi_{\tilde{x_0}}(\tilde{y})) = (\psi_{\tilde{x_0}} \circ \varphi_{\tilde{x_0}})(\tilde{y}) = \tilde{y}$ for every $\tilde{y} \in L^*$. Thus $\psi_{\tilde{x_0}} = \varphi_{\tilde{x_0}}^{-1}$ and $\psi_{\tilde{x_0}}$ is a bijection. Hence $\psi_{\tilde{x_0}}$ increasing bijection on L^* for every $\tilde{x_0} \neq \tilde{0}$.
Since $\tilde{x_0}$ is chosen arbitrarily, $\psi_{\tilde{x}} = \varphi_{\tilde{x}}^{-1}$ for all $\tilde{x} \neq \tilde{0}$ Thus for $\tilde{x} \neq \tilde{0}$ $I(\tilde{x}, \tilde{y}) = \psi_{\tilde{x}}(\tilde{y})$ and $J(\tilde{x}, \tilde{y}) = \psi_{\tilde{x}}^{-1}(\tilde{y})$.
Let $\tilde{x_1}, \tilde{x_2}$ not null such that $\tilde{x_1} \leq \tilde{x_2}$. Then $I(\tilde{x_1}, \tilde{y}) \leq I(\tilde{x_2}, \tilde{y})$ implies that $\psi_{\tilde{x_1}}(\tilde{y}) \leq \psi_{\tilde{x_2}}(\tilde{y})$ and $\psi_{\tilde{x_1}}^{-1}(\tilde{y}) \leq \psi_{\tilde{x_2}}^{-1}(\tilde{y})$ for all $\tilde{y} \in L^*$. And we have

$$\begin{aligned}
\psi_{\tilde{x_1}}^{-1} \leq \psi_{\tilde{x_2}}^{-1} &\Longrightarrow \psi_{\tilde{x_1}} \circ \psi_{\tilde{x_1}}^{-1} \leq \psi_{\tilde{x_1}} \circ \psi_{\tilde{x_2}}^{-1} \\
&\Longrightarrow \textbf{id} \leq \psi_{\tilde{x_1}} \circ \psi_{\tilde{x_2}}^{-1} \\
&\Longrightarrow \textbf{id} \leq \psi_{\tilde{x_1}} \circ \psi_{\tilde{x_2}}^{-1} \leq \psi_{\tilde{x_2}} \circ \psi_{\tilde{x_2}}^{-1} \\
&\Longrightarrow \textbf{id} \leq \psi_{\tilde{x_1}} \circ \psi_{\tilde{x_2}}^{-1} \leq \textbf{id}
\end{aligned}$$

Hence $\psi_{\tilde{x_1}} \circ \psi_{\tilde{x_2}}^{-1} \equiv \textbf{id}$ i.e $\psi_{\tilde{x_1}}(\tilde{y}) = \psi_{\tilde{x_2}}(\tilde{y})$ for all $\tilde{y} \in L^*$ Since $\tilde{x_1}$ and $\tilde{x_1}$ are arbitrarily chosen $\psi_{\tilde{x_1}} \equiv \psi_{\tilde{x_2}}$. Thus $I(\tilde{x}, \tilde{y}) = \psi(\tilde{y})$ and $J(\tilde{x}, \tilde{y}) = \psi^{-1}(\tilde{y})$ for some increasing bijection on L^*. □

Then from the obvious theorems we have the following result

Theorem 5. *$I \in \mathbb{I}$ is invertible w.r.t $*$ if and only if*

$$I(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi(\tilde{y}) & otherwise \end{cases}$$

where the function $\varphi : L^* \longrightarrow L^*$ is an increasing bijection

Let $\mathcal{K}$ the largest subgroup of the monoid $\mathbb{I}$
Now we propose yet another new generating method of intuitionistic fuzzy implications from intuitionistic fuzzy implications and show that this method imposes a semigroup structure on the set $\mathbb{I}$.

4 Semigroup structure on $\mathbb{I}$

Definition 6. *Let $I, J \in I$. Define $I \rhd J : L^{*2} \longrightarrow L^*$ as follows: $(I \rhd J)(\tilde{x}, \tilde{y}) = I(J(\tilde{1}, \tilde{x}), J(\tilde{x}, \tilde{y})), \tilde{x}, \tilde{y} \in L^*$.*

Theorem 6. *We have $I \rhd J$ is an intuitionistic fuzzy implication. i.e., $I \rhd J \in \mathbb{I}$.*

Proof. Let $I, J \in \mathbb{I}$ and $\tilde{x_1}, \tilde{x_2}, \tilde{y} \in L^*$.
Let $\tilde{x_1} \leq \tilde{x_2}$. Then $J(\tilde{x_1}, \tilde{y}) \geq J(\tilde{x_2}, \tilde{y})$ and $J(1, \tilde{x_1}) \leq J(1, \tilde{x_2})$

$$\begin{aligned}(I \rhd J)(\tilde{x_1}, \tilde{y}) &= I(J(\tilde{1}, \tilde{x_1}), J(\tilde{x_1}, \tilde{y})) \geq I(J(\tilde{1}, \tilde{x_1}), J(\tilde{x_2}, \tilde{y})) \\ &\geq I(J(\tilde{1}, \tilde{x_2}), J(\tilde{x_2}, \tilde{y})) = (I \rhd J)(\tilde{x_2}, \tilde{y})\end{aligned}$$

Thus $\rhd$ is decreasing in the first variable. Similarly, one can show that $\rhd$ is increasing in the second variable. Now we have
$(I \rhd J)(\tilde{0}, \tilde{0}) = I(J(\tilde{1}, \tilde{0}), J(\tilde{0}, \tilde{0})) = I(\tilde{0}, \tilde{1}) = \tilde{1}$.
$(I \rhd J)(\tilde{1}, \tilde{1}) = I(J(\tilde{1}, \tilde{1}), J(\tilde{1}, \tilde{1})) = I(\tilde{1}, \tilde{1}) = \tilde{1}$
$(I \rhd J)(\tilde{1}, \tilde{0}) = I(J(\tilde{1}, \tilde{1}), J(\tilde{1}, \tilde{0})) = I(\tilde{1}, \tilde{0}) = \tilde{0}$
Hence $I \rhd J$ is an intuitionistic fuzzy implication. □

Theorem 7. $(\mathbb{I}, \rhd)$ *is a semigroup.*

Proof. from the obvious theorem $\rhd$ is a binary operation on $\mathbb{I}$. Then it is enough to show that $\rhd$ is associative in **I**. Let $I, J, T \in \mathbb{I}$ and $\tilde{x}, \tilde{y} \in L^*$.
We have

$$\begin{aligned}(I \rhd (J \rhd T))(\tilde{x}, \tilde{y}) &= I((J \rhd T)(\tilde{1}, \tilde{x}), (J \rhd T)(\tilde{x}, \tilde{y})) \\ &= I(J(T(\tilde{1}, \tilde{1}), T(\tilde{1}, \tilde{x})), J(T(\tilde{1}, \tilde{x}), T(\tilde{x}, \tilde{y}))) \\ &= I(J(\tilde{1}, T(\tilde{1}, \tilde{x})), J(T(\tilde{1}, \tilde{x}), T(\tilde{x}, \tilde{y}))) \\ and, ((I \rhd J) \rhd T)(\tilde{x}, \tilde{y}) &= (I \rhd J)(T(\tilde{1}, \tilde{x}), T(\tilde{x}, \tilde{y})) \\ &= I(J(\tilde{1}, T(\tilde{1}, \tilde{x})), J(T(\tilde{1}, \tilde{x}), T(\tilde{x}, \tilde{y}))).\end{aligned}$$

Then $\rhd$ is associative in $\mathbb{I}$ and $(\mathbb{I}, \rhd)$ is a semigroup. □

Theorem 8. *Let $I, J \in \mathcal{K}$. Then $I \rhd J = I * J$.*

Proof. Let $I, J \in \mathcal{K}$ i.e., for some $\varphi, \psi \in \Theta$,
$I(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi(\tilde{y}) & otherwise \end{cases}$
and $J(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \psi(\tilde{y}) & otherwise \end{cases}$
Now we have

$$\begin{aligned}(I \rhd J)(\tilde{x}, \tilde{y}) &= I(J(\tilde{1}, \tilde{x}), J(\tilde{x}, \tilde{y})) \\ &= I(\psi(\tilde{x}), J(\tilde{x}, \tilde{y})) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi(\psi(\tilde{y})) & otherwise \end{cases}\end{aligned}$$

and $(I * J)(\tilde{x},\tilde{y}) = I(\tilde{x}, J(\tilde{x},\tilde{y})) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi(\psi(\tilde{y})) & otherwise \end{cases}$

Hence $I \rhd J = I * J$ □

Theorem 9. *For all $I \in \mathbb{I} T \in \mathcal{K}, T * (I \rhd T^{-1}) = (T * I) \rhd T^{-1}$*

Proof. Let $I \in \mathbb{I}$ and $T \in \mathcal{K}$ we know that $T(\tilde{x},\tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi(\tilde{y}) & otherwise \end{cases}$

for some $\varphi \in \Theta$. Also T^{-1} will be given by

$T^{-1}(\tilde{x},\tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi^{-1}(\tilde{y}) & otherwise \end{cases}$

if $\tilde{x} = \tilde{0}$. Then $(T * (I \rhd T^{-1}))(\tilde{0},\tilde{y}) = \tilde{1} = ((T * I) \rhd T^{-1})(\tilde{0},\tilde{y})$ if $\tilde{x} \neq \tilde{0}$. Then

$$\begin{aligned} (T * (I \rhd T^{-1}))(\tilde{x},\tilde{y}) &= T(\tilde{x}, (I \rhd T^{-1})(\tilde{x},\tilde{y})) \\ &= T(\tilde{x}, I(T^{-1}(\tilde{1},\tilde{x}), T^{-1}(\tilde{x},\tilde{y}))) \\ &= \varphi(I(\varphi^{-1}(\tilde{x}), \varphi^{-1}(\tilde{y}))) \end{aligned}$$

and

$$\begin{aligned} ((T \rhd I) * T^{-1})(\tilde{x},\tilde{y}) &= (T * I)(T^{-1}(\tilde{1},\tilde{x}), T^{-1}(\tilde{x},\tilde{y})) \\ &= T(T^{-1}(\tilde{1},\tilde{x}), I(T^{-1}(\tilde{1},\tilde{x}), T^{-1}(\tilde{x},\tilde{y}))) \\ &= T(\varphi(\tilde{x}), I(\varphi^{-1}(\tilde{x}), \varphi^{-1}(\tilde{y}))) \\ &= \varphi(I(\varphi^{-1}(\tilde{x}), \varphi^{-1}(\tilde{y}))) \end{aligned}$$

Hence we have proved that $(T * (I \rhd T^{-1}))(\tilde{x},\tilde{y}) = ((T * I) \rhd T^{-1})(\tilde{x},\tilde{y})$ for all $\tilde{x},\tilde{y} \in L^*$. □

5 Group action of $\mathcal{K}$ on $\mathbb{I}$

In this section we define the group action of $\mathcal{K}$ on $\mathbb{I}$. for that we first show some result that we need in the sequel.

Theorem 10. *The groups $(\mathcal{K}, *)$, $(\Theta, \circ)$ are isomorphic to each other*

Proof. Let $f : \Theta \longrightarrow \mathcal{K}$ defined by $f(\varphi) = I$ where

$$I(\tilde{x},\tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi(\tilde{y}) & otherwise \end{cases}$$

It is easy to see that the map f is one and onto. Let φ_1, $\varphi_2 \in \theta$ and $f(\varphi_1) = I_1$, $f(\varphi_2) = I_2$ Where $I_i(\tilde{x},\tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi_i(\tilde{y}) & otherwise \end{cases}$

for $i = 1, 2$

Now we have:

$$\begin{aligned}\big(f(\varphi_1) * f(\varphi_2)\big)(\tilde{x}, \tilde{y}) &= \big(I_1 * I_2\big)(\tilde{x}, \tilde{y}) \\ &= I_1\big)(\tilde{x}, I_2(\tilde{x}, \tilde{y}) \\ &= \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi_1(\varphi_2(\tilde{y})) & otherwise \end{cases} \\ &= f\big(\varphi_1 \circ \varphi_2\big)(\tilde{x}, \tilde{y})\end{aligned}$$

Thus f is an isomorphism. □

Definition 7. *Let $(G, *)$ be a group and H be a nonempty set. A function $\bullet : G \times H \longrightarrow H$ is called a group action if, for all $g_1, g_2 \in G$ and $h \in H$, $\bullet$ satisfies the following two conditions:*

1) $g_1 \bullet (g_2 \bullet h) = (g_1 * g_2) \bullet h$

2) $e \bullet h = h$ *where e is the identity of G.*

Definition 8. *Let $\bullet : \mathcal{K} \times I \longrightarrow I$ be a map defined by $(T, I) \longrightarrow T \bullet I = T * I * T^{-1}$.*

Lemma 2. *The function $\bullet$ is a group action of $\mathcal{K}$ on $\mathbb{I}$*

Proof. Let $T_1, T_2 \in \mathcal{K}$ and $I \in \mathbb{I}$.
1)

$$\begin{aligned}T_1 \bullet (T_2 \bullet I) &= T_1 * (T_2 \bullet I) * T_1^{-1} \\ &= T_1 * T_2 * I * T_2^{-1} * T_1^{-1} \\ &= (T_1 * T_2) * I * (T_1 * T_2)^{-1} \\ &= (T_1 * T_2) \bullet I.\end{aligned}$$

2) Similarly, $I_D \bullet I = I_D * I * I_D^{-1} = I$, since I_D is the identity of $(\mathbb{I}, *)$.
Thus $\bullet$ is a group action of $\mathcal{K}$ on $\mathbb{I}$. □

Definition 9. *Let $I, J \in \mathbb{I}$. Define $I \backsim J \Leftrightarrow J = T \bullet I$ for some $T \in \mathcal{K}$. In other words, $I \backsim J \Leftrightarrow J = T * I * T^{-1}$ for some $T \in \mathcal{K}$.*

Lemma 3. *The relation $\backsim$ is an equivalence relation and it partitions the set $\mathbb{I}$.*

Proof. We have for $I, J \in \mathbb{I}$
1 $I \backsim I$ because $I = I_D * I * I_D^{-1}$
2 And we have $I \backsim J \Rightarrow J = T * I * T^{-1}$ this implies that $I = T^{-1} * I * T$ then we take $H = T^{-1}$. Hence $J \backsim I$.
3 for the transitivity let $I \backsim J$ and $J \backsim K$ we can easily show that $I \backsim K$. □

Remark 2. *Let $I \in \mathbb{I}$. Then the equivalence class containing I will be of the form $[I] = \big\{J \in \mathbb{I} | J = T * I * T^{-1}$ for some $T \in \mathcal{K}\big\}$.*
Since any $T \in \mathcal{K}$ is of the form
$$T(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi(\tilde{y}) & otherwise \end{cases}$$
for some $\varphi \in \theta$, we have that, if $J \in [I]$, then $J(\tilde{x}, \tilde{y}) = \varphi(I(\tilde{x}, \varphi^{-1}(\tilde{y})))$ for all $\tilde{x}, \tilde{y} \in L^*$.

Now we define a new group action of $\mathcal{K}$ on $\mathbb{I}$.

Theorem 11. *Let $\sqcup : \mathcal{K} \times \mathbb{I} \longrightarrow \mathbb{I}$ be defined by $T \sqcup I = T * I$, $T \in \mathcal{K}, I \in \mathbb{I}$. The function $\sqcup$ is a left group action of $\mathcal{K}$ on $\mathbb{I}$.*

Proof. **i)** Let $T_1, T_2 \in \mathcal{K}$ and $I \in \mathbb{I}$. Then

$$\begin{aligned} T_1 \sqcup (T_2 \sqcup I) &= T_1 * (T_2 \sqcup I) \\ &= T_1 * (T_2 * I) \\ &= (T_1 * T_2) * I \\ &= (T_1 * T_2) \sqcup I \end{aligned}$$

ii)$I_D \sqcup I = I_D * I = I$ Thus $\sqcup$ is a left group action of $\mathcal{K}$ on $\mathbb{I}$ □

6 Bijective transformations of intuitionistic fuzzy implications

Definition 10. *Let $I : L^{*2} \longrightarrow L^*$ be a function and $\varphi, \psi, \mu \in \Theta$. We define the bijective transformation $J_{\varphi,\psi,\mu} : L^{*2} \longrightarrow L^*$ of I as follows:*

$$J_{\varphi,\psi,\mu}(\tilde{x}, \tilde{y}) = \varphi(I(\psi(\tilde{x}), \mu(\tilde{y})) \tag{1}$$

The following result shows that any bijective transformation of the form (1) can also generate intuitionistic fuzzy implications from intuitionistic fuzzy implications.

Theorem 12. *Let $I : L^{*2} \longrightarrow L^*$ be a function and $\varphi, \psi, \mu \in \Theta$. Let $J_{\varphi,\psi,\mu}$ be defined as in (1). Then the following statements are equivalent:*

i) I is an intuitionistic fuzzy implication

ii) $J_{\varphi,\psi,\mu}$ is an intuitionistic fuzzy implication

Proof. $\Rightarrow$) Let $\tilde{x_1}, \tilde{x_2}, \tilde{y} \in L^*$ such that $\tilde{x_1} \leq \tilde{x_2}$. Then we have $I(\tilde{x_2}, \tilde{y}) \leq I(\tilde{x_1}, \tilde{y})$ using the fact that $\varphi, \psi, \mu \in \Theta$ we defined $\varphi(I(\psi(\tilde{x_2}), \mu(\tilde{y})) \leq \varphi(I(\psi(\tilde{x_1}), \mu(\tilde{y}))$. This implies that $J_{\varphi,\psi,\mu}$ is decreasing for the first variable.

Similarly for the second variable.

And we have $J_{\varphi,\psi,\mu}(\tilde{0}, \tilde{1}) = \varphi(I(\psi(\tilde{0}), \mu(\tilde{1})) = \varphi(I(\tilde{0}, \tilde{1}) = \varphi(\tilde{1}) = \tilde{1}$,
$J_{\varphi,\psi,\mu}(\tilde{1}, \tilde{0}) = \varphi(I(\psi(\tilde{1}), \mu(\tilde{0})) = \varphi(I(\tilde{1}, \tilde{0}) = \varphi(\tilde{0}) = \tilde{0}$,
$J_{\varphi,\psi,\mu}(\tilde{1}, \tilde{1}) = \varphi(I(\psi(\tilde{1}), \mu(\tilde{1})) = \varphi(I(\tilde{1}, \tilde{1}) = \varphi(\tilde{1}) = \tilde{1}$.
Hence $J_{\varphi,\psi,\mu}$ is an intuitionistic fuzzy implication.
Conversely, let $J_{\varphi,\psi,\mu}$ an intuitionistic fuzzy implication. Then for $\tilde{x_1}, \tilde{x_2}, \tilde{y} \in L^*$ such that $\tilde{x_1} \leq \tilde{x_2}$.
We have $J_{\varphi,\psi,\mu}(\tilde{x_2}, \tilde{y}) \leq J_{\varphi,\psi,\mu}(\tilde{x_1}, \tilde{y})$
$\Longrightarrow \varphi(I(\psi(\tilde{x_2}), \mu(\tilde{y})) \leq \varphi(I(\psi(\tilde{x_1}), \mu(\tilde{y}))$ for some $\varphi, \psi, \mu \in \Theta$
$\Longrightarrow I(\psi(\tilde{x_2}), \mu(\tilde{y}) \leq I(\psi(\tilde{x_1}), \mu(\tilde{y})$ then I is a decreasing function for the first variable because $\varphi, \psi, \mu \in \Theta$. Similarly, I is increasing for the second variable.

Now we have $J_{\varphi,\psi,\mu}(\tilde{0},\tilde{1}) = \tilde{1} = \varphi(I(\psi(\tilde{0}),\mu(\tilde{1}))$ this implies that $\varphi(I(\tilde{0},\tilde{1})) = \tilde{1}$.
Hence $I(\tilde{0},\tilde{1}) = \tilde{1}$ because $\varphi(\tilde{1}) = \tilde{1} \quad \forall \varphi \in \theta$
$J_{\varphi,\psi,\mu}(\tilde{1},\tilde{1}) = \tilde{1} = \varphi(I(\psi(\tilde{1}),\mu(\tilde{1}))$ this implies that $\varphi(I(\tilde{1},\tilde{1})) = \tilde{1}$. Hence $I(\tilde{1},\tilde{1}) = \tilde{1}$
$J_{\varphi,\psi,\mu}(\tilde{1},\tilde{0}) = \tilde{0} = \varphi(I(\psi(\tilde{1}),\mu(\tilde{0}))$ this implies that $\varphi(I(\tilde{1},\tilde{0})) = \tilde{0}$. Hence $I(\tilde{1},\tilde{0}) = \tilde{0}$. □

From the obvious Theorem, it follows that one can always obtain intuitionistic fuzzy implications from given an intuitionistic fuzzy implication using (1).

Now, given $I, J \in \mathbb{I}$ we define

$$I_{\sim_{\varphi,\psi,\mu}} J \iff J = I_{\varphi,\psi,\mu} \tag{2}$$

for some $\varphi,\psi,\mu \in \Theta$. It can be easily seen that $\sim_{\varphi,\psi,\mu}$ is an equivalence relation, if $[I]_{\sim_{\varphi,\psi,\mu}}$ denotes the equivalence class of fuzzy implications containing I w.r.t. (2), then

$$\begin{aligned} [I]_{\sim_{\varphi,\psi,\mu}} &= \{J \in \mathbb{I} | J_{\sim_{\varphi,\psi,\mu}} I\} \\ &= \{J \in \mathbb{I} | J(\tilde{x},\tilde{y}) = \varphi(I(\psi(\tilde{x}),\mu(\tilde{y}))) for some \varphi,\psi,\mu \in \Theta\} \\ &= \{\varphi(I(\psi(\tilde{x}),\mu(\tilde{y}))) | \varphi,\psi,\mu \in \Theta\}. \end{aligned}$$

Now we propose two functions from $\mathcal{K} \times \mathbb{I} \longrightarrow \mathbb{I}$. One of these turns out to be a group action of $\mathcal{K}$ on $\mathbb{I}$, while the other is an anti-group action.

Definition 11. *Let $\Diamond : \mathbb{I} \times \mathcal{K} \longrightarrow \mathbb{I}$ be defined by $I \Diamond T = I * T$.*

Theorem 13. *$\Diamond$ is a right group action of $\mathcal{K}$ on $\mathbb{I}$.*

Proof. Let $I \in \mathbb{I}$ and $T_1, T_2 \in \mathcal{K}$.
$(I \Diamond T_1) \Diamond T_2 = (I * T_1) \Diamond T_2 = (I * T_1) * T_2 = I * (T_1 * T_2) = I \Diamond (T_1 * T_2)$.
$I \Diamond T_D = I * I_D = I$ for all $I \in \mathbb{I}$.
Thus $\Diamond$ is a right group action. □

Definition 12. *Define $\sim_\Diamond$ on $\mathbb{I}$ by $I \sim_\Diamond J \iff J = I \Diamond T = I * T$ for some $T \in \mathcal{K}$.*

It is easy to verify that $\sim_\Diamond$ is an equivalence relation.

Remark 3. *Let $I \in \mathbb{I}$. If $[I]_\Diamond$ denotes the equivalence class containing I, then*

$$\begin{aligned} [I]_\Diamond &= \{J \in \mathbb{I} | J_{\sim_\Diamond} I\} \\ &= \{J \in \mathbb{I} | J = I * T \textit{ for some } T \in \mathcal{K}\} \\ &= \{J \in \mathbb{I} | J(\tilde{x},\tilde{y}) = I(\tilde{x}, T(\tilde{x},\tilde{y})) \textit{ for some } T \in \mathcal{K}\} \\ &= \{J \in \mathbb{I} | J(\tilde{x},\tilde{y}) = I(\tilde{x}, \varphi(\tilde{y})) \textit{ for some } \varphi \in \Theta\} \\ &= \{I(\tilde{x}, \varphi(\tilde{y})) \textit{ for some } \varphi \in \Theta\}. \end{aligned}$$

Definition 13. ***(See [6])*** *Let $(G, *)$ be a group with identity e and S being a nonempty set. A map $\circ : G \times S \longrightarrow S$ is called anti-group action if for all $g_1, g_2 \in G, s \in S$ the map $\circ$ satisfies the following :*
(i) $g_1 \circ (g_2 \circ s) = (g_2 \circ g_1) \circ s$.
(ii) $e \circ s = s$.

Theorem 14. *Let $\sqsupset : \mathcal{K} \times \mathbb{I} \longrightarrow \mathbb{I}$ be defined by $T \sqsupset I = (I \rhd T) * T^{-1}$, $T \in \mathcal{K}, I \in \mathbb{I}$. Then $\sqsupset$ is an anti-group action of $\mathcal{K}$ on $\mathbb{I}$.*

Proof. **i)** Let $I \in \mathbb{I}$ and $T_1, T_2 \in \mathcal{K}$. Then

$$\begin{aligned} T_1 \sqsupset (T_2 \sqsupset I) = T_1(\sqsupset (I \rhd T_2) * T_2^{-1}) \\ = ((I \rhd T_2) * T_2^{-1} \rhd T_1) * T_1^{-1} \end{aligned}$$

Since $T_1, T_2 \in \mathcal{K}$. Then T_1, T_2 are of the following form

$$T_i(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi_i(\tilde{y}) & otherwise \end{cases}$$

$i = 1, 2$ for some $\varphi_i \in \theta$, if $\tilde{x} = \tilde{0}$. Then

$$\begin{aligned} (T_1 \sqsupset (T_2 \sqsupset I))(\tilde{x}, \tilde{y}) = &= ((I \rhd T_2) * T_2^{-1} \rhd T_1) * T_1^{-1})(\tilde{x}, \tilde{y}) \\ &= ((I \rhd T_2) * T_2^{-1} \rhd T_1)(\tilde{x}, T_1^{-1}(\tilde{x}, \tilde{y})) \\ &= ((I \rhd T_2) * T_2^{-1} \rhd T_1)(\tilde{x}, \varphi_1^{-1}(\tilde{y}))) \\ &= ((I \rhd T_2) * T_2^{-1})(T_1(\tilde{1}, \tilde{y}), T_1(\tilde{x}, \varphi_1^{-1}(\tilde{y}))) \\ &= ((I \rhd T_2) * T_2^{-1})(\varphi_1(x), \tilde{y})) \\ &= ((I \rhd T_2)(\varphi_1(x), T_2^{-1}(\varphi_1(\tilde{x}), \tilde{y}))) \\ &= (I \rhd T_2)(\varphi_1(\tilde{x}), \varphi_2^{-1}(\tilde{y})) \\ &= I(T_2(\tilde{1}, \varphi_1(\tilde{x})), T_2(\varphi_1(\tilde{x}), \varphi_2^{-1}(\tilde{y}))) \\ &= I(\varphi_2(\varphi_1(\tilde{x})), \tilde{y}), \end{aligned}$$

While

$$\begin{aligned} ((T_2 * T_1) \sqsupset I)(\tilde{x}, \tilde{y}) &= (I \rhd (T_2 * T_1) * (T_2 * T_1)^{-1})(\tilde{x}, \tilde{y}) \\ &= (I \rhd (T_2 * T_1) * (T_1^{-1} * T_2^{-1}))(\tilde{x}, \tilde{y}) \\ &= (I \rhd (T_2 * T_1) * T_1^{-1})(\tilde{x}, \varphi_2^{-1}(\tilde{y})) \\ &= (I \rhd (T_2 * T_1)(\tilde{x}, \varphi_1^{-1}(\varphi_2^{-1}(\tilde{y}))) \\ &= I((T_2 * T_1)(\tilde{1}, \tilde{x}), (T_2 * T_1)(\tilde{x}, \varphi_1^{-1}(\varphi_2^{-1}(\tilde{y})))) \\ &= I(\varphi_2(\varphi_1(\tilde{x})), \tilde{y}). \end{aligned}$$

Thus in all cases we have shown that $T_1 \sqsupset (T_2 \sqsupset I) = (T_2 * T_1) \sqsupset I$, for all $T_2, T_1 \in \mathcal{K}$ and $I \in \mathbb{I}$.
ii) Let $I \in \mathbb{I}$. Then $I_D \sqsupset I = (I \rhd I_D) * I_D^{-1} = I \rhd I_D = I$, hence $\sqsupset$ is an anti-group action. □

Definition 14. *Let $I, J \in \mathbb{I}$. Then the relation defined as follows is an equivalence relation: $I \sim_{\boxdot} J$ if and only if $J = T_1 \sqcup ((T_3 \sqsupset I) \Diamond T_2)$ for some $T_1, T_2, T_3 \in \mathcal{K}$.*

In fact, by expanding the above J as follows

$$\begin{aligned} J &= T_1 \sqcup ((T_3 \sqsupset I) \Diamond T_2) = T_1 * ((T_3 \sqsupset I) \Diamond T_2) \\ &= T_1 * ((T_3 \sqsupset I) * T_2) = T_1 * ((I \rhd T_3) * T_3^{-1} * T_2) \end{aligned}$$

Then $I \sim_{\boxdot} J$ if and only if $J = T_1 * ((I \rhd T_3) * T_3^{-1} * T_2)$ for some $T_1, T_2, T_3 \in \mathcal{K}$.

Theorem 15. *The equivalence classes of fuzzy implications as given in (16) are exactly the equivalence classes obtained from the relation $\sim_{\boxdot}$, i.e., for any $I \in \mathbb{I}$, $[I]_{\sim_{\varphi,\psi,\mu}} = [I]_{\sim_{\boxdot}}$.*

Proof. Let $I \in \mathbb{I}$. Then

$$
\begin{aligned}
[I]_{\sim_{\boxdot}} &= \{J \in \mathbb{I} | J \sim_{\boxdot} I\} \\
&= \{J \in \mathbb{I} | J = T_1 * ((I \rhd T_3) * T_3^{-1} * T_2) \quad for\, some\; T_1, T_2, T_3 \in \mathcal{K}\} \\
&= \{J \in \mathbb{I} | J(\tilde{x}, \tilde{y}) = (T_1 * ((I \rhd T_3) * T_3^{-1} * T_2)(\tilde{x}, \tilde{y}) \quad for\, all\; \tilde{x}, \tilde{y} \in L^*\} \\
&= \{J \in \mathbb{I} | J(\tilde{x}, \tilde{y}) = T_1(\tilde{x}, ((I \rhd T_3) * T_3^{-1} * T_2)(\tilde{x}, \tilde{y}) \quad for\, all \tilde{x}, \tilde{y} \in L^*\} \\
&= \{J \in \mathbb{I} | J(\tilde{x}, \tilde{y}) = T_1(\tilde{x}, ((I \rhd T_3)(\tilde{x}, (T_3^{-1} * T_2)(\tilde{x}, \tilde{y})))) \quad for\, all\; \tilde{x}, \tilde{y} \in L^*\} \\
&= \{J \in \mathbb{I} | J(\tilde{x}, \tilde{y}) = T_1(\tilde{x}, I(T_3(\tilde{1}, \tilde{x}), T_3(\tilde{x}, (T_3^{-1} * T_2)(\tilde{x}, \tilde{y})))) \quad for\, all\; \tilde{x}, \tilde{y} \in L^*\} \\
&= \{J \in \mathbb{I} | J(\tilde{x}, \tilde{y}) = T_1(\tilde{x}, I(T_3(\tilde{1}, \tilde{x}), T_2(\tilde{x}, \tilde{y})))) \quad for\, all\; \tilde{x}, \tilde{y} \in L^*\} \\
&= \left\{ J \in \mathbb{I} | J(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{1} & if \quad \tilde{x} = \tilde{0} \\ \varphi(I(\psi(\tilde{x}), \mu(\tilde{y})) & otherwise \end{cases} \right. \\
&= \{J \in \mathbb{I} | J(\tilde{x}, \tilde{y}) = \varphi(I(\psi(\tilde{x}), \mu(\tilde{y})) \quad for\, some\; \varphi, \psi, \mu \in \Theta\} \\
&= [I]_{\sim_{\varphi,\psi,\mu}}
\end{aligned}
$$

□

In other words, this result shows that any bijective transformation can be represented by a composition of group actions and an anti-group action of $\mathcal{K}$ on $\mathbb{I}$.

7 Conclusion

Our motivation for this study was to propose a binary operation $*$ on the set $\mathbb{I}$ of all intuitionistic fuzzy implications that would give a rich enough algebraic structure to glean newer and better perspectives on intuitionistic fuzzy implications. The operation $*$ proposed in this work not only gave a novel way of generating newer intuitionistic fuzzy implications from given ones, but also, for the first time, imposed a monoid structure on $\mathbb{I}$. By defining a suitable group action on $\mathbb{I}$ and the equivalence classes obtained therefrom. And we have shown that the bijective transformations given in (1) can be seen as a composition of group actions $\lozenge$, $\sqcup$ and $\sqsupset$.

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