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Extension of Hukuhara difference in intuitionistic fuzzy set theory

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Abstract: We proposed to give a sense of Hukuhara's difference in intuitionistic fuzzy theory. First we give the concept of elementary intuitionistic fuzzy theory, precisely the complete metric space $(\mathbb{F}_1, dp)$, the later helps us to make sense of the derivative of Hukuhara in the case intuitionistic.

Keywords: Hukuhara difference, Hukuhara derivative, Intuitionistic fuzzy number.
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1 Introduction

The Hukuhara difference is also motivated by the problem of inverting the standard Minkowski addition and multiplication, the idea of finding some inverse operation is crucial in interval and fuzzy arithmetic and analysis, with applications to the solution of equations, to the concept of differentiability, to interval and fuzzy integral and differential equations, etc, but in many applications it appears to have several limitations.

In [9] Luciano Stefanini propose generalization of difference as inverse operations (in some sense) to addition and multiplication, he considered the general setting of set-valued arithmetic and he suggested some generalizations (the classical Hukuhara difference is a special case) for compact sets, for compact convex sets (in particular compact intervals) and for fuzzy sets with compact and convex α-cuts (in particular fuzzy numbers).

On the other hand, in intuitionistic Fuzzy theory which introduced by K. Atanassov [1,2], the problem of inverting the addition and multiplication is always occurs for this reason we introduce the notion of Hukuhara difference in this theory which allows to introduce the concepts of differentiability.

The paper is organized as follows. In Section 2 we recall the generalized Hukuhara difference for compact sets precisely, The case of compact intervals is detailed in this Section . In Section 3 we recall the Hukuhara difference (H-difference) for fuzzy numbers and gives conditions for its existence. Section 4 presents basic concept of intuitionistic fuzzy sets and intuitionistic fuzzy number and we extend the results of Hukuhara difference in fuzzy theory to intuitionistic fuzzy theory. Finally, some concepts develop in Section 5 like continuity and differentiability.

2 General setting

Consider the space $\mathbb{X}$ with the induced topology and in particular the space $\mathbb{X} = \mathbb{R}^n$ $n \geq 1$ of real number equipped with standard addition and scalar multiplication operations. Following Diamond and Kloeden (see [3]), denote by $\mathcal{K}(\mathbb{X})$ and $\mathcal{K}_C(\mathbb{X})$ the spaces of nonempty compact and compact convex sets of $\mathbb{X}$. Given two subsets $A, B \subseteq \mathbb{X}$ and $k \in \mathbb{R}$, Minkowski addition and scalar multiplication are defined by $A + B = \{a + b | a \in A, b \in B\}$ and $kA = \{ka | a \in A\}$ and it is well known that addition is associative and commutative and with neutral element $\{0\}$. If $k = -1$, scalar multiplication gives the opposite $-A = (-1)A = \{-a | a \in A\}$ but, in general, $A + (-A) \neq \{0\}$, i.e. the opposite of A is not the inverse of A in Minkowski addition (unless $A = \{a\}$ is a singleton). Minkowski difference is $A - B = A + (-1)B = \{a - b | a \in A, b \in B\}$. A first implication of this fact is that, in general, even if it is true that $(A+C = B+C) \iff A = B$, addition/subtraction simplification is not valid, i.e. $(A + B) - B \neq A$.

To partially overcome this situation, Hukuhara [4] introduced the following H-difference :

$$A \ominus B = C \iff A = B + C \tag{1}$$

and an important property of $\ominus$ is that $A \ominus A = \{0\}$, $\forall A \in \mathcal{K}(\mathbb{X})$ and $(A + B) \ominus B = A$, $\forall A, B \in \mathcal{K}(\mathbb{X})$; H-difference is unique, but a necessary condition for $A \ominus B$ to exist is that A contains a translate $\{c\} + B$ of B with $c \in \mathbb{R}$. In general, $A - B \neq A \ominus B$.

From an algebraic point of view, the difference of two sets A and B may be interpreted both in terms of addition as in (1) or in terms of negative addition, i.e.

$$A \boxminus B = C \iff B = A + (-1)C \tag{2}$$

where $(-1)C$ is the opposite set of C. Conditions (1)and(2) are compatible to each other and this suggests a generalization of Hukuhara difference:

Definition 1. *Let $A, B \in \mathcal{K}(\mathbb{X})$; we define the generalized Hukuhara difference of A and B as the set $C \in \mathcal{K}(\mathbb{X})$ such that*

$$A \ominus_g B = C \iff \begin{cases} (i)\ A = B + C \\ or \\ (ii)\ B = A + (-1)C \end{cases}$$

Proposition 1. *((Unicity of $A \ominus_g B$)*
If $C = A \ominus_g B$ exists, it is unique and if also $A \ominus B$ exists then $A \ominus_g B = A \ominus B$.

2.1 The case of compact intervals in $\mathbb{R}$

For unidimensional compact intervals, the gH-difference always exists. In fact, let $A = [a^-, a^+]$ and $B = [b^-, b^+]$ be two intervals; the gH-difference is

$$[a^-, a^+] \ominus_g [b^-, b^+] = [c^-, c^+] \iff \begin{cases} (i) \begin{cases} a^- = & b^- + c^- \\ a^+ = & b^+ + c^+ \end{cases} \\ or \\ (ii) \begin{cases} b^- = & a^- - c^- \\ b^+ = & a^+ - c^+ \end{cases} \end{cases}$$

so that $[a^-, a^+] \ominus_g [b^-, b^+] = [c^-, c^+]$ is always defined by
$c^- = \min\{a^- - b^-, a^+ - b^+\}$,$c^+ = \max\{a^- - b^-, a^+ - b^+\}$ i.e

$$[a, b] \ominus_g [c, d] = [\min\{a - c, b - d\}, \max\{a - c, b - d\}]$$

Conditions (i) and (ii) are satisfied simultaneously if and only if the two intervals have the same length and $c^- = c^+$.

Also, the result is $\{0\}$ if and only if $a^- = b^-$ and $a^+ = b^+$.

Remark 1.

$$[a, b] + [c, d] = [a, b] + [c', d'] \Rightarrow [c, d] = [c', d'].$$

3 H-Difference of fuzzy numbers

A general fuzzy set over a given set (or space) $\mathbb{X}$ of elements (the universe) is usually defined by its membership function $\mu : \mathbb{X} \to \mathbb{T} \subseteq [0, 1]$ and a fuzzy (sub)set u of $\mathbb{X}$ is uniquely characterized by the pairs $(x, \mu_u(x))$ for each $x \in \mathbb{X}$; the value $\mu_u(x) \in [0, 1]$ is the membership grade of x to the fuzzy set u and μ_u is the membership function of a fuzzy set u over $\mathbb{X}$ (see [11] - [12] for the origins of Fuzzy Set Theory). The support of u is the (crisp) subset of points of $\mathbb{X}$ at which the membership grade $\mu_u(x)$ is positive: $supp(u) = \{x | x \in \mathbb{X}, \mu_u(x) > 0\}$. For $\alpha \in]0, 1]$, the α-level cut of u (or simply the α-cut)is defined by $[u]^\alpha = \{x | x \in \mathbb{X}, \mu_u(x) \geq \alpha\}$ and for $\alpha = 0$ (or $\alpha \to 0^+$) by the closure of the support $[u]^0 = cl\{x | x \in \mathbb{X}, \mu_u(x) > 0\}$.

We will consider the case $\mathbb{X} = \mathbb{R}^n$ with $n \geq 1$. A particular class of fuzzy sets u is when the support is a convex set and the membership function is quasi-concave (i.e $\mu_u((1 - t)x' + tx'') \geq min\{\mu_u(x'), \mu_u(x'')\}$) for every $x', x'' \in supp(u)$and $t \in [0, 1]$). Equivalently, μ_u is quasi-concave if the level sets $[u]^\alpha$ are convex sets for all $\alpha \in [0, 1]$. We will also require that the level-cuts $[u]^\alpha$ are closed sets for all $\alpha \in [0, 1]$ and that the membership function is normal, i.e. the core$[u]^1 = \{x | \mu_u(x) = 1\}$ is compact and nonempty.

The following properties characterize the normal, convex and upper semicontinuous fuzzy sets (in terms of the level-cuts):

(F1) $[u]^\alpha \in \mathcal{K}_C(\mathbb{X})$ for all $\alpha \in [0,1]$;

(F2) $[u]^\alpha \subseteq [u]^\beta$ for $\alpha \geq \beta$ (i.e. they are nested);

(F3) $[u]^\alpha = \bigcap_{k=1}^{+\infty} [u]^{\alpha_k}$ for all increasing sequences $\alpha_k \uparrow \alpha$ converging to α

Furthermore, any family$\{U^\alpha | \alpha \in [0,1]\}$ satisfying conditions **(F1)–(F3)** represents the level-cuts of a fuzzy set u having $[u]^\alpha = U^\alpha$.

We will denote by E^n the set of the fuzzy sets with the properties above (also called fuzzy numbers). The space E^1 of real fuzzy numbers is structured by an addition and a scalar multiplication, defined either by the level sets or, equivalently, by the Zadeh extension principle.

Let $u, v \in E^n$ have membership functions μ_u, μ_v and α-cuts $[u]^\alpha, [v]^\alpha, \alpha \in [0,1]$, respectively. where $[u]^\alpha = [u_-^\alpha, u_+^\alpha]$ and $[v]^\alpha = [v_-^\alpha, v_+^\alpha]$.

The addition $u + v \in E^n$ and the scalar multiplication $ku \in E^n$ have level cuts

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha = \{x + y | x \in [u]^\alpha, y \in [v]^\alpha\},$$
$$[ku]^\alpha = k[u]^\alpha = \{kx | x \in [u]^\alpha\}.$$

In the fuzzy or in the interval arithmetic contexts, equation $u = v + \omega$ is not equivalent to $\omega = u - v = u + (-1)v$ or to $v = u - \omega = u + (-1)\omega$ and this has motivated the introduction of the following Hukuhara difference [4, 5, 8].

Definition 2. *Given $u, v \in E^n$, the H-difference is defined by*

$$u \ominus v = \omega \Longleftrightarrow u = v + \omega$$

1) if $u \ominus v$ exist, it is unique,

2) $u \ominus u = \{0\}$,

In the unidimensional case $(n = 1)$, the α-cuts of H-difference are

$$[u \ominus v]^\alpha = \left[u_-^\alpha - v_-^\alpha, u_+^\alpha - v_+^\alpha\right]$$

where $[u]^\alpha = [u_-^\alpha, u_+^\alpha]$ and $[v]^\alpha = [v_-^\alpha, v_+^\alpha]$.

The conditions of the definition of $u \ominus v = w$ are

$$[w]^\alpha = [w_-^\alpha, w_+^\alpha] = [u]^\alpha \ominus [v]^\alpha$$

and

$$\begin{cases} \text{if } \ len([u]^\alpha) \geq len([v]^\alpha) \ \text{ for all } \ \alpha \in [0,1] \\ \qquad w_-^\alpha = u_-^\alpha - v_-^\alpha \\ \qquad w_+^\alpha = u_+^\alpha - v_+^\alpha \end{cases} \tag{3}$$

provided that w_-^α is nondecreasing with respect to α , w_+^α is nonincreasing with respect to α and $w_-^{(1)} \leq w_+^{(1)}$, where $len([u]^\alpha) = u_+^\alpha - u_-^\alpha$ is the length α-cuts of the u (similarly $len([v]^\alpha)$ for v.

Proposition 2. *Let $u, v \in E^1$, be two fuzzy numbers with α-cuts given by $[u]^\alpha$ and $[v]^\alpha$, respectively; the H-difference $u \ominus v \in E^1$ exists if and only if the following condition is satisfied*

$$\begin{cases} len([u]^\alpha) \geq len([v]^\alpha) \text{ for all } \alpha \in [0,1] \\ u_-^\alpha - v_-^\alpha \text{ is increasing with respect to } \alpha \\ u_+^\alpha - v_+^\alpha \text{ is decreasing with respect to } \alpha \end{cases}$$

Proof. See [9]. □

4 Extension of Hukuhara difference in intuitionistic fuzzy set theory

In the first, we recall some resultats and definitions of intuitionistic fuzzy theory. we denote by

$$\mathbb{F}_n = \mathbb{F}(\mathbb{R}^n) = \left\{ \langle u, v \rangle \ : \ \mathbb{R}^n \to [0,1]^2 , \mid \forall\, x \in \mathbb{R}^n,\ 0 \leq u(x) + v(x) \leq 1 \right\}$$

An element $\langle u, v \rangle$ of $\mathbb{F}_n$ is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v \rangle$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}^n$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous

(iv) $supp\, \langle u, v \rangle = \text{cl}\{x \in \mathbb{R}^n \ : \mid v(x) < 1\}$ is bounded.

For $\alpha \in [0,1]$ and $\langle u, v \rangle \in \mathbb{F}^n$, the upper and lower α-cuts of $\langle u, v \rangle$ are defined by

$$[\langle u, v \rangle]^\alpha = \{x \in \mathbb{R}^n : v(x) \leq 1 - \alpha\}$$

and

$$[\langle u, v \rangle]_\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$$

Remark 2. *If $\langle u, v \rangle \in IF_n$, so we can see $[\langle u, v \rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v \rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.*

Example. A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u, v \rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v :

$$u(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{if } a_2 \leq x \leq a_3, \\ 0 & \text{otherwise} \end{cases}$$

$$v(x)=\begin{cases}\dfrac{a_2-x}{a_2-a_1^{'}} & \text{if } a_1^{'}\leq x\leq a_2\\ \dfrac{x-a_2}{a_3^{'}-a_2} & \text{if } a_2\leq x\leq a_3^{'},\\ 1 & otherwise.\end{cases}$$

where $a_1^{'}\leq a_1\leq a_2\leq a_3\leq a_3^{'}$.

This TIFN is denoted by $\langle u,v\rangle=\left\langle a_1,a_2,a_3;a_1^{'},a_2,a_3^{'}\right\rangle$.

$$\begin{aligned}\Big[\langle u,v\rangle\Big]_\alpha &= \Big[a_1+\alpha(a_2-a_1),a_3-\alpha(a_3-a_2)\Big]\\ \Big[\langle u,v\rangle\Big]^\alpha &= \Big[a_1'+\alpha(a_2-a_1'),a_3'-\alpha(a_3'-a_2)\Big]\end{aligned}$$

We define $0_{\langle 1,0\rangle}\in \mathbb{F}_n$ as

$$0_{\langle 1,0\rangle}(t)=\begin{cases}\langle 1,0\rangle & t=0\\ \langle 0,1\rangle & t\neq 0\end{cases}$$

Let $\langle u,v\rangle$,$\langle u',v'\rangle\in \mathbb{F}_n$ and $\lambda\in\mathbb{R}$, we define the following operations by :

$$\Big(\langle u,v\rangle\oplus\langle u',v'\rangle\Big)(z)=\Big(\sup_{z=x+y}\min\left(u(x),u'(y)\right),\inf_{z=x+y}\max\left(v(x),v'(y)\right)\Big)$$

$$\lambda\langle u,v\rangle=\begin{cases}\langle \lambda u,\lambda v\rangle & \text{if } \lambda\neq 0\\ 0_{\langle 1,0\rangle} & \text{if } \lambda=0\end{cases}$$

For $\langle u,v\rangle$, $\langle z,w\rangle\in\mathbb{F}_n$ and $\lambda\in\mathbb{R}$, the addition and scaler multiplication are defined as follows

$$\begin{aligned}\Big[\langle u,v\rangle\oplus\langle z,w\rangle\Big]^\alpha &= \Big[\langle u,v\rangle\Big]^\alpha+\Big[\langle z,w\rangle\Big]^\alpha, & \Big[\lambda\langle z,w\rangle\Big]^\alpha &= \lambda\Big[\langle z,w\rangle\Big]^\alpha\\ \Big[\langle u,v\rangle\oplus\langle z,w\rangle\Big]_\alpha &= \Big[\langle u,v\rangle\Big]_\alpha+\Big[\langle z,w\rangle\Big]_\alpha, & \Big[\lambda\langle z,w\rangle\Big]_\alpha &= \lambda\Big[\langle z,w\rangle\Big]_\alpha\end{aligned}$$

Definition 3. *Let $\langle u,v\rangle$ an element of $\mathbb{F}_n$ and $\alpha\in[0,1]$, we define the following sets :*

$$\Big[\langle u,v\rangle\Big]_l^+(\alpha)=\inf\{x\in\mathbb{R}^n\mid u(x)\geq\alpha\},\quad \Big[\langle u,v\rangle\Big]_r^+(\alpha)=\sup\{x\in\mathbb{R}^n\mid u(x)\geq\alpha\}$$
$$\Big[\langle u,v\rangle\Big]_l^-(\alpha)=\inf\{x\in\mathbb{R}^n\mid v(x)\leq 1-\alpha\},\quad \Big[\langle u,v\rangle\Big]_r^-(\alpha)=\sup\{x\in\mathbb{R}^n\mid v(x)\leq 1-\alpha\}$$

Remark 3.

$$\begin{aligned}\Big[\langle u,v\rangle\Big]_\alpha &= \Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha),\Big[\langle u,v\rangle\Big]_r^+(\alpha)\Big]\\ \Big[\langle u,v\rangle\Big]^\alpha &= \Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha),\Big[\langle u,v\rangle\Big]_r^-(\alpha)\Big]\end{aligned}$$

Example. Let $\langle u,v\rangle=\langle 1,2,3.5;0.5,2,4\rangle$, $\langle u',v'\rangle=\langle 2,3,4;1,3,4\rangle$ and $\langle k,l\rangle=\langle u,v\rangle\oplus\langle u',v'\rangle$ Then $k(z)=\sup_{z=x+y}(\min(u(x),u'(y)))$ and $l(z)=\inf_{z=x+y}(\max(v(x),v'(y)))$.

Let us exhibit the computational procedure involve in above equation for membership function, first pick a value for z, then evaluate $\min(u(x),u'(y))$ for x and y which add up to $z=5$.

We have done this for certain values of x and y as shown in Table 1. It appear that the max occurs for $x = 2$ and $y = 3$, therefore $k(5) = 1$. Now do this for other values of z. Similarly for non-membership function, evaluate $\max(v(x), v'(y))$ for x and y which add up to $z = 5$. We have done this for certain values of x and y as shown in Table 2. The min occurs for $x = 2$ and $y = 3$ so that $l(5) = 0$ Now do this for other values of z. Finally, we get $\langle k, l\rangle = \langle 3, 5, 7.5; 1.5, 5, 8\rangle$ a TIFN.

x	$u(x)$	y	$u'(y)$	$\min(u(x), u'(y))$
1	0	4	0	0
2	1	3	1	1
2.5	0.666	2.5	0.5	0.5
4.5	0	0.5	0	0

Table 1: Finding membership function of sum of two TIFN

x	$v(x)$	y	$v'(y)$	$\max(col2, col4)$
3	0.5	2	0.5	0.5
2.5	0.25	2.5	0.25	0.25
2	0	3	0	0
1.5	0.333	3.5	0.5	0.5

Table 2: Finding non-membership function of sum of two TIFN

Proposition 3. *For all* α, $\beta \in [0, 1]$ *and* $\langle u, v\rangle \in I\!F_n$

(i) $\Big[\langle u, v\rangle\Big]_\alpha \subset \Big[\langle u, v\rangle\Big]^\alpha$

(ii) $\Big[\langle u, v\rangle\Big]_\alpha$ *and* $\Big[\langle u, v\rangle\Big]^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}^n$

(iii) if $\alpha \leq \beta$ *then* $\Big[\langle u, v\rangle\Big]_\beta \subset \Big[\langle u, v\rangle\Big]_\alpha$ *and* $\Big[\langle u, v\rangle\Big]^\beta \subset \Big[\langle u, v\rangle\Big]^\alpha$

(iv) If $\alpha_n \nearrow \alpha$ *then* $\Big[\langle u, v\rangle\Big]_\alpha = \bigcap_n \Big[\langle u, v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u, v\rangle\Big]^\alpha = \bigcap_n \Big[\langle u, v\rangle\Big]^{\alpha_n}$

Let M any set and $\alpha \in [0, 1]$ we denote by

$$M_\alpha = \{x \in \mathbb{R}^n \ : \ u(x) \geq \alpha\} \quad \text{and} \quad M^\alpha = \{x \in \mathbb{R}^n \ : \ v(x) \leq 1 - \alpha\}$$

Lemma 1. *[8] Let* $\Big\{M_\alpha,\ \alpha \in [0, 1]\Big\}$ *and* $\Big\{M^\alpha,\ \alpha \in [0, 1]\Big\}$ *two families of subsets of* $\mathbb{R}^n$ *satisfies (i)–(iv) in proposition 3, if* u *and* v *define by*

$$u(x) = \begin{cases} 0 & \text{if } x \notin M_0 \\ \sup\{\alpha \in [0, 1] \ : \ x \in M_\alpha\} & \text{if } x \in M_0 \end{cases}$$

$$v(x) = \begin{cases} 1 & \text{if } x \notin M^0 \\ 1 - \sup\{\alpha \in [0, 1] \ : \ x \in M^\alpha\} & \text{if } x \in M^0 \end{cases}$$

Then $\langle u, v\rangle \in I\!F_n$

4.1 Hukuhara difference of intuitionistic fuzzy numbers

Definition 4. *Let $\langle u,v\rangle$, $\langle u',v'\rangle \in I\!F_n$ the H-difference is the IFN $\langle z,w\rangle \in I\!F_n$, if it exists, such that*

$$\langle u,v\rangle \ominus \langle u',v'\rangle = \langle z,w\rangle \Longleftrightarrow \langle u,v\rangle = \langle u',v'\rangle \oplus \langle z,w\rangle$$

4.1.1 Support function and intuitionistic fuzzy H-difference

Let $A \neq \emptyset$ is compact of $\mathbb{R}^n$ and $S^{n-1} = \{p \mid p \in \mathbb{R}^n, \|p\| = 1\}$ is the unit sphere, the support function associated to A is $S_A : S^{n-1} \to \mathbb{R}$ defined by $S_A(p) = \sup\{\langle p,a\rangle | a \in A\},\ p \in S^{n-1}$, where $\langle p,a\rangle$ denotes the usual scalar product of a and p. The following properties are well known see [3] or [5] : Any function $s : S^{n-1} \to \mathbb{R}$ which is continuous (or, more generally, upper semicontinuous), positively homogeneous $s(tp) = ts(p),\ \forall p \in S^{n-1}$ and subadditive $s(p_1+p_2) \leq s(p_1) + s(p_2)\ \forall p_1,p_2 \in S^{n-1}$ is a support function of a compact convex set.

- If $A \in \mathcal{K}_C(\mathbb{R}^n)$ is a compact convex set, then it is characterized by its support function and
$$A = \{x \in \mathbb{R}^n \mid \langle p,x\rangle \leq s_A(p),\ \forall p \in S^{n-1}\}$$
- For $A,B \in \mathcal{K}_C(\mathbb{R}^n)$ and $\forall \in S^{n-1}$ we have $A \subseteq B \Rightarrow s_A(p) \leq s_B(p)$; $A = B \Longleftrightarrow s_A = s_B$
- $s_{kA+hB}(p) = ks_A(p) + hs_B(p), \forall\, h,k \geq 0$ and in particular $s_{A+B}(p) = s_A(p) + s_B(p)$.

An equivalent definition of $\langle z,w\rangle = \langle u,v\rangle \ominus \langle u',v'\rangle$ for multidimensional intuitionistic fuzzy numbers can be obtained in terms of support functions $S_{\langle z,w\rangle}(p,\alpha) = \Big(S_z(p,\alpha), S_{1-w}(p,\alpha)\Big)$, $\alpha \in [0,1]$ with

$$S_z(p,\alpha) = S_u(p,\alpha) - S_{u'}(p,\alpha) S_{1-w}(p,\alpha) = S_{1-v}(p,\alpha) - S_{1-v'}(p,\alpha),$$

where for an intuitionistic fuzzy numbers, the support functions are considered for each α-cut and defined to characterize the (compact) α-cut $[\langle u,v\rangle]^\alpha$ and $[\langle u,v\rangle]_\alpha$:

$$\begin{array}{rccc} S_{\langle u,v\rangle} & : & S^{n-1}\times[0,1] & \to & \mathbb{R}^2 \\ & & (p,\alpha) & \mapsto & \Big(S_u(p,\alpha), S_{1-v}(p,\alpha)\Big) \end{array}$$

defined by

$$S_u(p,\alpha) = \sup\{\langle p,x\rangle | x \in [\langle u,v\rangle]_\alpha \text{ for each } p \in S^{n-1}, \alpha \in [0,1]\}$$

$$S_{1-v}(p,\alpha) = \sup\{\langle p,x\rangle | x \in [\langle u,v\rangle]^\alpha \text{ for each } p \in S^{n-1}, \alpha \in [0,1]\}$$

As the functions of α, $S_u(p,.)$ and $S_{1-v}(p,.)$ are the support functions of two fuzzy number u and $1-v$ respectively and are non-increasing for all $p \in S^{n-1}$, due to the nesting property of the α-cuts.

Now, $S_{\langle z,w\rangle}$ is a correct support function if S_z and S_{1-w} are continuous (upper semi-continuous), positively homogeneous and sub-additive.

Consider $S_1 = S_u - S_{u'}$,$S_2 = S_{1-v} - S_{1-v'}$. Continuity and positive homogeneity of S_1 and S_2 are obvious. But S_1 and/or S_2 may fail to be sub-additive and the following Proposition, related to the definition of H-difference, is possible.

Proposition 4. *Let $S_{\langle u,v\rangle}$ and $S_{\langle u',v'\rangle}$ be the support functions of two intuitionistic fuzzy numbers $\langle u,v\rangle, \langle u',v'\rangle \in IF_n$. Consider $S_1 = S_u(p,.) - S_{u'}(p,.)$ and $S_2 = S_{1-v}(p,.) - S_{1-v'}(p,.)$. If S_1 and S_2 are both subadditive in p for all $\alpha \in [0,1]$ and are nonincreasing for all p, and $S_1 \leq S_2$ for all $\alpha \in [0,1]$, then $\langle u,v\rangle \ominus \langle u',v'\rangle$ exists.*

Proof. For $\alpha \in]0,1]$ consider the sets

$$M_\alpha = \{x \in \mathbb{R}^n \mid \langle p,x\rangle \leq S_1(p,\alpha) \text{ for all } p \in \mathbb{R}^n \text{with } \parallel p \parallel = \alpha\}$$
$$M^\alpha = \{x \in \mathbb{R}^n | \langle p,x\rangle \leq S_2(p,\alpha) \text{ for all } p \in \mathbb{R}^n \text{with } \parallel p \parallel = \alpha\}$$

and $M^0 = cl(\cup_{\alpha\in]0,1]} M^\alpha)$. Since S_1 and S_2 are subadditive for all $\alpha \in]0,1]$ then the sets M_α and M^α are compact convex sets in $\mathbb{R}^n$. The monotonicity condition $\forall p$ ensures that the property (ii) of Proposition 3 holds. Moreover, $S_1 \leq S_2$ then we have $M_\alpha \subset M^\alpha$. It remains to show property (iv) of Proposition 3 i.e $M_\alpha = \bigcap\limits_{k\geq 1} M_{\alpha_k}$ and $M^\alpha = \bigcap\limits_{k\geq 1} M^{\alpha_k}$ for all increasing sequences $\alpha_k \uparrow \alpha$ converging to $\alpha \in]0,1]$. As $M_\alpha \subseteq M_{\alpha_k}$ we have $M_\alpha \subseteq \bigcap\limits_{k\geq 1} M_{\alpha_k}$, let now $x \in \bigcap\limits_{k\geq 1} M_{\alpha_k}$ for all p having $\parallel p \parallel = \alpha$ and all $k = 1,2,\dots$ we have $\langle (\frac{\alpha_k}{\alpha})p, x\rangle \leq S_1((\frac{\alpha_k}{\alpha})p,\alpha)$ as S_1 is continuous (upper semi-continuous) $\langle p,x\rangle = \lim\langle (\frac{\alpha_k}{\alpha})p,x\rangle \leq \limsup S_1\Big((\frac{\alpha_k}{\alpha})p,\alpha\Big) = S_1(p,\alpha)$ as $k \to \infty$ and $x \in M_\alpha$, the same idea to prove $M^\alpha = \bigcap\limits_{k\geq 1} M^{\alpha_k}$. Finally By Lemma 1 the proof is complete. $\square$

It immediately follows a necessary and sufficient condition for $\langle u,v\rangle \ominus \langle u',v'\rangle$ to exist :

Proposition 5. *Let $\langle u,v\rangle, \langle u',v'\rangle \in I\!F_n$ be given with support functions $S_{\langle u,v\rangle}$ and $S_{\langle u',v'\rangle}$, then $\langle u,v\rangle \ominus \langle u',v'\rangle$ exists if and only if the two functions $S_1 = S_u(p,\alpha) - S_{u'}(p,\alpha)$ and $S_2 = S_{1-v}(p,\alpha) - S_{1-v'}(p,\alpha)$ are the support functions and are non-increasing with α for all p and $S_1 \leq S_2$ for all $\alpha \in [0,1]$.*

In the unidimensional case, the conditions of the definition of $\langle u,v\rangle \ominus \langle z,w\rangle = \langle k,l\rangle$ are

1. $\Big[\langle k,l\rangle\Big]_\alpha = \Big[\langle u,v\rangle\Big]_\alpha \ominus \Big[\langle z,w\rangle\Big]_\alpha$ and

$$\begin{cases} if\ \ len\Big(\Big[\langle u,v\rangle\Big]_\alpha\Big) \geq len\Big(\Big[\langle z,w\rangle\Big]_\alpha\Big) \text{ for all } \alpha \in [0,1] \\ \Big[\langle k,l\rangle\Big]_l^+(\alpha) = \Big[\langle u,v\rangle\Big]_l^+(\alpha) - \Big[\langle z,w\rangle\Big]_l^+(\alpha) \\ \Big[\langle k,l\rangle\Big]_r^+(\alpha) = \Big[\langle u,v\rangle\Big]_r^+(\alpha) - \Big[\langle z,w\rangle\Big]_r^+(\alpha) \end{cases} \tag{4}$$

provided that $\Big[\langle k,l\rangle\Big]_l^+(\alpha)$ is non-decreasing with respect to α, $\Big[\langle k,l\rangle\Big]_r^+(\alpha)$ is non-increasing with respect to α and $\Big[\langle k,l\rangle\Big]_l^+(1) \leq \Big[\langle k,l\rangle\Big]_r^+(1)$

2. $\Big[\langle k,l\rangle\Big]^\alpha = \Big[\langle u,v\rangle\Big]^\alpha \ominus \Big[\langle z,w\rangle\Big]^\alpha$ and

$$\begin{cases} if\ \ len\Big(\Big[\langle u,v\rangle\Big]^\alpha\Big) \geq len\Big(\Big[\langle z,w\rangle\Big]^\alpha\Big) \text{ for all } \alpha \in [0,1] \\ \Big[\langle k,l\rangle\Big]_l^-(\alpha) = \Big[\langle u,v\rangle\Big]_l^-(\alpha) - \Big[\langle z,w\rangle\Big]_l^-(\alpha) \\ \Big[\langle k,l\rangle\Big]_r^-(\alpha) = \Big[\langle u,v\rangle\Big]_r^-(\alpha) - \Big[\langle z,w\rangle\Big]_r^-(\alpha) \end{cases} \tag{5}$$

provided that $\Big[\langle k,l\rangle\Big]_l^-(\alpha)$ is non-decreasing with respect to α, $\Big[\langle k,l\rangle\Big]_r^-(\alpha)$ is nonincreasing with respect to α and $\Big[\langle k,l\rangle\Big]_l^-(1) \leq \Big[\langle k,l\rangle\Big]_r^-(1)$

Example 1. Let $\langle u,v\rangle = \langle 3,5,7.5;1.5,5,8\rangle$, $\langle u',v'\rangle = \langle 2,3,4;1,3,4\rangle$ be two TIFNs their level cuts are

$$\Big[\langle u,v\rangle\Big]_\alpha = \Big[3+2\alpha;7.5-2.5\alpha\Big] \ , \ \Big[\langle u',v'\rangle\Big]_\alpha = \Big[2+\alpha;4-\alpha\Big]$$
$$\Big[\langle u,v\rangle\Big]^\alpha = \Big[1.5+3.5\alpha;8-3\alpha\Big] \ , \ \Big[\langle u',v'\rangle\Big]^\alpha = \Big[1+2\alpha;4-\alpha\Big]$$

since

$$len\Big(\Big[\langle u,v\rangle\Big]_\alpha\Big) = 4.5(1-\alpha) \geq len\Big(\Big[\langle u',v'\rangle\Big]_\alpha\Big) = 2(1-\alpha)$$
$$len\Big(\Big[\langle u,v\rangle\Big]^\alpha) = 6.5(1-\alpha) \geq len\Big(\Big[\langle u',v'\rangle\Big]^\alpha\Big) = 3(1-\alpha)$$

then, we get the level cuts of (z,w) as follows :

$$\Big[\langle z,w\rangle\Big]_\alpha = \Big[1+\alpha;3.5-1.5\alpha\Big]$$
$$\Big[\langle z,w\rangle\Big]^\alpha = \Big[0.5+1.5\alpha;4-2\alpha\Big]$$

with $\Big[\langle z,w\rangle\Big]_l^-(1) \leq \Big[\langle z,w\rangle\Big]_r^-(1)$ and $\Big[\langle z,w\rangle\Big]_l^+(1) \leq \Big[\langle z,w\rangle\Big]_r^+(1)$.

The conditions (4) and (5) are satisfied, then $\langle u,v\rangle \ominus \langle u',v'\rangle = \langle z,w\rangle$ exists.

Example 2. The monotonicity of $\Big[\langle z,w\rangle\Big]_l^-(\alpha)$, $\Big[\langle z,w\rangle\Big]_r^-(\alpha)$, $\Big[\langle z,w\rangle\Big]_l^+(\alpha)$ and $\Big[\langle z,w\rangle\Big]_r^+(\alpha)$ is an important condition for the existence of $\langle u,v\rangle \ominus \langle u',v'\rangle = \langle z,w\rangle$ and is to be verified explicitly as in fact it may not be satisfied.

Consider $\langle u,v\rangle = \langle 5,9,11;4,9,13\rangle$, $\langle u',v'\rangle = \langle 12,15,19;8,15,20\rangle$.

We have

$$\Big[\langle u,v\rangle\Big]_\alpha = \Big[5+4\alpha;11-2\alpha\Big], \quad \Big[\langle u',v'\rangle\Big]_\alpha = \Big[11+3\alpha;19-4\alpha\Big]$$

then $\Big[\langle z,w\rangle\Big]_\alpha = \Big[-7+\alpha;-8+2\alpha\Big]$ and $\Big[\langle z,w\rangle\Big]_l^+(1) \leq \Big[\langle z,w\rangle\Big]_r^+(1)$ but $\Big[\langle z,w\rangle\Big]_r^+(\alpha)$ not decreasing as required by (4) .

Furthermore,

$$\Big[\langle u,v\rangle\Big]^\alpha = \Big[4+5\alpha;13-4\alpha\Big] \ , \ \Big[\langle u',v'\rangle\Big]^\alpha = \Big[8+7\alpha;20-5\alpha\Big]$$

then $\Big[\langle z,w\rangle\Big]^\alpha = \Big[-4-2\alpha;-7+\alpha\Big]$ and $\Big[\langle z,w\rangle\Big]_l^-(1) \leq \Big[\langle z,w\rangle\Big]_r^-(1)$ but $\Big[\langle z,w\rangle\Big]_l^-(\alpha)$ not increasing and $\Big[\langle z,w\rangle\Big]_r^-(\alpha)$ not decreasing as required by (5).

Theorem 1. *Let $\langle u,v\rangle$,$\langle z,w\rangle \in IF_1$ be two IFNs with α-cuts given by $\Big[\langle u,v\rangle\Big]_\alpha$, $\Big[\langle u,v\rangle\Big]^\alpha$ and $\Big[\langle z,w\rangle\Big]_\alpha$, $\Big[\langle z,w\rangle\Big]^\alpha$ respectively; the H-difference $\langle u,v\rangle \ominus \langle z,w\rangle \in IF_1$ exists if and only if the conditions are satisfied :*

$$(a)\begin{cases} len\Big(\Big[\langle u,v\rangle\Big]_{\alpha}\Big) \geq len\Big(\Big[\langle z,w\rangle\Big]_{\alpha}\Big) & \textit{for all } \alpha\in[0,1] \\ \Big[\langle u,v\rangle\Big]_{l}^{+}(\alpha)-\Big[\langle z,w\rangle\Big]_{l}^{+}(\alpha) & \textit{is increasing with respect to } \alpha \\ \Big[\langle u,v\rangle\Big]_{r}^{+}(\alpha)-\Big[\langle z,w\rangle\Big]_{r}^{+}(\alpha) & \textit{is decreasing with respect to } \alpha \end{cases}$$

and

$$(b)\begin{cases} len\Big(\Big[\langle u,v\rangle\Big]^{\alpha}\Big) \geq len\Big(\Big[\langle z,w\rangle\Big]^{\alpha}\Big) & \textit{for all } \alpha\in[0,1] \\ \Big[\langle u,v\rangle\Big]_{l}^{-}(\alpha)-\Big[\langle z,w\rangle\Big]_{l}^{-}(\alpha) & \textit{is increasing with respect to } \alpha \\ \Big[\langle u,v\rangle\Big]_{r}^{-}(\alpha)-\Big[\langle z,w\rangle\Big]_{r}^{-}(\alpha) & \textit{is decreasing with respect to } \alpha \end{cases}$$

Proof. We assume that (a) and (b) are satisfied then, we have

$$(a)\begin{cases} len\Big(\Big[\langle u,v\rangle\Big]_{\alpha}\Big) \geq len\Big(\Big[\langle z,w\rangle\Big]_{\alpha}\Big) & \text{for all } \alpha\in[0,1] \\ \Big[\langle u,v\rangle\Big]_{l}^{+}(\alpha)-\Big[\langle z,w\rangle\Big]_{l}^{+}(\alpha) & \text{is increasing with respect to } \alpha \\ \Big[\langle u,v\rangle\Big]_{r}^{+}(\alpha)-\Big[\langle z,w\rangle\Big]_{r}^{+}(\alpha) & \text{is decreasing with respect to } \alpha \end{cases}$$

with

$$\Big[\langle u,v\rangle\Big]_{\alpha}=\Big[\Big[\langle u,v\rangle\Big]_{l}^{+}(\alpha),\Big[\langle u,v\rangle\Big]_{r}^{+}(\alpha)\Big],\quad \Big[\langle z,w\rangle\Big]_{\alpha}=\Big[\Big[\langle z,w\rangle\Big]_{l}^{+}(\alpha),\Big[\langle z,w\rangle\Big]_{r}^{+}(\alpha)\Big]$$

from Proposition 2, we deduce that $\Big[\langle u,v\rangle\Big]_{\alpha}\ominus\Big[\langle z,w\rangle\Big]_{\alpha}$ exists in E^1. Furthermore, we have

$$(b)\begin{cases} len\Big(\Big[\langle u,v\rangle\Big]^{\alpha}\Big) \geq len\Big(\Big[\langle z,w\rangle\Big]^{\alpha}\Big) & \text{for all } \alpha\in[0,1] \\ \Big[\langle u,v\rangle\Big]_{l}^{-}(\alpha)-\Big[\langle z,w\rangle\Big]_{l}^{-}(\alpha) & \text{is increasing with respect } to\ \alpha \\ \Big[\langle u,v\rangle\Big]_{r}^{-}(\alpha)-\Big[\langle z,w\rangle\Big]_{r}^{-}(\alpha) & \text{is decreasing with respect } to\ \alpha \end{cases}$$

with

$$\Big[\langle u,v\rangle\Big]^{\alpha}=\Big[\Big[\langle u,v\rangle\Big]_{l}^{-}(\alpha),\Big[\langle u,v\rangle\Big]_{r}^{-}(\alpha)\Big],\quad \Big[\langle z,w\rangle\Big]^{\alpha}=\Big[\Big[\langle z,w\rangle\Big]_{l}^{-}(\alpha),\Big[\langle z,w\rangle\Big]_{r}^{-}(\alpha)\Big]$$

from Proposition 2, we deduce that $\Big[\langle z,w\rangle\Big]^{\alpha}\ominus\Big[\langle u,v\rangle\Big]^{\alpha}$ exists in E^1
In addition, the conditions (4) and (5) are satisfied then $\langle u,v\rangle\ominus\langle z,w\rangle\in\mathbb{F}_1$ exists. $\square$

Proposition 6. $\Big($***Unicity of*** $\langle u,v\rangle\ominus\langle u',v'\rangle\Big)$
If $\langle z,w\rangle=\langle u,v\rangle\ominus\langle u',v'\rangle$ *exists, it is unique.*

Proof. We have $\langle u,v\rangle=\langle u',v'\rangle\oplus\langle z,w\rangle$, if there exists $\langle z',w'\rangle\in\mathbb{F}_1$ such that

$$\langle u,v\rangle=\langle u',v'\rangle\oplus\langle z',w'\rangle$$

then

$$\Big[\langle u',v'\rangle \oplus \langle z,w\rangle\Big]_\alpha = \Big[\langle u,v\rangle\Big]_\alpha, \quad \Big[\langle u',v'\rangle \oplus \langle z,w\rangle\Big]^\alpha = \Big[\langle u,v\rangle\Big]^\alpha$$

$$\Big[\langle u',v'\rangle \oplus \langle z',w'\rangle\Big]_\alpha = \Big[\langle u,v\rangle\Big]_\alpha, \quad \Big[\langle u',v'\rangle \oplus \langle z',w'\rangle\Big]^\alpha = \Big[\langle u,v\rangle\Big]^\alpha$$

with the remark 1 obtining

$$\Big[\langle z,w\rangle\Big]^\alpha = \Big[\langle z',w'\rangle\Big]^\alpha$$

$$\Big[\langle z,w\rangle\Big]_\alpha = \Big[\langle z',w'\rangle\Big]_\alpha$$

This proves the uniqueness. $\square$

Proposition 7. *Given $\langle u,v\rangle$, $\langle u',v'\rangle \in I\!F_1$, the H-difference $\ominus$ has the following properties:*

1. $\langle u,v\rangle \ominus \langle u,v\rangle = 0_{\langle 1,0\rangle}$;
2. $\Big(\langle u,v\rangle \oplus \langle u',v'\rangle\Big) \ominus \langle u',v'\rangle = \langle u,v\rangle$.

5 Continuity-differentiability in the intuitionistic fuzzy metric space

In this section we introduce the notions of continuity and differentiability in the Intuitionistic fuzzy metric space $\Big(I\!F_1, d_p\Big)$

5.1 Intuitionistic fuzzy metric space $\Big(I\!F_1, d_p\Big)$

On the space $I\!F_1$ we will consider the following L_p-metric,

$$d_p\Big(\langle u,v\rangle, \langle z,w\rangle\Big) = \frac{1}{4}\bigg(\int_0^1 \Big|\Big[\langle u,v\rangle\Big]_r^+(\alpha) - \Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big|^p d\alpha + \int_0^1 \Big|\Big[\langle u,v\rangle\Big]_l^+(\alpha) - \Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big|^p d\alpha + \int_0^1 \Big|\Big[\langle u,v\rangle\Big]_r^-(\alpha) - \Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big|^p d\alpha + \int_0^1 \Big|\Big[\langle u,v\rangle\Big]_l^-(\alpha) - \Big[\langle z,w\rangle\Big]_l^-(\alpha)\Big|^p d\alpha\bigg)^{\frac{1}{p}}$$

and

$$d_\infty\Big(\langle u,v\rangle, \langle z,w\rangle\Big) = \frac{1}{4}\bigg(\sup_{0<\alpha\le 1}\Big|\Big[\langle u,v\rangle\Big]_r^+(\alpha) - \Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big| + \sup_{0<\alpha\le 1}\Big|\Big[\langle u,v\rangle\Big]_l^+(\alpha) - \Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big| + \sup_{0<\alpha\le 1}\Big|\Big[\langle u,v\rangle\Big]_r^-(\alpha) - \Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big| + \sup_{0<\alpha\le 1}\Big|\Big[\langle u,v\rangle\Big]_l^-(\alpha) - \Big[\langle z,w\rangle\Big]_l^-(\alpha)\Big|\bigg)$$

Proposition 8. *[8] $\Big(I\!F_1, d_p\Big)$ is a metric space.*

Theorem 2. *[8] $\Big(I\!F_1, d_p\Big)$ is a complete space, for $p \in [0, +\infty]$.*

5.2 Continuity

Definition 5. *Let $F : I\!F_1 \to I\!F_1$ be an intuitionistic fuzzy valued mapping and $\langle u, v\rangle \in I\!F_1$. Then F is called intuitionistic fuzzy continuous in $\langle u, v\rangle$ if and only if :*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall \langle z, w\rangle \in I\!F_1)(d_p\Big(\langle u, v\rangle , \langle z, w\rangle \Big) < \delta) \Rightarrow d_p\Big(F(\langle u, v\rangle), F(\langle z, w\rangle)\Big) < \varepsilon.$$

Definition 6. *Let $F : [a, b] \to I\!F_1$ be an intuitionistic fuzzy valued mapping and $t_0 \in [a, b]$. Then F is called intuitionistic fuzzy continuous in t_0 if and only if :*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall t \in [a, b] \textit{such as} \mid t - t_0 \mid < \delta) \Rightarrow d_p\Big(F(t), F(t_0)\Big) < \varepsilon.$$

Definition 7. *F is called intuitionistic fuzzy continuous if and only if is intuitionistic fuzzy continuous in every point of $[a, b]$.*

5.3 Differentiability

Definition 8. *A mapping $F : (a, b) \to I\!F_1$ is said to be Hukuhara derivable at t_0 if there exist $F'(t_0) \in I\!F_1$ such that both limits:*

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) \ominus F(t_0)}{\Delta t} \quad \textit{and} \quad \lim_{\Delta t \to 0^+} \frac{F(t_0) \ominus F(t_0 - \Delta t)}{\Delta t} \tag{6}$$

exist and they are equal to $F'(t_0) = \langle u'(t_0), v'(t_0)\rangle$, which is called the Hukuhara derivative of F at t_0. At the end points of $[a, b]$ we consider only the one-sided derivatives.

Definition 9. *Let $F : [a, b] \to I\!F_1$ be an intuitionistic fuzzy valued mapping. Let $P : [a, b] \to I\!F_1$ be a Hukuhara derivative mapping at every $t \in (a, b)$. P is said to be a primitive of F if the Hukuhara derivative of P equals F for every $t \in (a, b)$, that is, $P'(t) = F(t)$.*

Theorem 3. *Let $F : [a, b] \to I\!F_1$ be differentiable. Denote $F^\alpha(t) = [F(t)]^\alpha = [\lambda_\alpha(t), \lambda^\alpha(t)]$, $F_\alpha(t) = [F(t)]_\alpha = [\mu_\alpha(t), \mu^\alpha(t)]$. Then $\lambda_\alpha(t), \lambda^\alpha(t), \mu_\alpha(t)$ and $\mu^\alpha(t)$ are differentiable*

$$[F(t)']^\alpha = [\lambda'_\alpha(t), \lambda^{\alpha\prime}(t)], \qquad [F(t)']_\alpha = [\mu'_\alpha(t), \mu^{\alpha\prime}(t)] \tag{7}$$

Proof. We prove that for F^α, and its similarly for F_α. Now

$$\Big[F(t + h) \ominus F(t)\Big]^\alpha = \Big[\lambda_\alpha(t + h) - \lambda_\alpha(t), \lambda^\alpha(t + h) - \lambda^\alpha(t)\Big]$$
$$\Big[F(t) \ominus F(t - h)\Big]^\alpha = \Big[\lambda_\alpha(t) - \lambda_\alpha(t - h), \lambda^\alpha(t) - \lambda^\alpha(t - h)\Big]$$

divided by h, the result is obtained by passage to the limit ($h \longrightarrow 0$). □

Proposition 9. *Let $F : [a, b] \to I\!F_1$ and $G : [a, b] \to I\!F_1$ be two Hukuhara derivable mappings. If F and G are both primitives of the same mapping and there exists $F(t) \ominus G(t)$ for every $t \in (a; b)$, then $F(t) = G(t) \oplus C$, being $C \in I\!F_1$.*

Proof. Let $F(t) = G(t) \oplus C(t)$, By taking the Hukuhara derivative at both sides, we have that $F'(t) = G'(t) \oplus C'(t)$, and hence $C'(t) = 0_{\langle 1,0\rangle}$ for every $t \in (a, b)$ which implies that C is constant. □

Theorem 4. *If $F : [a,b] \to I\!F_1$ is differentiable then it is continuous with respect to the metric d_∞.*

Proof. Let $t, t+h \in [a,b]$ with $h > 0$.

$$d_\infty\Big(F(t+h), F(t)\Big) = d_\infty\Big(F(t+h) \ominus F(t), 0_{\langle 1,0\rangle}\Big)$$
$$\leq h d_\infty\Big(\frac{F(t+h) \ominus F(t)}{h}, F'(t)\Big) + h d_\infty(F'(t), 0_{\langle 1,0\rangle}),$$

where h is so small that the H-difference $F(t+h) \ominus F(t)$ exists.

When $h \to 0$ the right-hand side goes to 0 and hence F is right continuous. The left continuity is similarly proven. □

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