Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 23, 2017, No. 5, 1–6

Properties of the intuitionistic fuzzy implication $\rightarrow_{188}$

Krassimir Atanassov[1], Eulalia Szmidt[2], Janusz Kacprzyk[2] and Nora Angelova[3]

[1] Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria,
and
Intelligent Systems Laboratory, Prof. Asen Zlatarov University
1, Yakim Yakimov Blvd., Burgas-8010, Bulgaria
e-mail: krat@bas.bg

[2] Systems Research Institute, Polish Academy of Sciences,
ul. Newelska 6, 01-447 Warsaw, Poland
and
Warsaw School of Information Technology,
ul. Newelska 6, 01-447 Warsaw, Poland
e-mails: {szmidt,kacprzyk}@ibspan.waw.pl

[3] Department of Computer Informatics
Faculty of Mathematics and Informatics
University of Sofia "St. Kliment Ohridski"
5, James Bourchier Blvd., Sofia-1164, Bulgaria

In memory of Prof. Lotfi Zadeh (1921 – 2017)

Received: 27 October 2017 **Accepted:** 26 November 2017

Abstract. In [5], the new intuitionistic fuzzy implication $\rightarrow_{188}$ is defined and some of its properties are studied. Here, new propertirs of the new implication are studied.

Keywords: Implication, Intuitionistic fuzzy implication, Intuitionistic fuzzy logic.
AMS Classification: 03E72.

1 Introduction

In [5], the intuitionistic fuzzy implication $\rightarrow_{188}$ is introduced and some of its properties are studied. Here, we continue the previous research.

Initially, we remind that in intuitionistic fuzzy logic (see [1, 2]), each proposition, variable or formula is evaluated with two degrees – "truth degree" or "degree of validity" $\mu(p)$ and "falsity degree" or "degree of non-validity" $\nu(p)$. Thus, to each one of these objects, e.g., p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p), \nu(p) \in [0,1] \text{ and } \mu(p) + \nu(p) \leq 1.$$

Let

$$\pi(p) = 1 - \mu(p) - \nu(p).$$

The above function determines the degree of uncertainty (indeterminacy).

Let an evaluation function V be defined over a set of propositions $\mathcal{S}$, in such a way that for $p \in \mathcal{S}$:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Hence the function $V : \mathcal{S} \rightarrow [0,1] \times [0,1]$ gives the truth and falsity degrees of all elements of $\mathcal{S}$.

We assume that the evaluation function V assigns to the logical truth T

$$V(T) = \langle 1, 0 \rangle,$$

and to the logical falsity F

$$V(F) = \langle 0, 1 \rangle.$$

Here, we define only the operations "negation", "disjunction" and "conjunction", originally introduced in [1, 2], that have classical logic analogues, as follows:

$$V(\neg_1 p) = \langle \nu(p), \mu(p) \rangle,$$

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle,$$

$$V(p \wedge q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle.$$

Below, for simplicity, we write $\neg$ instead of $\neg_1$.

For the needs of the discussion below, we define the notions of Intuitionistic Fuzzy Tautology (IFT, see, e.g. [1, 2]) and tautology.

Formula A is an IFT if and only if (iff) for every evaluation function V, if $V(A) = \langle a, b \rangle$, then,

$$a \geq b,$$

while it is a (classical) tautology if and only if for every evaluation function V, if $V(A) = \langle a, b \rangle$, then,

$$a = 1,\ b = 0.$$

Below, when it is clear, we will omit notation "$V(A)$", using directly "A" instead of the intuitionistic fuzzy evaluation of A.

In [3], we called the object $\langle \mu(p), \nu(p) \rangle$ an Intuitionistic Fuzzy Pair (IFP).

For brevity, in a lot of places, instead of the IFP $\langle \mu(A), \nu(A) \rangle$ we will use the IFP $\langle a, b \rangle$, where $a, b \in [0,1]$ and $a + b \leq 1$.

It is also suitable, if $\langle a, b \rangle$ and $\langle c, d \rangle$ are IFPs, to have

$$\langle a, b \rangle \leq \langle c, d \rangle \text{ iff } a \leq c \text{ and } b \geq d$$

and

$$\langle a, b \rangle \geq \langle c, d \rangle \text{ iff } a \geq c \text{ and } b \leq d.$$

If an IFP is an IFT, we call it Intuitionistic Fuzzy Tautological Pair (IFTP) and if it is a tautology – Tautological Intuitionistic Fuzzy Pair (TIFP).

In [5], the intuitionistic fuzzy implication $\rightarrow_{188}$ is defined by:

$$x \rightarrow_{188} y = \neg x \vee y = \langle \min(b, c), ad \rangle.$$

2 Main results

Here, we show that the implication $\rightarrow_{188}$ generates a new negation with the form

$$\neg_{54} \langle a, b \rangle = \langle a, b \rangle \rightarrow_{188} \langle 0, 1 \rangle = \langle 0, a \rangle.$$

For brevity, below we will write $\rightarrow$ instead of $\rightarrow_{188}$.

In [5], it is checked the validity of G.F. Rose's formula [13, 15] that has the form:

$$((\neg\neg x \rightarrow_{188} x) \rightarrow_{188} (\neg\neg x \vee \neg x)) \rightarrow_{188} (\neg\neg x \vee \neg x)$$

when negation is the classical negation $\neg_1$. Now, we prove

Theorem 1. *Rose's formula is an IFT for* $\neg_{54}$.

Proof. Sequentially, we obtain:

$$\begin{aligned} &((\neg_{54}\neg_{54} x \rightarrow_{188} x) \rightarrow_{188} (\neg_{54}\neg_{54} x \vee \neg_{54} x)) \rightarrow_{188} (\neg_{54}\neg_{54} x \vee \neg_{54} x) \\ &= (((\langle 0,0 \rangle \rightarrow_{188} \langle a, b \rangle) \rightarrow_{188} (\langle 0,0 \rangle \vee \langle 0, a \rangle)) \rightarrow_{188} (\langle 0,0 \rangle \vee \langle 0, a \rangle) \\ &= (\langle 0,0 \rangle \rightarrow_{188} (\langle 0,0 \rangle)) \rightarrow_{188} \langle 0,0 \rangle \\ &= \langle 0,0 \rangle \rightarrow_{188} \langle 0,0 \rangle) = \langle 0,0 \rangle. \end{aligned}$$

that is an IFTP. □

Second, we check C. A. Meredith's axiom (see, e.g., [12]).

Theorem 2. *For every five formulas A, B, C, D and E,* C. A. *Meredith's axiom*

$$((((A \rightarrow B) \rightarrow (\neg C \rightarrow \neg D)) \rightarrow C) \rightarrow E) \rightarrow ((E \rightarrow A) \rightarrow (D \rightarrow A))$$

is an IFT for $\neg_1$ *and for* $\neg_{54}$.

Proof. Let $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$, $V(C) = \langle e, f\rangle$, $V(D) = \langle g, h\rangle$, $V(E) = \langle i, j\rangle$, where $a, b, \ldots, j \in [0, 1]$ and $a + b \leq 1$, $c + d \leq 1$, $e + f \leq 1$, $g + h \leq 1$ and $i + j \leq 1$. Then

$$V(((((A \to B) \to (\neg_{54} C \to \neg_{54} D)) \to C) \to E) \to ((E \to A) \to (D \to A)))$$

$$= (((((\langle a, b\rangle \to \langle c, d\rangle) \to (\langle 0, e\rangle \to \langle 0, g\rangle)) \to \langle e, f\rangle) \to \langle i, j\rangle)$$

$$\to ((\langle i, j\rangle \to \langle a, b\rangle) \to (\langle g, h\rangle \to \langle a, b\rangle))$$

$$= ((((\langle \min(b, c), ad\rangle \to \langle 0, 0\rangle) \to \langle e, f\rangle) \to \langle i, j\rangle) \to (\langle \min(a, j), bi\rangle \to \langle \min(a, h), bg\rangle)$$

$$= (((\langle 0, 0\rangle \to \langle e, f\rangle) \to \langle i, j\rangle) \to \langle \min(bi, a, h), \min(a, j)bg\rangle$$

$$= (\langle 0, 0\rangle \to \langle i, j\rangle) \to \langle \min(bi, a, h), \min(a, j)bg\rangle$$

$$= (\langle 0, 0\rangle \to \langle i, j\rangle) \to \langle \min(bi, a, h), \min(a, j)bg\rangle$$

$$= \langle 0, 0\rangle.$$

The first case is proved by analogy.

The next assertions are proved by the same manner so we will omit their proofs. □

The axioms of the intuitionistic logic (see, e.g., [14]) are the following.

(IL1) $A \to A$,

(IL2) $A \to (B \to A)$,

(IL3) $A \to (B \to (A \wedge B))$,

(IL4) $(A \to (B \to C)) \to (B \to (A \to C))$,

(IL5) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

(IL6) $A \to \neg\neg A$,

(IL7) $\neg(A \wedge \neg A)$,

(IL8) $(\neg A \vee B) \to (A \to B)$,

(IL9) $\neg(A \vee B) \to (\neg A \wedge \neg B)$,

(IL10) $(\neg A \wedge \neg B) \to \neg(A \vee B)$,

(IL11) $(\neg A \vee \neg B) \to \neg(A \wedge B)$,

(IL12) $(A \to B) \to (\neg B \to \neg A)$,

(IL13) $(A \to \neg B) \to (B \to \neg A)$,

(IL14) $\neg\neg\neg A \to \neg A$,

(IL15) $\neg A \to \neg\neg\neg A$,

(IL16) $\neg\neg(A \to B) \to (A \to \neg\neg B)$,

(IL17) $(C \to A) \to ((C \to (A \to B)) \to (C \to B))$.

Theorem 3. *All axioms of the intuitionistic logic are IFTs for* $\to_{188}$ *and for* $\neg_{54}$.

The axioms of A. Kolmogorov (see, e.g., [16]) are the following.

(K1) $A \to (B \to A)$,

(K2) $(A \to (A \to B)) \to (A \to B))$,

(K3) $(A \to (B \to C)) \to (B \to (A \to C))$,

(K4) $(B \to C) \to ((A \to B) \to (A \to C))$,

(K5) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

Theorem 4. *All axioms of A. Kolmogorov are IFTs for* $\to_{188}$ *and for* $\neg_{54}$.

The axioms of J. Łukasiewicz and A. Tarski (see, e.g., [16]) are the following.

(LT1) $A \to (B \to A)$,

(LT2) $(A \to B) \to ((B \to C) \to (A \to C))$,

(LT3) $\neg A \to (\neg B \to (B \to A))$,

(LT4) $((A \to \neg A) \to A) \to A$.

Theorem 5. *All axioms of J. Łukasiewicz and A. Tarski are IFTs for* $\to_{188}$ *and for* $\neg_{54}$.

3 Conclusion

In a next reseach, we will study validity of Klir and Yuan's axioms for the intuitionistc fuzzy implications $\to_{187}$ (introduced in [4]) and $\to_{188}$ and other properties of these implications. Meantime, in [7], another implication - $\to_{189}$, related to the two our implications, was introduced and in [8] its properties had been studied. All these research show that intuitionistc fuzzy sets and logics in the sense, described in [2] correspond to the ideas of Brouwer's intuitionism (see [9, 10, 11]).

Acknowledgements

The authors are thankful for the support provided by the Bulgarian National Science Fund under Grant Ref. No. DFNI-I-02-5 "InterCriteria Analysis: A New Approach to Decision Making."

References

[1] Atanassov, K. (1988) Two variants of intuitionistic fuzzy propositional calculus. *Preprint IM-MFAIS-5-88*, Sofia, Reprinted: *Int J Bioautomation*, 2016, 20(S1), S17–S26.

[2] Atanassov, K. (2017) *Intuitionistic Fuzzy Logics*, Springer, Cham.

[3] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013) On intuitionistic fuzzy pairs, *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[4] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2017) On intuitionistic fuzzy implication $\rightarrow_{187}$, *Notes on Intuitionistic Fuzzy Sets*, 23(2), 37–43.

[5] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2017) On intuitionistic fuzzy implication $\rightarrow_{188}$, *Notes on Intuitionistic Fuzzy Sets*, 23(1), 6–13.

[6] Atanassov, K., Szmidt, E., & Angelova, N. (2017) Properties of the intuitionistic fuzzy implication $\rightarrow_{187}$, *Notes on Intuitionistic Fuzzy Sets*, 23(3), 3–8.

[7] Atanassova, L. (2017) Intuitionistic fuzzy implication $\rightarrow_{189}$, *Notes on Intuitionistic Fuzzy Sets*, 23(1), 14–20.

[8] Atanassova, L. (2017) Properties of the intuitionistic fuzzy implication $\rightarrow_{189}$. *Notes on Intuitionistic Fuzzy Sets*, 23(4), 10–14

[9] Van Atten, M. (2004) *On Brouwer*, Wadsworth, Behnout.

[10] Brouwer, L. E. J. (1975) *Collected Works*, Vol. 1, North Holland, Amsterdam.

[11] Van Dalen, D. (Ed.) (1981) Brouwer's Cambridge Lectures on Intuitionism, Cambridge Univ. Press, Cambridge.

[12] Mendelson, E. (1964) *Introduction to Mathematical Logic*, Princeton, NJ: D. Van Nostrand.

[13] Plisko, V. (2009) A survey of propositional realizability logic, *The Bulletin of Symbolic Logic*, 15(1), 1–42.

[14] Rasiova H. & Sikorski, R. (1963) *The mathematics of Metamathematics*, Pol. Acad. of Sci., Warszawa.

[15] Rose, G. F. (1953) Propositional calculus and realizability, *Transactions of the American Mathematical Society*, 75, 1–19.

[16] Tabakov, M. (1986) *Logics and Axiomatics*, Nauka i Izkustvo, Sofia (in Bulgarian).