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Characterization of compact subset of intuitionistic fuzzy sets

S. Melliani, R. Ettoussi, M. Elomari and L. S. Chadli

LMACS, Laboratoire de Mathématiques Appliquées & Calcul Scientifique
Sultan Moulay Slimane University, PO Box 523, 23000 Beni Mellal, Morocco
e-mail: said.melliani@gmail.com

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Abstract: In this paper we introduce two classes of metrics for spaces of intuitionistic fuzzy sets IF_n. The spaces shown to be complete. We use the support function to embed the metric space (IF_n, d_p) into Banach space, and we give a sufficient and necessary condition for characterization of compacts and locally compact subsets of space IF_n.
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1 Introduction

Applications of fuzzy set theory very often involve the metric space (E^n, d_p), for $1 \leq p \leq \infty$ of normal fuzzy convex fuzzy sets over $\mathbb{R}^n$, where d_p extends the Hausdorff metric. This metric has been found very convenient in studying, for example, fuzzy differential equations (Kaleva [6]), dynamical systems (Kloeden [7]). A characterization of compact subset is discussed for the metric space of normal fuzzy convex fuzzy sets on the space $\mathbb{R}^n$ the metric for which the supremum over the Hausdorff distance between corresponding level sets [3], also for $1 \leq p < \infty$, Diamond and Kloeden [4] are discussed the characterization of compact and locally compact subset.

As the intuitionistic fuzzy sets is a generalization of fuzzy sets so we propose in this paper to introduce the metrics on the space of intuitionistic fuzzy numbers IF_n. The first metric d_p is based upon L_p metrics and the second metric d_∞ extends the Hausdorff metric. Our principal result is that, for each $1 \leq p \leq \infty$ the metric spaces (IF_n, d_p) are complete, A characterization of the compact subsets in these spaces is also given in terms of boundedness and p-mean equileft-continuity.

Various definitions and preliminaries are set out in Section 2. In section 3, the space IF_n^L embedded into $\mathcal{C}([0,1] \times S^{n-1})$ (Banach space of continuous functions on $[0,1] \times S^{n-1}$) by using the support function. Finally we present a characterization of compacts and locally compact subsets of IF_n.

2 Preliminaries

Definition 2.1. *An intuitionistic fuzzy set A in X ([1], [2]) is a set of ordered triples*

$$A = \{< x, \mu_A(x), \nu_A(x) >, x \in X\}$$

where $\mu_A, \nu_A : X \to [0,1]$ are functions such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \ \ , \ \ \forall x \in X$$

For each x the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and degree of nonmembership of the element $x \in X$ to $A \subset X$, respectively. For each element $x \in X$ we can compute the so-called, the intuitionistic fuzzy index of x in A defined as follows

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

Of course, a fuzzy set is a particular case of the intuitionistic fuzzy set with $\nu_A(x) = 1 - \mu_A(x)$. Hereinafter $X = \mathbb{R}^n$. We denote by

$$\text{IF}_n = \text{IF}(\mathbb{R}^n) = \left\{ \langle u, v \rangle \ : \ \mathbb{R}^n \to [0,1]^2 \, , |\forall \, x \in \mathbb{R}^n \ |0 \leq u(x) + v(x) \leq 1 \right\} .$$

An element $\langle u, v \rangle$ of IF_n is said to be an intuitionistic fuzzy number if it satisfies the following conditions:

(i) $\langle u, v \rangle$ is normal i.e there exists x_0, $x_1 \in \mathbb{R}^n$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous

(iv) $supp \, \langle u, v \rangle = \text{cl}\{x \in \mathbb{R}^n \ : | \ v(x) < 1\}$ is bounded.

So we denote the collection of all intuitionistic fuzzy number by IF_n.

For $\alpha \in [0,1]$ and $\langle u, v \rangle \in \text{IF}^n$, the upper and lower α-cuts of $\langle u, v \rangle$ are defined by

$$[\langle u, v \rangle]^\alpha = \{x \in \mathbb{R}^n : v(x) \leq 1 - \alpha\}$$

and

$$[\langle u, v \rangle]_\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$$

Remark 2.1. *If $\langle u, v \rangle \in IF_n$, so we can see $[\langle u, v \rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v \rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.*

We define $0_{\langle 1,0\rangle} \in IF_n$ as

$$0_{\langle 1,0\rangle}(t) = \begin{cases} \langle 1,0\rangle & t=0 \\ \langle 0,1\rangle & t\neq 0 \end{cases}.$$

Let $\langle u,v\rangle$,$\langle u',v'\rangle \in \mathrm{IF}_n$ and $\lambda \in \mathbb{R}$, we define the following operations by:

$$\Big(\langle u,v\rangle \oplus \langle u',v'\rangle\Big)(z) = \Big(\sup_{z=x+y} \min\left(u(x), u'(y)\right), \inf_{z=x+y} \max\left(v(x), v'(y)\right)\Big)$$

$$\lambda\langle u,v\rangle = \begin{cases} \langle \lambda u, \lambda v\rangle & \text{if } \lambda \neq 0 \\ 0_{\langle 1,0\rangle} & \text{if } \lambda = 0 \end{cases}.$$

For $\langle u,v\rangle$, $\langle z,w\rangle \in \mathrm{IF}_n$ and $\lambda \in \mathbb{R}$, the addition and scaler-multiplication are defined as follows

$$\begin{aligned} \Big[\langle u,v\rangle \oplus \langle z,w\rangle\Big]^\alpha &= \Big[\langle u,v\rangle\Big]^\alpha + \Big[\langle z,w\rangle\Big]^\alpha, \\ \Big[\lambda\langle z,w\rangle\Big]^\alpha &= \lambda\Big[\langle z,w\rangle\Big]^\alpha \end{aligned} \tag{1}$$

$$\begin{aligned} \Big[\langle u,v\rangle \oplus \langle z,w\rangle\Big]_\alpha &= \Big[\langle u,v\rangle\Big]_\alpha + \Big[\langle z,w\rangle\Big]_\alpha. \\ \Big[\lambda\langle z,w\rangle\Big]_\alpha &= \lambda\Big[\langle z,w\rangle\Big]_\alpha \end{aligned} \tag{2}$$

Definition 2.2. *Let $\langle u,v\rangle$ be an element of IF_n and $\alpha \in [0,1]$, we define the following sets:*

$$\begin{aligned} \Big[\langle u,v\rangle\Big]_l^+(\alpha) &= \inf\{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^+(\alpha) = \sup\{x \in \mathbb{R}^n \mid u(x) \geq \alpha\} \\ \Big[\langle u,v\rangle\Big]_l^-(\alpha) &= \inf\{x \in \mathbb{R}^n \mid v(x) \leq 1-\alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^-(\alpha) = \sup\{x \in \mathbb{R}^n \mid v(x) \leq 1-\alpha\} \end{aligned}$$

Proposition 2.1. *For all α, $\beta \in [0,1]$ and $\langle u,v\rangle \in IF_n$*

(i) $\Big[\langle u,v\rangle\Big]_\alpha \subset \Big[\langle u,v\rangle\Big]^\alpha$.

(ii) $\Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha$ *are nonempty compact convex sets in $\mathbb{R}^n$.*

(iii) If $\alpha \leq \beta$ then $\Big[\langle u,v\rangle\Big]_\beta \subset \Big[\langle u,v\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\beta \subset \Big[\langle u,v\rangle\Big]^\alpha$.

(iv) If $\alpha_n \nearrow \alpha$ then $\Big[\langle u,v\rangle\Big]_\alpha = \bigcap_n \Big[\langle u,v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u,v\rangle\Big]^\alpha = \bigcap_n \Big[\langle u,v\rangle\Big]^{\alpha_n}$.

Let M be any set and $\alpha \in [0,1]$. We denote by

$$M_\alpha = \{x \in \mathbb{R}^n \,:\, u(x) \geq \alpha\} \quad \text{and} \quad M^\alpha = \{x \in \mathbb{R}^n \,:\, v(x) \leq 1-\alpha\}.$$

Lemma 2.1. *[9] let* $\Big\{M_\alpha,\ \alpha\in[0,1]\Big\}$ *and* $\Big\{M^\alpha,\ \alpha\in[0,1]\Big\}$ *two families of subsets of* $\mathbb{R}^n$ *satisfies (i)–(iv) in Proposition 2.1, if* u *and* v *define by*

$$u(x)=\begin{cases}0 & \text{if } x\notin M_0\\ \sup\{\alpha\in[0,1]\ :\ x\in M_\alpha\} & \text{if } x\in M_0\end{cases}$$
$$v(x)=\begin{cases}1 & \text{if } x\notin M^0\\ 1-\sup\{\alpha\in[0,1]\ :\ x\in M^\alpha\} & \text{if } x\in M^0\end{cases}$$

Then $\langle u,v\rangle\in IF_n$.

Lemma 2.2. *Let* I *a dense subset of* $[0,1]$*, if* $\Big[\langle u,v\rangle\Big]_\alpha=\Big[\langle u',v'\rangle\Big]_\alpha$ *and* $\Big[\langle u,v\rangle\Big]^\alpha=\Big[\langle u',v'\rangle\Big]^\alpha$*, for all* $\alpha\in I$ *then* $\langle u,v\rangle=\langle u',v'\rangle$.

2.1 Metric on IF_n

We consider the mapping

$$d_p\left(\langle u,v\rangle,\langle u',v'\rangle\right)=\left(\int_0^1 d_H^p\left([\langle u,v\rangle]_\alpha,[\langle u',v'\rangle]_\alpha\right)d\alpha\right)^{1/p}+\left(\int_0^1 d_H^p\left([\langle u,v\rangle]^\alpha,[\langle u',v'\rangle]^\alpha\right)d\alpha\right)^{1/p},$$

where d_H is the Hausdorff metric and $p\in[1,\infty[$.

$$d_\infty\left(\langle u,v\rangle,\langle u',v'\rangle\right)=\sup_{0\le\alpha\le1} d_H\left([\langle u,v\rangle]_\alpha,[\langle u',v'\rangle]_\alpha\right)+\sup_{0\le\alpha\le1} d_H\left([\langle u,v\rangle]^\alpha,[\langle u',v'\rangle]^\alpha\right).$$

Theorem 2.1. d_p *defines a metric on* IF_n *for* $p\in[1,\infty]$.

Proof. Symmetry and the triangle inequality are trivial.

It remains to show that, if $d_p\left(\langle u,v\rangle,\langle u',v'\rangle\right)=0$ then $\langle u,v\rangle=\langle u',v'\rangle$.

Suppose that $d_p\left(\langle u,v\rangle,\langle u',v'\rangle\right)=0$, then $d_H^p\left([\langle u,v\rangle]_\alpha,[\langle u',v'\rangle]_\alpha\right)=0$ and $d_H^p\left([\langle u,v\rangle]^\alpha,[\langle u',v'\rangle]^\alpha\right)=0$, except where α describes some A negligible for the Lebesgue-measuring, which is complementary dense.

As, d_H is a metric on the space $\mathcal{K}_c(\mathbb{R}^n)$ so $[\langle u,v\rangle]_\alpha=[\langle u',v'\rangle]_\alpha$ a.e and $[\langle u,v\rangle]^\alpha=[\langle u',v'\rangle]^\alpha$ a.e. According to the Lemma 2.2 the equalities hold for all α, and hence $\langle u,v\rangle=\langle u',v'\rangle$.

A similar reasoning proves that d_∞ is a metric. □

Theorem 2.2. (IF_n,d_p) *is a complete metric space.*

Proof. Let $\left([\langle u_n,v_n\rangle]\right)_n$ be a sequence of Cauchy in IF_n, for $\varepsilon>0$ there exist an integer n_0 such that for n, $q\ge n_0$ we have

$$\begin{aligned} d_p\left(\langle u_n,v_n\rangle,\langle u_q,v_q\rangle\right) &= \left(\int_0^1 d_H^p\left([\langle u_n,v_n\rangle]_\alpha,[\langle u_q,v_q\rangle]_\alpha\right)d\alpha\right)^{1/p}\\ &+\left(\int_0^1 d_H^p\left([\langle u_n,v_n\rangle]^\alpha,[\langle u_q,v_q\rangle]^\alpha\right)d\alpha\right)^{1/p}\\ &\le \varepsilon\end{aligned}$$

by the completed of the space L^p, hence

$$d_H\left([\langle u_n, v_n\rangle]_\alpha, [\langle u_q, v_q\rangle]_\alpha\right) \underbrace{\longrightarrow}_{n,q\to\infty} 0$$

and

$$d_H\left([\langle u_n, v_n\rangle]^\alpha, [\langle u_q, v_q\rangle]^\alpha\right) \underbrace{\longrightarrow}_{n,q\to\infty} 0.$$

In addition, the space $(\mathcal{K}_c(\mathbb{R}^n), d_H)$ is a complete metric space, where $\mathcal{K}_c(\mathbb{R}^n)$ is the set of all compact convex subset of $\mathbb{R}^n$, so $[\langle u_n, v_n\rangle]_\alpha \underbrace{\longrightarrow}_{n\to\infty} [\langle u, v\rangle]_\alpha$ and $[\langle u_n, v_n\rangle]^\alpha \underbrace{\longrightarrow}_{n\to\infty} [\langle u, v\rangle]^\alpha$.

Thus the sequence $\langle u_n, v_n\rangle$ converge to the limit $\langle u, v\rangle$, the construction of the $\langle u, v\rangle$ via Lemma 2.1. $\square$

3 The embedding theorem

We denote S^{n-1} the unit sphere in $\mathbb{R}^n$. Let IF_n^L the space of $\langle u, v\rangle \in IF_n$ with lipschitzian α-level sets $[\langle u, v\rangle]_\alpha$ and $[\langle u, v\rangle]^\alpha$, i.e., with

$$d_H\left([\langle u, v\rangle]_\alpha, [\langle u, v\rangle]_\beta\right) \leq K_1|\alpha - \beta|$$

and

$$d_H\left([\langle u, v\rangle]^\alpha, [\langle u, v\rangle]^\beta\right) \leq K_2|\alpha - \beta|$$

for all $\alpha, \beta \in [0, 1]$ and $K_1, K_2 \in \mathbb{R}^+$.

The addition and scalar multiplication defined by (1), (2) define a linear structure on IF_n, but does not make IF_n a vector space. So for this reason we will use the embedding theorem [8] for embed the subset IF_n^L of the space IF_n into Banach space $\mathcal{C}([0, 1] \times S^{n-1})$ by using the support function $\langle u, v\rangle^* = j(\langle u, v\rangle)$ where $\langle u, v\rangle^*$ is the support function of $\langle u, v\rangle$ defined by

$$\langle u, v\rangle^*(\alpha, x) = \sup_{a\in[\langle u,v\rangle]^\alpha} \langle a, x\rangle \text{ for all } (\alpha, x) \in I \times S^{n-1}.$$

Then for all $\langle u, v\rangle \in IF_n$, there corresponds a support function $\langle u, v\rangle^* = j(\langle u, v\rangle)$ is well-defined and satisfies the following properties

1. $\langle u, v\rangle^*$ is uniformly bounded on $I \times S^{n-1}$,

$$|\ \langle u, v\rangle^*(\alpha, x)\ | \leq \sup_{a\in[\langle u,v\rangle]^0} ||a|| \text{ for all } \alpha \in I \text{ and all } x \in S^{n-1};$$

2. $\langle u, v\rangle^*(., x)$ is nonincreasing and leftcontinuous in α for each $x \in S^{n-1}$;

3. $\langle u, v\rangle^*(\alpha, .)$ is Lipschitz continuous in x uniformly in $\alpha \in I$

$$|\ \langle u, v\rangle^*(\alpha, x) - \langle u, v\rangle^*(\alpha, y)\ | \leq \Big(\sup_{a\in[\langle u,v\rangle]^0} ||a||\Big)||x - y||$$

for all $\alpha \in I$ and all $x, y \in S^{n-1}$;

4. For each $\alpha \in I$ and $\langle u,v\rangle, \langle u',v'\rangle \in \mathrm{IF}_n$, according to Proposition 2.1 property (i), the following inequality holds

$$d_H\Big([\langle u,v\rangle]_\alpha, [\langle u'v'\rangle]_\alpha\Big) \leq d_H\Big([\langle u,v\rangle]^\alpha, [\langle u',v'\rangle]^\alpha\Big) = \sup_{x\in S^{n-1}} \big|\langle u,v\rangle^* (\alpha,x) - \langle u',v'\rangle^* (\alpha,x)\big|.$$

The following theorem gives the desired embedding, which we will used in the next section to characterize compacts subsets of IF_n.

Theorem 3.1. *There exists a function $j : IF_n^L \to \mathcal{C}\left([0,1]\times S^{n-1}\right)$ such that*

1. $d_\infty\Big(\langle u,v\rangle, \langle u',v'\rangle\Big) \leq 2||j\Big(\langle u,v\rangle\Big) - j\Big(\langle u',v'\rangle\Big)||_\infty$*,*
2. $j\Big(\langle u,v\rangle \oplus \langle u',v'\rangle\Big) = j\Big(\langle u,v\rangle\Big) + j\Big(\langle u',v'\rangle\Big)$*,*
3. $j\Big(\lambda\langle u,v\rangle\Big) = \lambda j\Big(\langle u,v\rangle\Big)$*, $\lambda \geq 0$.*

Proof. Let $\langle u,v\rangle$, $\langle u',v'\rangle \in \mathrm{IF}_n$ We define the function j by

$$j\left(\langle u,v\rangle\right) = \begin{cases} \max\limits_{a\in[\langle u,v\rangle]^\alpha} \langle a,x\rangle & if\ \alpha > 0 \\ \max\limits_{a\in supp\{\langle u,v\rangle\}} \langle a,x\rangle & if\ \alpha = 0 \end{cases}.$$

Hence, the support function verified the property (4) which allows to write

$$\begin{aligned} d_\infty\left(\langle u,v\rangle, \langle u',v'\rangle\right) &= \sup_{0\leq\alpha\leq1} d_H\left([\langle u,v\rangle]_\alpha, [\langle u',v'\rangle]_\alpha\right) + \sup_{0\leq\alpha\leq1} d_H\left([\langle u,v\rangle]^\alpha, [\langle u',v'\rangle]^\alpha\right) \\ &\leq 2\sup_{0\leq\alpha\leq1} d_H\left([\langle u,v\rangle]^\alpha, [\langle u',v'\rangle]^\alpha\right) \\ &\leq 2||j\Big(\langle u,v\rangle\Big) - j\Big(\langle u',v'\rangle\Big)||_\infty. \end{aligned}$$

For the Properties 2 and 3 we refer to [8]. □

4 Compactness in d_p topology

Definition 4.1. *Let $\langle u,v\rangle \in IF_n$. If for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \langle u,v\rangle) > 0$, such that for all $0 \leq h < \delta$,*

$$\int_h^1 d_H\left([\langle u,v\rangle]^\alpha, [\langle u,v\rangle]^{\alpha-h}\right)^p d\alpha \leq \varepsilon^p,$$

say that $\langle u,v\rangle$ is p-mean left-continuous. If for nonempty $U \subset IF_n$ this holds uniformly in $\langle u,v\rangle \in U$, we say U is p-mean equi-left-continuous. If, in addition, U is uniformly support bounded i.e (if there exists a $K \in \mathbb{R}^+$ such that $\sup\limits_{a\in[\langle u,v\rangle]^0} ||a|| \leq K$), then U is said to have the p-Blaschke property. Also, this property translates as

$$\int_h^1 \Big(\langle u,v\rangle^* (\alpha-h,x) - \langle u,v\rangle^* (\alpha,x)\Big)^p d\alpha \leq \varepsilon^p.$$

For all $0 \leq h \leq \delta$, $x \in S^{n-1}$ and $\langle u,v\rangle^* \in U^* = j(U)$.

In the limit $p = \infty$ this concept is just the Blaschke property of the previous section.

Lemma 4.1. *Any $\langle u, v\rangle \in IF_n$, is p-mean left-continuous.*

Proof. Let $\alpha \in [0,1]$ and suppose $\{\alpha_n\}$ is a nondecreasing sequence converging to α. Then, $[\langle u,v\rangle]^\alpha = \bigcap\limits_{n\geq 1} [\langle u,v\rangle]^{\alpha_n}$ and $[\langle u,v\rangle]_\alpha = \bigcap\limits_{n\geq 1} [\langle u,v\rangle]_{\alpha_n}$ so, $d_H\left([\langle u,v\rangle]_{\alpha_n}, [\langle u,v\rangle]_\alpha\right) \to 0$ and $d_H\left([\langle u,v\rangle]^{\alpha_n}, [\langle u,v\rangle]^\alpha\right) \to 0$, and the result follows from left-continuity on the compact interval $[0,1]$ □

Theorem 4.1. *A closed set U of (IF_n, d_p) $(1 \leq p < \infty)$, is compact iff U has the p-Blaschke property.*

Proof. **Necessity.** Let U be a compact set in (IF_n, d_p). If U were not uniformly support bounded, then there would exist a sequence of compact convex sets in $\mathbb{R}^n$ (i.e $(V_j)_{j\in\mathbb{N}} \subset \mathcal{K}_c(\mathbb{R}^n)$), $V_j = supp\{\langle u_j, v_j\rangle\}$, $\langle u_j, v_j\rangle \in U$, such that $d_H(V_j, \{0\}) > j$. Clearly $\{V_j\}$ has no subsequence with limit in $\mathcal{K}_c(\mathbb{R}^n)$. But since U is compact, there is a subsequence $\langle u_{j_k}, v_{j_k}\rangle$ converging to $\langle u,v\rangle \in U$, and $lim_k V_{j_k} = supp\{\langle u,v\rangle\}$ which is impossible. Hence U must be uniformly support bounded.

Let $\varepsilon > 0$ and let $\langle u_1, v_1\rangle, ..., \langle u_n, v_n\rangle \in \mathrm{IF}_n$ be a $\frac{1}{3}\varepsilon$-cover of U, that is for any $\langle u,v\rangle \in U$ one of the sequence elements $\langle u_i, v_i\rangle$ satisfies $d_p(\langle u,v\rangle, \langle u_i, v_i\rangle) < \frac{1}{3}\varepsilon$.

Such a sequence exists by the compactness of U.

By Lemma 4.1, $\langle u_1, v_1\rangle, ..., \langle u_k, v_k\rangle$ are p-mean left-continuous and so there exists $\eta = \min\limits_{1\leq i\leq k} \delta(\varepsilon, \langle u_i, v_i\rangle) > 0$ such that $\int_h^1 d_H\left([\langle u_i,v_i\rangle]^\alpha, [\langle u_i,v_i\rangle]^{\alpha-h}\right)^p d\alpha < (\frac{1}{3}\varepsilon)^p$ for $i = 1, ..., k$ and $0 \leq h \leq \eta$. Thus for $\langle u,v\rangle \in U$, the triangle inequality gives

$$\begin{aligned}
\left(\int_h^1 d_H\left([\langle u,v\rangle]^\alpha, [\langle u,v\rangle]^{\alpha-h}\right)^p d\alpha\right)^{1/p} &\leq d_p(\langle u,v\rangle, \langle u_i,v_i\rangle) \\
&+ \left(\int_h^1 d_H\Big([\langle u_i,v_i\rangle]^\alpha, [\langle u_i,v_i\rangle]^{\alpha-h}\Big)^p d\alpha\right)^{1/p} \\
&+ d_p\Big(\langle u,v\rangle, \langle u_i,v_i\rangle\Big) \\
&\leq \varepsilon
\end{aligned}$$

so U is p-mean equileft-continuous.

Sufficiency. Let $\{\langle u_k, v_k\rangle\}$ be a sequence in U and $\{\langle u_k, v_k\rangle^*\}$ the corresponding sequence in U^*. Let $D_1 = \{\alpha_i \in I,\ i = 1,2,3,...\}$, $D_2 = \{x_j \in S^{n-1},\ j = 1,2,3,...\}$ be countable dense subsets of I and S^{n-1} respectively. The usual diagonalisation construction gives a subsequence $\{\langle u_{k_k}, v_{k_k}\rangle^*\}$ and a function $g: D_1 \times D_2 \to \mathbb{R}$ such that $\langle u_{k_k}, v_{k_k}\rangle^*(\alpha_i, x_j) \to g(\alpha_i, x_j)$ uniformly in $(\alpha_i, x_j) \in D_1 \times D_2$ as $k \to \infty$. For notational simplicity write $\langle w,z\rangle_k^* = \langle u_{k_k}, v_{k_k}\rangle^*$,$\langle w,z\rangle_k = \langle u_{k_k}, v_{k_k}\rangle$.

Since U is uniformly support bounded, there exists $K > 0$ such that

$$|\langle w,z\rangle_k^*(\alpha_i, x) - \langle w,z\rangle_k^*(\alpha_i, y)| \leq \Big(\sup_{a\in[\langle w,z\rangle_k]^0} ||a||\Big)||x-y|| = K||x-y||$$

for all $\alpha_i \in D_1$ and any $\langle w, z\rangle^* \in U^*$. That is, the $\langle w, z\rangle_k^* (\alpha_i, .)$ are equicontinuous on S^{n-1}, uniformly in $\alpha_i \in D_1$. Hence the sequence $\{\langle w, z\rangle_k^* (\alpha_i, x)\}$ converges for each $\alpha_i \in D_1$ and $x \in S^{n-1}$, by Theorem 3.1, Property 1, also this convergence in the d_∞ and hence d_p norms, and we denote the limits by $g(\alpha_i, x)$. As in [3] (see also [5]) such convergence is uniform in S^{n-1}, and moreover is uniform in D_1 as well, for the sup norm, and thus for d_p norm.

From the properties of the $\langle w, z\rangle_k^* \in U^*$ it follows that

1. $|g(\alpha_i, x)| \leq K$ for all $\alpha_i \in D_1$ and $x \in S^{n-1}$;
2. $|g(\alpha_i, x) - g(\alpha_i, y)| \leq K||x - y||$ for all $\alpha_i \in D_1$;
3. $g(\alpha_i, x) \leq g(\beta_i, x)$ for all $\beta_i \leq \alpha_i$ in D_1 and $x \in S^{n-1}$.

Then for each $(\alpha_i, x) \in I \times S^{n-1}$, define

$$g(\alpha, x) = \lim_{\alpha_i \to \alpha^-} g(\alpha_i, x),\ \alpha_i \in D_1.$$

Each such exists because $g(., x)$ is nonincreasing in $\alpha_i \in D_1$ and bounded. This defines g on all of $I \times S^{n-1}$ and in such a way that the three properties, immediately above, hold for g on all of $I \times S^{n-1}$. These, together with the left-continuity of $g(., x)$, show that $g(., .)$ is the support function of a well-defined intuitionistic fuzzy set $\langle w, z\rangle$ whose support lies in $\bigcup_{\langle u,v\rangle \in U} [\langle u, v\rangle]^0$. It remains to show that $d_p\Big(\langle w, z\rangle_k, \langle w, z\rangle \Big) \to 0$ as $k \to \infty$.

By p-mean equi-left-continuity, for a monotonic nondecreasing sequence $\alpha_i = \alpha - h_i \in D_1$

$$\int_{h_i}^{1} d_H\Big([\langle w, z\rangle_k]^{\alpha - h_i}, [\langle w, z\rangle_k]^{\alpha}\Big)^p d\alpha < (\frac{1}{2}\varepsilon)^p$$

provided $0 \leq h_i < \delta$ for $\delta = \delta(\varepsilon)$, uniformly in $\langle w, z\rangle_k \in U$. But for $k > N(\frac{1}{2}\varepsilon)$, $g(\alpha_i, x) - \frac{1}{2}\varepsilon < \langle w, z\rangle_k^* (\alpha_i, x) < g(\alpha_i, x)$ uniformly in S^{n-1} and since g is nonincreasing,

$$g(\alpha, x) - \frac{1}{2}\varepsilon \leq g(\alpha_i, x) - \frac{1}{2}\varepsilon < \langle w, z\rangle_k^* (\alpha_i, x) < g(\alpha, x) + \frac{1}{2}\varepsilon.$$

Thus, $d_H\Big([\langle w, z\rangle_k]^{\alpha_i}, [\langle w, z\rangle]^{\alpha}\Big) = \sup_{x \in S^{n-1}} |\,\langle w, z\rangle_k^* (\alpha_i, x) - g(\alpha, x)| < \frac{1}{2}\varepsilon$. Hence,

$$\begin{aligned} d_p(\langle w, z\rangle_k, \langle w, z\rangle) &\leq \Big(\int_{h_i}^{1} d_H\Big([\langle w, z\rangle_k]^{\alpha}, [\langle w, z\rangle_k]^{\alpha - h_i}\Big)^p d\alpha\Big)^{\frac{1}{p}} \\ &\quad + \Big(\int_{h_i}^{1} d_H\Big([\langle w, z\rangle_k]^{\alpha - h_i}, [\langle w, z\rangle]^{\alpha}\Big)^p d\alpha\Big)^{\frac{1}{p}} \\ &< \varepsilon \end{aligned}$$

for all $K > N(\frac{1}{2}\varepsilon)$.

□

5 Locally compact in $(\mathbf{IF}_n, d_p)$

Theorem 5.1. *The space* (IF_n, d_p)*,* $1 \leq p < \infty$*, is locally compact. Moreover, a subset* U *is locally compact iff every uniformly support bounded and closed subset of* U *is p-Blaschke.*

Proof. **For sufficiency**, let $U \subset \mathrm{IF}_n$ be such that any uniformly support bounded and closed set is p-Blaschke, and take $\langle u, v\rangle \in U$. Since $\langle u, v\rangle$ has compact support, there exists $K > 0$ such that $d_p\left(\langle u, v\rangle, 0_{\langle 1,0\rangle}\right) \leq K$. Then, $N_\varepsilon(\langle u, v\rangle) = \{\langle u', v'\rangle : d_p(\langle u, v\rangle, \langle u', v'\rangle) < \varepsilon\}$ form a neighborhood basis of $\langle u, v\rangle$, and for every $\langle w, z\rangle \in N_\varepsilon(\langle u, v\rangle)$, $d_p\left(\langle w, z\rangle, 0_{\langle 1,0\rangle}\right) \leq d_p(\langle w, z\rangle, \langle u, v\rangle) + d_p\left(\langle u, v\rangle, 0_{\langle 1,0\rangle}\right) \leq K + \varepsilon$. So $N_\varepsilon(\langle u, v\rangle)$ is uniformly support bounded, and hence p-Blaschke. So $cl(N_\varepsilon(\langle u, v\rangle))$ is compact, and U is locally compact.
For necessity, we have the space (IF_n, d_p), $1 \leq p < \infty$, is locally compact, since the same argument shows every point of the metric space has a compact neighborhood. Since, for $1 \leq p < \infty$, the space is also separable, $\mathrm{IF}_n = \bigcup_{k \geq 1} U_k$ where $U_1 \subseteq U_2 ... \subseteq U_k \subseteq U_{k+1} ...$ and the U_k are p-Blaschke. So any closed subset of U that is uniformly support bounded lies in one of the U_k, for some sufficiently large k, and is thus p-mean equileft-continuous, and so p-Blaschke. □

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