Notes on Intuitionistic Fuzzy Sets
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A new operation over intuitionistic fuzzy pairs

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To Prof. Janusz Kacprzyk for his 75-th Anniversary!

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Abstract: A new operation defined over intuitionistic fuzzy pairs is introduced. Some of its basic propertires are studied.
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1 Introduction

The concept of an Intuitionistic Fuzzy Pair (IFP) was introduced in [5] as follows: the ordered pair $\langle a, b\rangle$ is an IFP if and only if (IFF) $a, b, a+b \in [0, 1]$. In [4,5], many intuitionistic fuzzy operations on IFPs are defined, e.g., negation and several operations of conjunctive and disjunctive type:

$$\begin{aligned}
\neg\langle a,b\rangle &= \langle b,a\rangle,\\
\langle a,b\rangle \wedge \langle c,d\rangle &= \langle \min(a,c), \max(b,d)\rangle,\\
\langle a,b\rangle \vee \langle c,d\rangle &= \langle \max(a,c), \min(b,d)\rangle,\\
\langle a,b\rangle + \langle c,d\rangle &= \langle a+c-ac, bd\rangle,\\
\langle a,b\rangle . \langle c,d\rangle &= \langle ac, b+d-bd\rangle,\\
\langle a,b\rangle @ \langle c,d\rangle &= \left\langle \frac{a+c}{2}, \frac{b+d}{2}\right\rangle.
\end{aligned}$$

In [1, 2] a lot of other intuitionistic fuzzy operations of conjunctive and disjunctive type are defined, as well. All of them have been generated by the existing intuitionistic fuzzy implications and by the intuitionistic fuzzy negations, respectively generated by them.

In the next section of the present paper, we will introduce a new operation, which is of an essentially different type.

2 Main results

Let everywhere below, the two IFPs $x = \langle a,b\rangle$ and $y = \langle c,d\rangle$ be given. We define:

$$\langle a,b\rangle \bullet \langle c,d\rangle = \langle a(1-c), b(1-d)\rangle. \tag{1}$$

Theorem 1. *The operation $\bullet$ is defined correctly.*

Proof. Let x and y be two IFPs. For them we see directly, that

$$\begin{aligned}
&0 \le a(1-c), \qquad\qquad b(1-d) \le 1,\\
&0 \le a(1-c) + b(1-d) \le a+b \le 1.
\end{aligned}$$

□

From (1) we see immediately that the following equalities are valid:

$$\begin{aligned}
\langle 0,1\rangle \bullet \langle 0,1\rangle &= \langle 0,0\rangle,\\
\langle 0,1\rangle \bullet \langle 0,0\rangle &= \langle 0,1\rangle,\\
\langle 0,1\rangle \bullet \langle 1,0\rangle &= \langle 0,1\rangle,\\
\langle 0,0\rangle \bullet \langle 0,1\rangle &= \langle 0,0\rangle,\\
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\langle 0,0\rangle \bullet \langle 1,0\rangle &= \langle 0,0\rangle,\\
\langle 1,0\rangle \bullet \langle 0,1\rangle &= \langle 1,0\rangle,\\
\langle 1,0\rangle \bullet \langle 0,0\rangle &= \langle 1,0\rangle,\\
\langle 1,0\rangle \bullet \langle 1,0\rangle &= \langle 0,0\rangle,
\end{aligned}$$

and for each IFP $\langle a,b\rangle$:

$$\langle a, b\rangle \bullet \langle a, b\rangle = \langle a(1-a), b(1-b)\rangle,$$
$$\langle a, b\rangle \bullet \langle 0, 1\rangle = \langle a, 0\rangle,$$
$$\langle a, b\rangle \bullet \langle 0, 0\rangle = \langle a, b\rangle,$$
$$\langle a, b\rangle \bullet \langle 1, 0\rangle = \langle 0, b\rangle,$$
$$\langle 0, 1\rangle \bullet \langle a, b\rangle = \langle 0, 1-b\rangle,$$
$$\langle 0, 0\rangle \bullet \langle a, b\rangle = \langle 0, 0\rangle,$$
$$\langle 1, 0\rangle \bullet \langle a, b\rangle = \langle 1-a, 0\rangle.$$

Theorem 2. *For every three IFPs x, y and $z = \langle e, f\rangle$, the following equalities are valid:*

(a) $(x \wedge y) \bullet z = (x \bullet z) \wedge (y \bullet z)$,

(b) $(x \vee y) \bullet z = (x \bullet z) \vee (y \bullet z)$,

(c) $(x + y) \bullet z \leq (x \bullet z) + (y \bullet z)$,

(d) $(x.y) \bullet z \geq (x \bullet z).(y \bullet z)$,

(e) $(x@y) \bullet z = (x \bullet z)@(y \bullet z)$.

where $\leq$ and $\geq$ correspond to the standard order defined in the context of the Intuitionistic Fuzzy Sets theory [3].

Proof. For Equality (a) we have sequentially:

$$\begin{aligned}
(x \wedge y) \bullet z &= (\langle a, b\rangle \wedge \langle c, d\rangle) \bullet \langle e, f\rangle \\
&= \langle \min(a, c), \max(b, d)\rangle \bullet \langle e, f\rangle \\
&= \langle \min(a, c)(1-e), \max(b, d)(1-f)\rangle \\
&= \langle \min((1-e)a, (1-e)c), \max((1-f)b, (1-f)d)\rangle \\
&= \langle a(1-e), b(1-f)\rangle \wedge \langle c(1-e), d(1-f)\rangle \\
&= (\langle a, b\rangle \bullet \langle e, f\rangle) \wedge (\langle c, d\rangle \bullet \langle e, f\rangle) \\
&= (x \bullet z) \wedge (y \bullet z).
\end{aligned}$$

Equality (b) is proved by the same way.
For Equality (c) we have sequentially:

$$\begin{aligned}
(x + y) \bullet z &= (\langle a, b\rangle + \langle c, d\rangle) \bullet \langle e, f\rangle \\
&= \langle a + c - ac, bd\rangle \bullet \langle e, f\rangle \\
&= \langle (a + c - ac)(1-e), bd(1-f)\rangle \\
&= \langle (a + c)(1-e) - ac(1-e), bd(1-f)\rangle \\
&\leq \langle (a + c)(1-e) - ac(1-e)^2, bd(1-f)^2\rangle \\
&= \langle a(1-e), b(1-f)\rangle + \langle c(1-e), d(1-f)\rangle \\
&= (x \bullet z) + (y \bullet z).
\end{aligned}$$

Equality (d) is proved in the same way.
For Equality (e) we have sequentially:

$$\begin{aligned}(x@y) \bullet z &= (\langle a, b\rangle @ \langle c, d\rangle) \bullet \langle e, f\rangle \\ &= \langle \frac{a+c}{2}, \frac{b+d}{2}\rangle \bullet \langle e, f\rangle \\ &= \langle \frac{a+c}{2}(1-e), \frac{b+d}{2}(1-f)\rangle \\ &= \langle \frac{a(1-e)+c(1-e)}{2}, \frac{b(1-f)+d(1-f)}{2}\rangle \\ &= \langle a(1-e), b(1-f)\rangle @ \langle c(1-e), d(1-f)\rangle \\ &= (x \bullet z)@(y \bullet z).\end{aligned}$$

This completes the proof. □

Theorem 3. *For every three IFPs x, y and $z = \langle e, f\rangle$, the following equalities are valid.*

(a) $x \bullet (y \wedge z) = (x \bullet y) \vee (x \bullet z)$,

(b) $x \bullet (y \vee z) = (x \bullet y) \wedge (x \bullet z)$,

(c) $x \bullet (y + z) = (x \bullet y).(x \bullet z)$,

(d) $x \bullet (y.z) = (x \bullet y) + (x \bullet z)$,

(e) $x \bullet (y@z) = (x \bullet y)@(x \bullet z)$.

Proof. For Equality (b) we have sequentially:

$$\begin{aligned}x \bullet (y \vee z) &= \langle a, b\rangle \bullet (\langle c, d\rangle \vee \langle e, f\rangle) \\ &= \langle a, b\rangle \bullet \langle \max(c, e), \min(d, f)\rangle \\ &= \langle a(1 - \max(c, e)), b(1 - \min(d, f))\rangle \\ &= \langle a \ \min(1-c, 1-e), b \ \max(1-d, 1-f)\rangle \\ &= \langle \min(a(1-c), a(1-e), \max(b(1-d), b(1-f)\rangle \\ &= \langle a(1-c), b(1-d)\rangle \wedge \langle a(1-e), b(1-f)\rangle \\ &= (x \bullet y) \wedge (x \bullet z).\end{aligned}$$

Equalities (a) and (c)–(e) are proved in the same way. □

Theorem 4. *For every three IFPs x, y and $z = \langle e, f\rangle$, the following equality is valid:*

$$(x \bullet y) \bullet z = (x \bullet z) \bullet y.$$

Proof. We have sequentially:

$$\begin{aligned}(x \bullet y) \bullet z &= (\langle a, b\rangle \bullet \langle c, d\rangle) \bullet \langle e, f\rangle \\ &= \langle a(1-c), b(1-d)\rangle \bullet \langle e, f\rangle \\ &= \langle a(1-c)(1-e), b(1-d)(1-f)\rangle \\ &= \langle a(1-e)(1-c), b(1-f)(1-d)\rangle \\ &= \langle a(1-e), b(1-f)\rangle \bullet \langle c, d\rangle \\ &= (x \bullet z) \bullet y. \qquad \square\end{aligned}$$

Theorem 5. *For every two IFPs x and y, the following equality is valid:*

$$\neg(x \bullet y) = \neg x \bullet \neg y.$$

Proof. We have sequentially:

$$\begin{aligned}\neg(x \bullet y) &= \neg(\langle a, b\rangle \bullet \langle c, d\rangle) \\ &= \neg\langle a(1-c), b(1-d)\rangle \\ &= \langle b(1-d), a(1-c)\rangle \\ &= \langle b, a\rangle \bullet \langle d, c\rangle) \\ &= \neg x \bullet \neg y. \qquad \square\end{aligned}$$

3 Discussion and conclusion

In the present research, a new operation on IFPs is introduced. In the future, we will study its relations to other intuitionistic fuzzy operations, as well as the intuitionistic fuzzy operators from modal and level types. It is also important to study the semantics of the introduced operation. This will be a subject of further research but here we would like to propose the following—very preliminary—interpretation.

The $\bullet$ operation defined in (1) may be seen as a "hesitation-increasing" tool. Namely, following the concept of hesitation in standard IFS theory [3], let us define hesitation degree h of an IFP $\langle a, b\rangle$ as $h = 1 - a - b$. Then, in order to increase the hesitation degree of $\langle a, b\rangle$, one can apply the $\bullet$ operation with the second argument, another IFP $\langle c, d\rangle$, expressing how much the hesitation degree should increase and in what proportion by changing a and b. Notice that the hesitation degree h' of $\langle c, d\rangle \bullet \langle c, d\rangle$ equals $1 - a - b + ac + bd$, i.e., $h' = h + ac + bd$. Thus, the increase of the hesitation degree is proportional to c and d, and at the same time it is proportional to a and b. In particular, if $\langle c, d\rangle = \langle 0, 0\rangle$, then no change of hesitation occurs while for $\langle c, d\rangle = \langle 1, 0\rangle$ the hesitation degree may also remain unchanged (for $a = 0$) but may also rise to 1 (for $a = 1$).

In such a context, the properties of the $\bullet$ operation shown in Theorem 2 have an interesting interpretation. For example, the first property states that the increase of hesitation for a pair of IFPs combined using conjunction operation does not depend on the order of operations: if one first applies the conjunction and then increases the hesitation, the result would be the same as if the hesitation of the conjuncts is first increased and only then conjunction is applied. Conversely, Theorem 4 states that if a series of hesitation increases is to be carried out, then the order in which they are applied does not matter.

Another question is what may be the practical use of such a "hesitation-increasing" operation. It also requires further studies but one of the possibilities is to employ it in the context of group decision making, notably in consensus reaching process support, [6]. Namely, in such a context, an IFP may correspond to a quantitative representation of the decision maker's preferences. During the decision process, a decision maker may want to change the expression of his or her preferences. Then one of the possible directions of such a change may consist in increasing the hesitation degree what may be implied by some contradictory information gathered during the process, or by the willingness to contribute to reaching consensus within the group. Preferences with higher hesitation degree would be usually assumed to be a signal for a given decision maker's readiness to accept the preferences of other group members.

Thus, as stated earlier, the newly proposed operation certainly requires further theoretical studies but there are promising directions for this research also in view of potential practical applications.

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