Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy group algebra

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Abstract: This paper introduces the concept of an intuitionistic fuzzy group algebra associated with a finite group G and an intuitionistic fuzzy group A on G. We establish its structural properties, showing that it simultaneously behaves as an intuitionistic fuzzy algebra and an intuitionistic fuzzy G-module. Extending author's earlier results on the semi-simplicity of intuitionistic fuzzy G-modules, we explore links to complete reducibility and injectivity. Further, we study intersections, (α, β)-cuts, and homomorphic images of such algebras, and define intuitionistic fuzzy group algebra homomorphisms. Finally, we prove that the class of all intuitionistic fuzzy group algebras forms a category.
Keywords: Intuitionistic fuzzy G-module, Intuitionistic fuzzy group, Intuitionistic fuzzy group algebra, Intuitionistic fuzzy group algebra homomorphism, Category.
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1 Introduction

The notion of a fuzzy set in a set was introduced by Zadeh [28], and since then this concept has been applied to many mathematical branches. Rosenfeld [14] applied the notion of fuzzy sets to algebra and introduced the notion of fuzzy subgroups. The literature of various fuzzy algebraic concepts has been growing very rapidly. In particular, Negoita and Ralescu [13] introduced and examined the notion of fuzzy submodule of a module. Since then different types of fuzzy

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submodules were investigated in the last two decades. Shery Fernadez introduced and studied the notion of fuzzy G-modules in [9]. Recently, Abraham *et al.* [1] defined fuzzy group algebra and studied some basic properties of it.

One of the interesting generalizations of the theory of fuzzy sets is the theory of intuitionistic fuzzy sets introduced by Atanassov [2–4]. Biswas [6] was the first one to introduce the notion of intuitionistic fuzzy subgroup of a group. Using the Atanassov's idea, Davvaz *et al.* [8] established the intuitionistic fuzzification of the concept of submodule in a module and introduced the notion of intuitionistic fuzzy submodule of a module which was further studied by many authors (for example see [5, 11, 15, 17, 25]). The notion of intuitionistic fuzzy G-modules was introduced by Sharma *et al.* in [26]. Many properties like representation, reducibility, complete reducibility, semi-simplicity, fundamental theorems of isomorphisms, injectivity, projectivity, etc., of intuitionistic fuzzy G-modules have been discussed in [18–24, 27].

We turn our attention towards intuitionistic fuzzification of Maschke's Theorem on semi-simplicity of group algebra. The primary objective in this process was to introduce the intuitionistic fuzzy version of a group algebra. In this paper, we introduce the concept of an intuitionistic fuzzy group algebra using a finite group G and an intuitionistic fuzzy group A on G. We had observed the basic properties of an intuitionistic fuzzy group algebra, including its behaviour as an intuitionistic fuzzy algebra and as an intuitionistic fuzzy G-module. The intersections and (α, β)-cuts of an intuitionistic fuzzy group algebra, image and inverse image of the intuitionistic fuzzy group algebra under the group algebra homomorphism induced by a group homomorphism will be analyzed. This will lead to desirable results asserting the main objectives.

2 Preliminaries

For the sake of convenience, the existing concepts which will be used in this paper are mainly taken from [3, 5–7, 19, 20, 26]. Throughout the paper, M will always be a G-module over the field K (a subfield of the field of complex numbers).

Definition 2.1. ([3]) Let X be a non-empty set. An intuitionistic fuzzy set (IFS) A of X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and degree of non-membership of the element $x \in X$ respectively and for any $x \in X$, we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Remark 2.2. When $\mu_A(x)+\nu_A(x) = 1$, i.e., $\nu_A(x) = 1-\mu_A(x)$ for any $x \in X$, then A is called a fuzzy set. For convenience, we write the IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition 2.3. ([3]) Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two IFSs of X, then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)\ \forall\ x \in X$;

(ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)\ \forall\ x \in X$;

(iii) $A^c = (\mu_{A^c}, \nu_{A^c})$, where $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)\ \forall\ x \in X$;

(iv) $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$, where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_{A\cap B}(x) = \mu_A(x) \vee \mu_B(x)$;

(v) $A \cup B = (\mu_{A\cup B}, \nu_{A\cup B})$, where $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$ and $\nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x)$.

Definition 2.4. ([6]) Let $(G, \cdot)$ be a group, and let A be an IFS in G. Then A is called an intuitionistic fuzzy group (IFG) if, for all $g_1, g_2, g \in G$, the following hold:

(i) $\mu_A(g_1g_2) \geq \min\{\mu_A(g_1), \mu_A(g_2)\}$; (ii) $\mu_A(g^{-1}) = \mu_A(g)$;

(iii) $\nu_A(g_1g_2) \leq \max\{\nu_A(g_1), \nu_A(g_2)\}$; (iv) $\nu_A(g^{-1}) = \nu_A(g)$.

Definition 2.5. ([27]) Let $f : G_1 \to G_2$ be a group homomorphism. If A and B be respectively an intuitionistic fuzzy group on G_1 and G_2, then f is called a weak intuitionistic fuzzy homomorphism of A into B, when $f(A) \subseteq B$. The homomorphism f is an intuitionistic fuzzy homomorphism of A onto B if $f(A) = B$. Further, if $f : G_1 \to G_2$ be an isomorphism, then f is called a weak intuitionistic fuzzy isomorphism if $f(A) \subseteq B$ and f is an intuitionistic fuzzy isomorphism, when $f(A) = B$.

Definition 2.6. ([7]) Let G be a group and M be a vector space over a field K. Then M is called a G-module if for every $g \in G$ and $m \in M$, there exists a product (called the action of G on M), $g \cdot m \in M$ that satisfies the following axioms:

(i) $1_G.m = m, \forall\, m \in M$ (1_G being the identity of G);

(ii) $(gh) \cdot m = g \cdot (h \cdot m), \forall\, m \in M, g, h \in G$;

(iii) $g.(k_1m_1 + k_2m_2) = k_1(g.m_1) + k_2(g.m_2), \forall\, k_1, k_2 \in K; m_1, m_2 \in M$ and $g \in G$.

A subspace of M, which itself is a G-module with the same action is called G-submodule. A nonzero G-module M is irreducible if the only G-submodules of M are M and $\{0_M\}$. Otherwise, it is reducible. A nonzero G-module M is completely reducible if for every G-submodule N of M there exists a G-submodule N^* of M such that $M = N \oplus N^*$. A G-module M is semi-simple if there exists a family of irreducible G-submodules M_i such that $M = \bigoplus_{i=1}^{n} M_i$.

Definition 2.7. ([7]) Let M and M^* be G-modules. A mapping $f :$ M $\to M^*$ is a G-module homomorphism if:
(i) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$
(ii) $f(g \cdot m) = g \cdot f(m), \forall k_1, k_2 \in K; m, m_1, m_2 \in M$ and $g \in G$.

Definition 2.8. ([26]) Let G be a group and M be a G-module over K, which is a subfield of C. Then an intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of M such that for all $a, b \in K, g \in G; m, x, y \in M$, the following conditions are satisfied

(i) $\mu_A(ax + by) \geq \min\{\mu_A(x), \mu_A(y)\}$; (ii) $\nu_A(ax + by) \leq \max\{\nu_A(x), \nu_A(y)\}$;

(iii) $\mu_A(g \cdot m) \geq \mu_A(m)$; (iv) $\mu_A(g \cdot m) \leq \mu_A(m)$.

Let G be a group and K be a field. The K vector space having G as Hamel basis is called the group algebra denoted by $K[G]$. It contains elements of the form $a = \sum_{g \in G} a_g g, a_g \in K$ and $a_g = 0$ for all but a finite number of elements of G. The addition and multiplication in $K[G]$ are defined by the following operations.

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

and

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) = \sum_{x\in G}\left(\left(\sum_{g\in G} a_g b_{g^{-1}x}\right)x\right).$$

With these two operations $K[G]$ is a K-algebra with identity element $1_{K[G]} = \sum_{g\in G} a_g g$, where $a_g = 1_K$ if $g = 1_G$ and $a_g = 0$, otherwise. The action of an element x of G on $K[G]$ is defined by

$$\left(\sum_{g\in G} a_g g\right) x = \sum_{g\in G} a_g gx = \sum_{g\in G} a_{gx^{-1}} g.$$

Then $K[G]$ can be considered as a G-module. It may be noted that if H is a subgroup of G, then $K[H]$ is a subalgebra of $K[G]$ [10].

Recall that, for $x = \sum_{g\in G} a_g g \in K(G)$, $\mathrm{supp}(x) = \{g \in G : a_g \neq 0\}$ is called the support of x. Then $\mathrm{supp}(hx) = h(\mathrm{supp}(x))$, for all $h \in G$ (see [12, Exercise 1, page 107]).

3 Intuitionistic fuzzy group algebra (IFGA)

Definition 3.1. Let G be a finite group, and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy group on G. Then the intuitionistic fuzzy group algebra (IFGA) induced by A is an intuitionistic fuzzy set $K[A] = (\mu_{K[A]}, \nu_{K[A]})$ on the group algebra $K[G]$ defined as

$$(\mu_{K[A]}(x), \nu_{K[A]}(x)) = \begin{cases} (1,0), & \text{if } \mathrm{supp}(x) = \varnothing \\ (\min\{\mu_A(g) : g \in \mathrm{supp}(x)\}, \max\{\nu_A(g) : g \in \mathrm{supp}(x)\}), & \text{if } \mathrm{supp}(x) \neq \varnothing, \end{cases}$$

where $\mathrm{supp}(x) = \{g \in G : a_g \neq 0\}$ denotes the support of the element $x = \sum_{g\in G} a_g g \in K[G]$.

Remark 3.2. Every IFG on G can be used to construct an IFGA on $K[G]$, the restriction of which to G yields the original IFG. The mapping f defined by $f(g) = 1 \cdot g$ is an isomorphism from G into $K[G]$. This f is an IF-homomorphism from any IFG A of G to the IFGA $K[A]$ of $K[G]$. It is evident from the fact that:

$$\mu_{f(A)}(1 \cdot g) = \mu_A(g) = \mu_{K[A]}(1 \cdot g) \text{ and } \nu_{f(A)}(1 \cdot g) = \nu_A(g) = \nu_{K[A]}(1 \cdot g).$$

Example 3.3. Let $K = \mathbb{Z}_2 = \{0, 1\}$ and $G = \{1, x, x^2\}$ be a cyclic group of order 3. Then

$$K[G] = \{a_1.1 + a_2.x + a_3.x^2 | a_i \in K\} = \{0, 1, x, x^2, 1 + x, 1 + x^2, x + x^2, 1 + x + x^2\}.$$

Let $A = (\mu_A, \nu_A)$ be an IFS on G defined by

$$\mu_A(g) = \begin{cases} 1, & \text{if } g = 1 \\ 0.6, & \text{if } g = x, x^2 \end{cases} \quad \text{and} \quad \nu_A(g) = \begin{cases} 0, & \text{if } g = 1 \\ 0.3, & \text{if } g = x, x^2. \end{cases}$$

It is easy to check that A is an IFG on G. Also, it is easy to verify that the IFS $K[A]$ on $K[G]$ is defined by

$$\mu_{K[A]}(a_1.1 + a_2.x + a_3.x^2) = \begin{cases} 1, & \text{if } a_1 \in \{0, 1\}, a_2 = a_3 = 0 \\ 0.6, & \text{otherwise} \end{cases}$$

and

$$\nu_{K[A]}(a_1.1 + a_2.x + a_3.x^2) = \begin{cases} 0, & \text{if } a_1 \in \{0,1\}, a_2 = a_3 = 0 \\ 0.3, & \text{otherwise} \end{cases}$$

is an IFGA of A over the group algebra $K[G]$.

Furthermore, it may be noted that the map $f : G \to K[G]$ defined by $f(g) = 1 \cdot g$ is an isomorphism from G into $K[G]$. Also, then f is an IF-homomorphism from A into $K[A]$. Moreover, the restriction of $K[A]$ to $G \subseteq K[G]$ equals to A, for

$$\mu_{K[A]}(1 \cdot g) = \begin{cases} 1, & \text{if } g = 1 \\ 0.6, & \text{if } g = x, x^2 \end{cases} \quad \text{and} \quad \nu_{K[A]}(1 \cdot g) = \begin{cases} 0, & \text{if } g = 1 \\ 0.3, & \text{if } g = x, x^2. \end{cases}$$

Example 3.4. Let $K = \mathbb{Z}_2 = \{0, 1\}$ and $G = \{e, r, s, rs\}$ be the Klein four group. Then

$$\begin{aligned} K(G) &= \{a_1.e + a_2.r + a_3.s + a_4.rs \mid a_i \in K\} \\ &= \{0, e, r, s, rs, e + r, e + s, e + rs, r + s, r + rs, s + rs, e + r + s, \\ &\quad e + r + rs, e + s + rs, r + s + rs, e + r + s + rs\}. \end{aligned}$$

Let $A = (\mu_A, \nu_A)$ be an IFS on G defined by

$$\mu_A(g) = \begin{cases} 1, & \text{if } g = e \\ 0.8, & \text{if } g = r \\ 0.7, & \text{if } g = s, rs \end{cases} \quad \text{and} \quad \nu_A(g) = \begin{cases} 0, & \text{if } g = e \\ 0.1, & \text{if } g = r \\ 0.2, & \text{if } g = s, rs \end{cases}$$

It is easy to check that A is an IFG on G. Also, it is easy to verify that the IFS $K[A]$ on $K[G]$ is defined by

$$\mu_{K[A]}(a_1.e + a_2.r + a_3.s + a_4.rs) = \begin{cases} 1, & \text{if } a_1 \in \{0,1\}, a_2 = a_3 = a_4 = 0 \\ 0.8, & \text{if } (a_2 = 1, a_1 = a_3 = a_4 = 0) \\ & \quad \text{or } (a_1 = a_2 = 1, a_3 = a_4 = 0) \\ 0.7, & \text{otherwise} \end{cases}$$

and

$$\nu_{K[A]}(a_1.e + a_2.r + a_3.s + a_4.rs) = \begin{cases} 0, & \text{if } a_1 \in \{0,1\}, a_2 = a_3 = a_4 = 0 \\ 0.1, & \text{if } (a_2 = 1, a_1 = a_3 = a_4 = 0) \\ & \quad \text{or } (a_1 = a_2 = 1, a_3 = a_4 = 0) \\ 0.2, & \text{otherwise} \end{cases}$$

is an IFGA of A over the group algebra $K[G]$.

Proposition 3.5. *For any IFG A on a finite group G and field K, the IFGA $K[A]$ on $K[G]$ is an intuitionistic fuzzy algebra (IFA). In general, IFGAs are IFAs.*

Proof. Consider an IFG A on a finite group G. Then, for $x = \sum_{g \in G} a_g g, y = \sum_{g \in G} b_g g \in K[G]$ and $a, b \in K$, we have

$$\begin{aligned}
\mu_{K[A]}(ax + by) &= \mu_{K[A]}\left[a\sum_{g\in G} a_g g + b\sum_{g\in G} b_g g\right] = \mu_{K[A]}\left[\sum_{g\in G} aa_g g + \sum_{g\in G} bb_g g\right] \\
&= \mu_{K[A]}\left[\sum_{g\in G}(aa_g + bb_g)g\right] \\
&= \min\{\mu_A(g) : aa_g + bb_g \neq 0\} \\
&\geq \min\{\mu_A(g) : a_g \neq 0, b_g \neq 0\} \\
&\geq \min\{\min\{\mu_A(g) : a_g \neq 0\}, \min\{\mu_A(g) : b_g \neq 0\}\} \\
&= \min\{\mu_{K[A]}(x), \mu_{K[A]}(y)\}.
\end{aligned}$$

Similarly, we can show that $\nu_{K[A]}(ax + by) \leq \max\{\nu_{K[A]}(x), \nu_{K[A]}(y)\}$.
Also, for $x = \sum_{g\in G} a_g g, y = \sum_{h\in G} b_h h \in K[G]$, we have

$$\begin{aligned}
\mu_{K[A]}(xy) &= \mu_{K[A]}\left[\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right)\right] \\
&= \mu_{K[A]}\left[\sum_{t\in G}\left(\sum_{g\in G}(a_g b_{g^{-1}t})\right) t\right] \\
&= \mu_{K[A]}\left[\sum_{t\in G} c_t t\right] \text{, where } c_t = \sum_{g\in G}(a_g b_{g^{-1}t}) \\
&= \min\{\mu_A(t) : c_t \neq 0\} \\
&= \min\{\mu_A(t) : \sum(a_g b_{g^{-1}t}) \neq 0\} \\
&\geq \left[\min\{\mu_A(g), \mu_A(g^{-1}t)\} : a_g \neq 0, b_{g^{-1}t} \neq 0\right] \\
&= [\min\{\mu_A(g), \mu_A(h)\} : a_g \neq 0, b_h \neq 0] \\
&\geq \min\{\min\{\mu_A(g) : a_g \neq 0\}, \min\{\mu_A(h) : b_h \neq 0\}\} \\
&= \min\{\mu_{K[A]}(x), \mu_{K[A]}(y)\}.
\end{aligned}$$

Similarly, we can show that $\nu_{K[A]}(xy) \leq \max\{\nu_{K[A]}(x), \nu_{K[A]}(y)\}$.
This concludes the proof that IFGAs are IFAs. □

Proposition 3.6. *The IFGA $K[A]$ is an intuitionistic fuzzy G-module on $K[G]$, if $K[G]$ is considered as a G-module.*

Proof. By Proposition 3.5, it is evident that for all $a, b \in K, x, y \in K[G]$, we have

$$\mu_{K[A]}(ax + by) \geq \min\{\mu_{K[A]}(x), \mu_{K[A]}(y)\}$$

and

$$\nu_{K[A]}(ax + by) \leq \max\{\nu_{K[A]}(x), \nu_{K[A]}(y)\}.$$

Next, let $m = \sum_{x\in G} a_x x \in K[G]$ and $g \in G$, we have

$$\begin{aligned}
\mu_{K[A]}(g \cdot m) &= \mu_{K[A]}\left(g \cdot \sum_{x\in G} a_x x\right) \\
&= \mu_{K[A]}\left[(1\cdot g)\sum_{x\in G}(a_x x)\right] \\
&= \mu_{K[A]}\left[\sum_{x\in G}(a_{xg^{-1}} x)\right] \\
&= \min\{\mu_A(x) : a_{xg^{-1}} \neq 0\} \\
&\geq \min\{\mu_A(x) : a_x \neq 0\} \\
&= \mu_{K[A]}(m).
\end{aligned}$$

Similarly, we can show that $\nu_{K[A]}(g \cdot m) \leq \nu_{K[A]}(m)$.
This shows that all IFGAs behave as an intuitionistic fuzzy G-module over $K[G]$. □

Proposition 3.7. *If A and B are two IFGs defined on a group G, then $K[A\cap B] = K[A]\cap K[B]$ as the IFGAs on $K[G]$.*

Proof. For IFGs A and B on group G, $A \cap B$ is also an IFG (see Proposition (3.5) of [6]). For any $x = \sum_{g\in G} a_g g \in K[G]$, we have

$$\begin{aligned}
\mu_{K[A\cap B]}(x) &= \mu_{K[A\cap B]}\left(\sum_{g\in G} a_g g\right) \\
&= \min\{\mu_{A\cap B}(g) : a_g \neq 0\} \\
&= [\min\{\mu_A(g), \mu_B(g)\} : a_g \neq 0] \\
&= \min\left\{\min\{\mu_A(g) : a_g \neq 0\}, \min\{\mu_B(g) : a_g \neq 0\}\right\} \\
&= \min\{\mu_{K[A]}(x), \mu_{K[B]}(x)\} \\
&= \mu_{K[A]\cap K[B]}(x).
\end{aligned}$$

Similarly, we can show that $\nu_{K[A\cap B]}(x) = \nu_{K[A]\cap K[B]}(x)$.
Hence, $K[A\cap B] = K[A]\cap K[B]$. □

Proposition 3.8. *If A and B two IFGs on a group G with $A \subseteq B$, then on the group algebra $K[G]$ the IFGAs satisfy, $K[A] \subseteq K[B]$.*

Proof. By Definition, $A \subseteq B$ gives $\mu_A(g) \leq \mu_B(g)$ and $\nu_A(g) \geq \nu_B(g)$ for every $g \in G$.
For any $x = \sum_{g\in G} a_g g \in K[G]$, we have

$$\begin{aligned}
\mu_{K[A]}(x) &= \mu_{K[A]}\left(\sum_{g\in G} a_g g\right) \\
&= \min\{\mu_A(g) : a_g \neq 0\} \\
&\leq \min\{\mu_B(g) : a_g \neq 0\} \\
&= \mu_{K[B]}(x).
\end{aligned}$$

Similarly, we can show that $\nu_{K[A]}(x) \geq \nu_{K[B]}(x)$. Hence, $K[A] \subseteq K[B]$. □

Proposition 3.9. *For $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \le 1$. The (α,β)-cut set of an IFGA $K[A]$ is the group algebra of the corresponding (α,β)-cut set of IFG A, i.e., $(K[A])_{(\alpha,\beta)} = K[A_{(\alpha,\beta)}]$.*

Proof. Let $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \le 1$ and $x = \sum_{g \in G} a_g g \in K[G]$, we have:

$$\begin{aligned} x \in (K[A])_{(\alpha,\beta)} &\Leftrightarrow \mu_{K[A]}(x) \ge \alpha \text{ and } \nu_{K[A]}(x) \le \beta \\ &\Leftrightarrow \min\{\mu_A(g) : a_g \ne 0\} \ge \alpha \text{ and } \max\{\nu_A(g) : a_g \ne 0\} \le \beta \\ &\Leftrightarrow \mu_A(g) \ge \alpha \text{ and } \nu_A(g) \le \beta, \text{ for all } a_g \ne 0 \text{ in } x = \sum_{g \in G} a_g g \\ &\Leftrightarrow g \in A_{(\alpha,\beta)}, \text{ for all } a_g \ne 0 \text{ in } x = \sum_{g \in G} a_g g \\ &\Leftrightarrow x = \sum_{g \in G} a_g g \in K(A_{(\alpha,\beta)}). \end{aligned}$$

Hence, $(K[A])_{(\alpha,\beta)} = K[A_{(\alpha,\beta)}]$. □

Definition 3.10. ([7]) Let $\psi : G_1 \to G_2$ be a group homomorphism. The induced K-algebra homomorphism $f : K[G_1] \to K[G_2]$ is defined by

$$f(\textstyle\sum_{g \in G_1} a_g g) = \sum_{g \in G_1} a_g \psi(g), a_g \in K.$$

Theorem 3.11. *Let $f : K[G_1] \to K[G_2]$ be a group algebra homomorphism induced by a group homomorphism $\psi : G_1 \to G_2$. Then the inverse image under f of an IFGA $K[B]$ of $K[G_2]$ is an IFGA of $K[G_1]$, where B is an IFG on G_2.*

Proof. Let $\psi : G_1 \to G_2$ be a group homomorphism. Then the induced K-algebra homomorphism $f : K[G_1] \to K[G_2]$ is defined by:

$$f(\textstyle\sum_{g \in G_1} a_g g) = \sum_{g \in G_1} a_g \psi(g), \text{ for all } \sum_{g \in G_1} a_g g \in K[G_1].$$

Let $K[B] = (\mu_{K[B]}, \nu_{K[B]})$ be an IFGA on $K[G_2]$ corresponding to the IFG $B = (\mu_B, \nu_B)$ on G_2. Then:

$$\begin{aligned} \mu_{f^{-1}(K[B])}\left(\sum_{g \in G_1} a_g g\right) &= \mu_{K[B]}\left(f\left(\sum_{g \in G_1} a_g g\right)\right) \\ &= \mu_{K[B]}\left(\sum_{g \in G_1} a_g \psi(g)\right) \\ &= \min\{\mu_B(\psi(g)) : a_g \ne 0\} \\ &= \min\{\mu_{\psi^{-1}(B)}(g) : a_g \ne 0\} \\ &= \mu_{K[\psi^{-1}(B)]}\left(\sum_{g \in G_1} a_g g\right). \end{aligned}$$

Similarly, we can show that $\nu_{f^{-1}(K[B])}\left(\sum_{g \in G_1} a_g g\right) = \nu_{K[\psi^{-1}(B)]}\left(\sum_{g \in G_1} a_g g\right)$.

Therefore, $f^{-1}(K[B]) = K[\psi^{-1}(B)]$. Since $\psi^{-1}(B)$ is an IFG on G_1 (see Theorem (4.4) of [16]), it follows that $K[\psi^{-1}(B)]$ is an IFGA on $K[G_1]$. Hence, $f^{-1}(K[B])$ is an IFGA on $K[G_1]$. □

Theorem 3.12. *Let $f : K[G_1] \to K[G_2]$ be a group algebra homomorphism induced by a group homomorphism $\psi : G_1 \to G_2$, and $A = (\mu_A, \nu_A)$ be an IFG on G_1. Then the image of the IFGA $K[A]$ under f is an IFGA $K[\psi(A)]$ of $K[G_2]$.*

Proof. Let $\psi : G_1 \to G_2$ be a group homomorphism. Then the induced K-algebra homomorphism $f : K[G_1] \to K[G_2]$ is defined by:

$$f(\textstyle\sum_{g\in G_1} a_g g) = \sum_{g\in G_1} a_g \psi(g), \text{ for all } \sum_{g\in G_1} a_g g \in K[G_1].$$

Let $K[A] = (\mu_{K[A]}, \nu_{K[A]})$ be an IFGA on $K[G_1]$ corresponding to the IFG $A = (\mu_A, \nu_A)$ of G_1. Then:

$$\begin{aligned}
\mu_{f(K[A])}\left(\sum_{g\in G_2} a_g g\right) &= \sup\left\{\mu_{K[A]}\left(\sum_{h\in G_1} b_h h\right) : f\left(\sum_{h\in G_1} b_h h\right) = \sum_{g\in G_2} a_g g\right\} \\
&= \sup\left\{\min\{\mu_A(h) : b_h \neq 0\} : \sum_{h\in G_1} b_h \psi(h) = \sum_{g\in G_2} a_g g\right\} \\
&= \sup\left\{\min\{\mu_A(h) : b_h \neq 0\} : a_g = \sum_{h\in G_1, \psi(h)=g} b_h, \forall g \in G_2\right\} \\
&= \sup\left\{\min\{\mu_A(h) : a_g \neq 0\},\ \text{where } \psi(h) = g\right\} \\
&= \min\left\{\sup\{\mu_A(h) : \psi(h) = g\} : a_g \neq 0\right\} \\
&= \min\left\{\mu_{\psi(A)}(g) : a_g \neq 0\right\} \\
&= \mu_{K[\psi(A)]}\left(\sum_{g\in G_2} a_g g\right).
\end{aligned}$$

Similarly, we can show that $\nu_{f[K(A)]}\left(\sum_{g\in G_2} a_g g\right) = \nu_{K[\psi(A)]}\left(\sum_{g\in G_2} a_g g\right)$.
Therefore, $f(K[A]) = K(\psi(A))$. Since $\psi(A)$ is an IFG on G_2 (see Theorem (4.2) of [16]), it follows that $K[\psi(A)]$ is an IFGA on $K[G_2]$, i.e., $f[K(A)]$ is an IFGA of $K[G_2]$. ☐

4 Intuitionistic fuzzy group algebra homomorphism

In this section, we introduce the concept of an intuitionistic fuzzy group algebra homomorphism (IFGA-homomorphism), which serves as an analogue to the classical group algebra homomorphism induced by a group homomorphism. Subsequently, we construct a category of intuitionistic fuzzy group algebras based on this notion.

Definition 4.1. Let G_1 and G_2 be groups, and let K be a field. Suppose A is an IFG of G_1 and B is an IFG of G_2. Let $K[A]$ and $K[B]$ denote the corresponding IFGAs over $K[G_1]$ and $K[G_2]$, respectively. Then the map $f : K[A] \to K[B]$ is called an *intuitionistic fuzzy group algebra homomorphism* (IFGA-homomorphism) if the following hold:

(i) f is an induced K-algebra homomorphism

(ii) $\mu_{K[B]}(f(\alpha)) \geq \mu_{K[A]}(\alpha)$ and $\nu_{K[B]}(f(\alpha)) \leq \nu_{K[A]}(\alpha)$, $\forall \alpha = \sum_{g\in G_1} a_g g \in K[G_1]$.

To avoid confusion, we denote an IFGA-homomorphism by $\overline{f} : K[A] \to K[B]$, while its underlying K-algebra homomorphism is written as $f : K[G_1] \to K[G_2]$.

Example 4.2. Let $G_1 = \mathbb{Z}_2 = \{0, 1\}$ under addition modulo 2, and $G_2 = \mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3 be finite groups. Let $K = \mathbb{R}$. Define IFSs $A = (\mu_A, \nu_A)$ of G_1 and $B = (\mu_B, \nu_B)$ of G_2 as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.7, & \text{if } x = 1 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.2, & \text{if } x = 1. \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.9, & \text{if } x = 1 \\ 0.6, & \text{if } x = 2 \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.05, & \text{if } x = 1 \\ 0.3, & \text{if } x = 2. \end{cases}$$

Consider the group homomorphism $\psi : \mathbb{Z}_2 \to \mathbb{Z}_3$ given by $\psi(0) = 0$, $\psi(1) = 1$. This is clearly a homomorphism since $\psi(x+y \bmod 2) \equiv \psi(x)+\psi(y) \pmod 3$. Extend ψ linearly to a K-algebra homomorphism $f : K[\mathbb{Z}_2] \to K[\mathbb{Z}_3]$ by

$$f\left(\sum_{g\in G_1} a_g g\right) = \sum_{g\in G_1} a_g \psi(g).$$

For example, if $\alpha = 2 \cdot 0 + 3 \cdot 1 \in K[\mathbb{Z}_2]$, then

$$f(\alpha) = 2 \cdot \psi(0) + 3 \cdot \psi(1) = 2 \cdot 0 + 3 \cdot 1 = 3 \cdot 1 \in K[\mathbb{Z}_3].$$

Now, since

$$\mu_B(\psi(0)) = \mu_B(0) = 1 \geq \mu_A(0) = 1, \quad \nu_B(\psi(0)) = 0 \leq 0,$$

$$\mu_B(\psi(1)) = \mu_B(1) = 0.9 \geq 0.7 = \mu_A(1), \quad \nu_B(\psi(1)) = 0.05 \leq 0.2 = \nu_A(1),$$

the intuitionistic fuzzy inequalities hold for group elements.

By the linearity of f, these inequalities extend to all elements of $K[\mathbb{Z}_2]$, hence for any $\alpha \in K[\mathbb{Z}_2]$ we have

$$\mu_{K[B]}(f(\alpha)) \geq \mu_{K[A]}(\alpha), \quad \nu_{K[B]}(f(\alpha)) \leq \nu_{K[A]}(\alpha).$$

Thus, $\overline{f} : K[A] \to K[B]$ is an IFGA-homomorphism according to Definition 4.1.

Theorem 4.3. *Let $K[A], K[B], K[C]$ be IFGAs corresponding to IFGs A on G_1, B on G_2, and C on G_3, respectively. If $\overline{f} : K[A] \to K[B]$ and $\overline{h} : K[B] \to K[C]$ are IFGA-homomorphisms, then the composition $\overline{h} \circ \overline{f} : K[A] \to K[C]$ is also an IFGA-homomorphism.*

Proof. Since $\overline{f}$ and $\overline{h}$ are intuitionistic fuzzy group algebra homomorphisms, they satisfy the three conditions of Definition 4.1.

(i) Algebra homomorphism property. Both $f : K[G_1] \to K[G_2]$ and $h : K[G_2] \to K[G_3]$ are K-algebra homomorphisms. The composition of two K-algebra homomorphisms is again a K-algebra homomorphism; hence

$$h \circ f : K[G_1] \to K[G_3]$$

satisfies

$$(h\circ f)(x+y) = (h\circ f)(x)+(h\circ f)(y), \;\; (h\circ f)(\lambda x) = \lambda(h\circ f)(x), \;\; (h\circ f)(xy) = (h\circ f)(x)(h\circ f)(y)$$

for all $x, y \in K[G_1]$ and $\lambda \in K$.

(ii) Induced by a group homomorphism. By Definition 3.12, f is induced by a group homomorphism $\psi : G_1 \to G_2$ and h is induced by a group homomorphism $\phi : G_2 \to G_3$. That is, for every $\alpha = \sum_{g\in G_1} a_g g \in K[G_1]$,

$$f(\alpha) = \sum_{g\in G_1} a_g \, \psi(g) \in K[G_2],$$

and for every $\beta = \sum_{u\in G_2} b_u u \in K[G_2]$,

$$h(\beta) = \sum_{u\in G_2} b_u \, \phi(u) \in K[G_3].$$

The composition $h\circ f$ is therefore induced by the composition of the group maps $\phi\circ\psi : G_1 \to G_3$, since

$$(h \circ f)\left(\sum_{g\in G_1} a_g g\right) = h\left(\sum_{g\in G_1} a_g \psi(g)\right) = \sum_{g\in G_1} a_g \, \phi(\psi(g)) = \sum_{g\in G_1} a_g \, (\phi \circ \psi)(g).$$

As ϕ and ψ are group homomorphisms, so is $\phi \circ \psi$; thus $(h \circ f)$ is induced by a group homomorphism.

(iii) Preservation of intuitionistic fuzzy structure. Because $\overline{f}$ and $\overline{h}$ are intuitionistic fuzzy homomorphisms, they satisfy for all $\alpha \in K[G_1]$ and all $\beta \in K[G_2]$:

$$\mu_{K[B]}\big(f(\alpha)\big) \geq \mu_{K[A]}(\alpha), \quad \nu_{K[B]}\big(f(\alpha)\big) \leq \nu_{K[A]}(\alpha),$$

and

$$\mu_{K[C]}\big(h(\beta)\big) \geq \mu_{K[B]}(\beta), \quad \nu_{K[C]}\big(h(\beta)\big) \leq \nu_{K[B]}(\beta).$$

Taking $\beta = f(\alpha)$ and composing these inequalities yields, for every $\alpha \in K[G_1]$,

$$\mu_{K[C]}\big((h \circ f)(\alpha)\big) = \mu_{K[C]}\big(h(f(\alpha))\big) \geq \mu_{K[B]}\big(f(\alpha)\big) \geq \mu_{K[A]}(\alpha),$$

and similarly

$$\nu_{K[C]}\big((h \circ f)(\alpha)\big) = \nu_{K[C]}\big(h(f(\alpha))\big) \leq \nu_{K[B]}\big(f(\alpha)\big) \leq \nu_{K[A]}(\alpha).$$

Thus $h \circ f$ preserves the intuitionistic fuzzy membership and non-membership inequalities.

Combining (i)–(iii), we conclude that $\overline{h \circ f} = \overline{h}\circ\overline{f}$ satisfies all the conditions of Definition 4.1 and therefore is an intuitionistic fuzzy group algebra homomorphism. $\square$

Proposition 4.4. *The collection of all intuitionistic fuzzy group algebras over group algebras, together with intuitionistic fuzzy group algebra homomorphisms as morphisms, forms a category.*

Proof. From Theorem 4.3, the composition of two IFGA-homomorphisms is again an IFGA-homomorphism. Moreover, for every IFGA $K[A]$, the identity map $\mathrm{id}_{K[A]} : K[A] \to K[A]$ is trivially an IFGA-homomorphism, since it satisfies all three conditions of Definition 4.1.

Thus, the class of IFGAs together with these IFGA-homomorphisms satisfies the axioms of a category: associativity of morphism composition and the existence of identity morphisms. □

Remark 4.5. The above category of IFGAs generalizes the usual category of group algebras. Indeed, if the IFG $A = (\mu_A, \nu_A)$ on a group G is taken to be *crisp*, i.e.,

$$\mu_A(g) = \begin{cases} 1, & \text{if } g \in G, \\ 0, & \text{otherwise}, \end{cases} \qquad \nu_A(g) = 1 - \mu_A(g),$$

then the IFGA $K[A]$ coincides with the classical group algebra $K[G]$. In this case, the notion of an IFGA-homomorphism reduces to the ordinary K-algebra homomorphism induced by a group homomorphism.

5 Conclusions

This work develops the theory of intuitionistic fuzzy group algebras, establishing their dual role as intuitionistic fuzzy algebras and intuitionistic fuzzy G-modules. The properties of these algebras as intuitionistic fuzzy G-modules provide a foundation for studying their semi-simplicity, marking an important step toward the intuitionistic fuzzification of Maschke's Theorem on the semi-simplic ty of group algebras. By further examining intersections, (α, β)-cuts, and homomorphic images, as well as introducing intuitionistic fuzzy group algebra homomorphisms, the study situates these structures within a categorical framework. Overall, these results open new directions for exploring the interplay between intuitionistic fuzzy algebra, module theory, and categorical structures.

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