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On intuitionistic fuzzy modal feeble topological structures with modal operator of second type

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Abstract: Two new Intuitionistic Fuzzy Modal Feeble Topological Structures (IFMFTSs) are introduced of fifth and sixth types. Examples for these structures are given.
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1 Introduction

In a series of papers [6–9], based on the books [11, 13, 15], the concept of an Intuitionistic Fuzzy Modal Topological Structure (IFMTS) is introduced by the author. In these papers, two examples of IFMTSs and a lot of examples of Intuitionistic Fuzzy Modal Feeble Topological Structure (IFMFTS) are given. In them, the word "feeble" is used, because some of the axioms of the standard topological structures are replaced with weaker ones, but the concept of weak topology has different sense.

In the present paper, a new–fifth–type of an IFMFTS will be defined and illustrated.

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2 Preliminaries

First, following [3, 5], we mention that each IFS A^* has the form

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where E is a fixed universe, $A \subseteq E$ and for each $x \in E$ both functions satisfy the condition:

$$\mu_A(x) + \nu_A(x) \leq 1.$$

As usually, instead of A^* for brevity below we will use A.

Over the IFSs a lot of operations, relations and operators are defined. The most used among them, which we will need below are the following (see [3, 5]):

$$\begin{array}{lcl}
A \subseteq B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \ \&\ \nu_A(x) \geq \nu_B(x)); \\
A \supseteq B & \text{iff} & B \subset A; \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \ \&\ \nu_A(x) = \nu_B(x)); \\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}.
\end{array}$$

Here, for the first time in the frames of our research over intuitionistic fuzzy topological structures, we will use one of the most extended operators from the second type:

$$\boxdot_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \geq, \delta \in [0, 1]$ and

$$\max(\alpha, \beta) + \gamma + \delta \leq 1. \tag{1}$$

In contrast to the modal operators of the first type, that are the intuitionistic fuzzy versions of the standard modal logic operators (see, e.g., [10, 12, 14]), the operators of the second type do not have direct analogues in modal logic.

The first topological operators over IFSs were described in details in [3, 5]. These operators are analogues of the topological operators "closure" and "interior" and they have the forms

$$\mathcal{C}(A) = \{\langle x, \sup_{y\in E} \mu_A(y), \inf_{y\in E} \nu_A(y)\rangle | x \in E\},$$

$$\mathcal{I}(A) = \{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}.$$

It can be easily seen that operators $\mathcal{C}$ and $\mathcal{I}$ are direct modifications of the intuitionistic fuzzy operations "union" ($\cup$) and "intersection" ($\cap$).

3 Definitions of the fifth type of intuitionistic fuzzy modal feeble topological structures

First, let us define:

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$
$$U^* = \{\langle x, 0, 0\rangle | x \in E\},$$
$$E^* = \{\langle x, 1, 0\rangle | x \in E\}.$$

Let for each set X, $\mathcal{P}(X) = \{Y | Y \subseteq X\}$. Then

$$\mathcal{P}(E^*) = \{A | A \subseteq E^*\},$$

where A is an IFS over the universe E.

Here, for the first time, we introduce IFMFTSs of the fifth type (IFMFTS5s).

The first of the IFMFTS5 is the object that we will denote as cl-IFMFTS5 with the form

$$\langle \mathcal{P}(E^*), \mathcal{O}, \bullet, \circ \rangle,$$

where E is a fixed universe, $\bullet$, $\blacksquare$: $\mathcal{P}(E^*) \times \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ are the operations and for two IFSs $A, B \in \mathcal{P}(E^*)$ satisfying the equalities

$$A \bullet B = \neg(\neg A \blacksquare \neg B),$$
$$A \blacksquare B = \neg(\neg A \bullet \neg B),$$

$\mathcal{O} : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is an operator of a closure type, $\circ : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is a modal operator, and for every two IFSs $A, B \in \mathcal{P}(E^*)$:

C1 $\mathcal{O}(A \bullet B) = \mathcal{O}(A) \bullet \mathcal{O}(B)$,

C2 $A \subseteq \mathcal{O}(A)$,

C3 $\mathcal{O}(A) = \mathcal{O}(\mathcal{O}(A))$,

C4 $O^* = \mathcal{O}(O^*)$,

C5 $\circ\mathcal{O}(A) = \mathcal{O}(\circ A)$,

C6 $\circ(A \blacksquare B) = \circ A \blacksquare \circ B$,

C7 $\circ \circ A = \circ A$.

The difference of this definition compared to the definition from [6] is that the two conditions

C8 $\circ A \subseteq A$,

C9 $\circ E^* = E^*$

here are omitted, i.e., the present structure is weaker (feebler) than the standard cl-IFMTS.

Theorem 1. $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxdot_{\alpha,\beta,\gamma,\delta}\rangle$ *is a cl-IFMFTS5 with the modal operator of the second type* $\boxdot_{\alpha,\beta,\gamma,\delta}$*, where* $\alpha, \beta, \gamma, \delta \in [0,1]$ *and they satisfy* (1).

Proof. Conditions C1–C4 are checked for validity in [6]. So, we check the remaining three conditions(C5–C7) as follows.

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\delta}\mathcal{C}(A) &= \boxdot_{\alpha,\beta,\gamma,\delta}\{\langle x, \sup_{y\in E}\mu_A(y), \inf_{y\in E}\nu_A(y)\rangle | x \in E\}\\ &= \{\langle x, \alpha\sup_{y\in E}\mu_A(y) + \gamma, \beta\inf_{y\in E}\nu_A(y) + \delta\rangle | x \in E\}\\ &= \{\langle x, \sup_{y\in E}(\alpha\mu_A(y) + \gamma), \inf_{y\in E}(\beta\nu_A(y) + \delta)\rangle | x \in E\}\\ &= \mathcal{C}(\{\langle x, \alpha\mu_A(y) + \gamma, \beta\nu_A(y) + \delta\rangle | x \in E\})\\ &= \mathcal{C}(\boxdot_{\alpha,\beta,\gamma,\delta}A);\end{aligned}$$

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\delta}(A\cap B) &= \boxdot_{\alpha,\beta,\gamma,\delta}\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}\\ &= \{\langle x, \alpha\min(\mu_A(x), \mu_B(x)) + \gamma, \beta\max(\nu_A(x), \nu_B(x)) + \delta\rangle | x \in E\}\\ &= \{\langle x, \min(\alpha\mu_A(x) + \gamma, \alpha\mu_B(x) + \gamma), \max(\beta\nu_A(x) + \delta, \beta\nu_B(x) + \delta)\rangle | x \in E\}\\ &= \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta, \rangle | x \in E\}\\ &\qquad \cap \{\langle x, \alpha\mu_B(x) + \gamma, \beta\nu_B(x) + \delta, \rangle | x \in E\}\\ &= \boxdot_{\alpha,\beta,\gamma,\delta}A \cap \boxdot_{\alpha,\beta,\gamma,\delta}B.\end{aligned}$$

Let $\varepsilon, \zeta, \eta, \theta \in [0,1]$ and $\max(\varepsilon, \eta) + \zeta + \theta \leq 1$. Then

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\delta}\boxdot_{\varepsilon,\zeta,\eta,\theta}A &= \boxdot_{\alpha,\beta,\gamma,\delta}\{\langle x, \varepsilon\mu_A(x) + \eta, \zeta\nu_A(x) + \theta\rangle | x \in E\}\\ &= \{\langle x, \alpha(\varepsilon\mu_A(x) + \eta) + \gamma, \beta(\zeta\nu_A(x) + \theta) + \delta\rangle | x \in E\}\\ &= \{\langle x, \alpha\varepsilon\mu_A(x) + \alpha\eta + \gamma, \beta\zeta\nu_A(x) + \beta\theta + \delta\rangle | x \in E\}\\ &= \boxdot_{\alpha\varepsilon,\beta\zeta,\alpha\eta+\gamma,\beta\theta+\delta}A.\end{aligned}$$

□

The second of the IFMFTS5 is the object that we will denote as in-IFMFTS5 with the form

$$\langle \mathcal{P}(E^*), \mathcal{O}, \blacksquare, *\rangle,$$

where the notations are as above, but now $\mathcal{O} : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is an operator of an interior type, $* : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is a modal operator so that for each IFS A:

$$*A = \neg \circ \neg A, \tag{2}$$

$$\circ A = \neg * \neg A, \tag{3}$$

and for every two IFSs $A, B \in \mathcal{P}(E^*)$:

I1 $\mathcal{O}(A \blacksquare B) = \mathcal{O}(A) \blacksquare \mathcal{O}(B)$,

I2 $\mathcal{O}(A) \subseteq A$,

I3 $\mathcal{O}(A) = \mathcal{O}(\mathcal{O}(A))$,

I4 $E^* = \mathcal{O}(E^*)$,

I5 $*\mathcal{O}(A) = \mathcal{O}(*A)$,

I6 $*(A \bullet B) = *A \bullet *B$,

I7 $**A = *A$.

The difference of this definition compared to the definition from [6] is that the two conditions

I8 $A \subseteq *A$,

I9 $*O^* = O^*$

here are omitted, i.e., the present structure is again weaker (feebler) than the standard in-IFMTS.

Now, we can see that:

$$\begin{aligned}\neg\boxdot_{\alpha,\beta,\gamma,\delta}\neg A &= \neg\boxdot_{\alpha,\beta,\gamma,\delta}\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}\\ &= \neg\{\langle x, \alpha\nu_A(x) + \gamma, \beta\mu_A(x) + \delta\rangle | x \in E\}\\ &= \{\langle x, \beta\mu_A(x) + \delta, \alpha\nu_A(x) + \gamma\rangle | x \in E\}\\ &= \boxdot_{\beta,\alpha,\delta,\gamma}A,\end{aligned}$$

i.e., this modal operator is self-dual and it satisfies equalities (2) and (3). Now, we can prove the following theorem.

Theorem 2. $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \boxdot_{\alpha,\beta,\gamma,\delta}\rangle$ *is an in-IFMFTS5 with the modal operator of second type* $\boxdot_{\alpha,\beta,\gamma,\delta}$*, where* $\alpha, \beta, \gamma, \delta \in [0, 1]$ *and they satisfy* (1).

Proof. Conditions I1–I4 are checked for validity in [6]. Condition (I7) coincides with condition (C7), and its check was described in the proof of Theorem 1. Hence, it remains to check the rest two conditions (I5 and I6), as follows.

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\delta}\mathcal{I}(A) &= \boxdot_{\alpha,\beta,\gamma,\delta}\{\langle x, \inf_{y\in E}\mu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x \in E\}\\ &= \{\langle x, \alpha\inf_{y\in E}\mu_A(y) + \gamma, \beta\sup_{y\in E}\nu_A(y) + \delta\rangle | x \in E\}\\ &= \{\langle x, \inf_{y\in E}(\alpha\mu_A(y) + \gamma), \sup_{y\in E}(\beta\nu_A(y) + \delta)\rangle | x \in E\}\\ &= \mathcal{I}(\{\langle x, \alpha\mu_A(y) + \gamma, \beta\nu_A(y) + \delta\rangle | x \in E\})\\ &= \mathcal{I}(\boxdot_{\alpha,\beta,\gamma,\delta}A);\end{aligned}$$

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\delta}(A\cup B) &= \boxdot_{\alpha,\beta,\gamma,\delta}\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}\\ &= \{\langle x, \alpha\max(\mu_A(x), \mu_B(x)) + \gamma, \beta\min(\nu_A(x), \nu_B(x)) + \delta\rangle | x \in E\}\\ &= \{\langle x, \max(\alpha\mu_A(x) + \gamma, \alpha\mu_B(x) + \gamma), \min(\beta\nu_A(x) + \delta, \beta\nu_B(x) + \delta)\rangle | x \in E\}\\ &= \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta\rangle | x \in E\} \cup \{\langle x, \alpha\mu_B(x) + \gamma, \beta\nu_B(x) + \delta\rangle | x \in E\}\\ &= \boxdot_{\alpha,\beta,\gamma,\delta}A \cup \boxdot_{\alpha,\beta,\gamma,\delta}B.\end{aligned}$$

□

4 Definitions of the sixth type of intuitionistic fuzzy modal feeble topological structures

If we modify condition C9 to the form

C9′ $\circ E^* \subseteq E^*$,

then we obtain the sixth type of a cl-IFMFTS (cl-IFMFTS6). It can be illustrated with the following theorem.

Theorem 3. *$\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxdot_{\alpha,\beta,0,\delta} \rangle$ is a cl-IFMFTS6 with the modal operator of the second type $\boxdot_{\alpha,0,\gamma,\delta}$, where $\alpha, \gamma, \delta \in [0, 1]$ and they satisfy the inequality*

$$\max(\alpha, \beta) + \delta \leq 1.$$

Proof. Conditions C1–C7 were already checked for validity. So, it remains that we check only the validity of condition C9′. Indeed,

$$\boxdot_{\alpha,\beta,0,\delta} E^* = \{\langle x, \alpha, \delta \rangle | x \in E\} \subseteq \{\langle x, 1, 0 \rangle | x \in E\} = E^*.$$

We see immediately, that $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxdot_{\alpha,0,1-\alpha,0} \rangle$ is a cl-IFMFTS6 for which condition C9 is valid. □

Finally, if we modify condition I9 to the form

I9′ $O^* \subseteq *O^*$,

then we obtain the sixth type of an in-IFMFTS (in-IFMFTS6). It can be illustrated with the following assertion that is checked as above.

Theorem 4. *$\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \boxdot_{\alpha,\beta,\gamma,0} \rangle$ is an in-IFMFTS6 with the modal operator of the second type $\boxdot_{\alpha,\beta,\gamma,0}$, where $\alpha, \beta, \gamma \in [0, 1]$ and they satisfy the inequality*

$$\max(\alpha, \beta) + \gamma \leq 1.$$

5 Conclusion

The results from the papers [6–9] and from the present one show the necessity of a new notation in describing the IFMTS and this will be an object of investigation in the near future. Also, in future, we will study new structures in which operations "union" and "intersection" have other forms in sense of [1, 2]. These new forms of the operations will generate new intuitionistic fuzzy topological operators and probably, new stuctures.

A classification of the different types of intuitionistic fuzzy topological structures is further anticipated as a next leg of the present research.

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