Notes on Intuitionistic Fuzzy Sets
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On intuitionistic fuzzy semiprime submodules

P. K. Sharma

Post-Graduate Department of Mathematics, D.A.V.College
Jalandhar, Punjab, India
e-mail: pksharma@davjalandhar.com

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Abstract: The purpose of this paper is to extend the notion of ordinary semiprime submodules to intuitionistic fuzzy semiprime submodules. Also we introduce and study new properties of intuitionistic fuzzy semiprime submodules. Many related results are obtained.
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1 Introduction

After the formulation of fuzzy sets theory by Zadeh [20], there have been various extensions to this basic idea. One of the prominent generalizations of fuzzy sets theory is the theory of intuitionistic fuzzy sets introduced by Atanassov [1]. In fuzzy sets, each element is associated with the degree of membership in a given set, while in the intuitionistic fuzzy sets each element is associated with two degrees, one being the degree of membership and the other being the degree of non-membership of the element to the set. With respect to fuzzy sets, the degree of membership of each element lies in the range of 0 and 1, this is additionally the situation for the degree of non-membership of every element in the intuitionsitic fuzzy sets, and the sum of these two degrees is not more than 1. For more information on intuitionistic fuzzy sets theory, we suggest the reader to refer to [1–3]. Fuzzy sets are intuitionistic fuzzy sets, however the converse is not really obvious [2]. There are circumstances where intuitionistic fuzzy set theory is more effective than fuzzy set theory (see [7]). Some mathematicians applied this idea to generalize different notions of algebra, for instance Biswas [6] was the first to introduce the intuitionistic fuzzification of the algebraic structures and developed the concept of intuitionistic fuzzy subgroup of a group. Hur et al. in [9] defined and studied intuitionistic fuzzy subrings and ideals of a ring. Bakhadach et al. in [4] studied the notion of intuitionistic prime ideals of a ring with the help of intuitionistic fuzzy points. Sharma et al. in [19] studied the notion of intuitionistic fuzzy prime radical and intuitionistic fuzzy primary ideal of a Γ-ring. Davvaz et al. [8] applied this idea to modules. They presented the idea of an intuitionistic fuzzy submodule of a module and concentrated on related properties. Many authors, like Basnet in [5] studied the basic properties of some of the concepts of intuitionistic fuzzy algebraic structure of rings and modules like sum and product of ideals, essential submodules, closed and complements of modules etc. Isaac et al. in [10] proved some characterizations like sum, union and intersection of two intuitionistic fuzzy submodules of an R-module. Sharma in [12] strengthened some of the elementary results on intuitionistic fuzzy submodules by using their level cut sets. Rahman and Saikia in [11] studied homomorphic behavior of intuitionistic fuzzy submodules. Sharma et al. in [17] introduced the concept of residual quotient of intuitionistic fuzzy subsets of rings and modules. Using the concept of residual quotient, some important characterization of intuitionistic fuzzy annihilator of subsets of ring and module has been derived. They have also studied some of the properties of intuitionistic fuzzy prime submodules with the help of residual quotient in [18]. The concepts like intuitionistic L-fuzzy primary and P-primary submodules were defined in [14] by Sharma et al. and studied their properties. Decomposition theorem of an intuitionistic fuzzy ideal in terms of intuitionistic fuzzy prime submodules was discussed by Sharma et al. in [16]. The topological structure on the spectrum of intuitionistic L-fuzzy prime submodules had been discussed by Sharma et al. in [15]. In [13], the author introduced the concept of intuitionistic fuzzy multiplication modules and studied some of their properties.

The motivation behind this paper is to study intuitionistic fuzzy semiprime submodules, which is a natural generalization of intuitionistic fuzzy semiprime ideals. We examine the connection between intuitionistic fuzzy prime submodules and intuitionistic fuzzy semiprime submodules. Apart from discussing many related results, an attempt has been made to study the structure of residual quotient $(A : \chi_M)$ of the intuitionistic fuzzy semiprime submodule A. A significant characterization of intuitionistic fuzzy semiprime submodules will be examined.

2 Preliminaries

Throughout this paper R is a commutative ring with non-zero identity, M is a unitary R-module with zero element θ. In order to make this paper easier to follow, we recall in this section various notions and results from intuitionistic fuzzy commutative algebra theory which can be found in [5, 9, 14, 17, 18].

Given a nonempty set X, an intuitionistic fuzzy subset A is an ordered function (μ_A, ν_A) from X to $[0,1] \times [0,1]$. We denote by $IFS(X)$ the set of all intuitionistic fuzzy subsets of X. For $A, B \in IFS(X)$ we write $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$. For fixed $p, q \in [0,1]$ such that $p+q \leq 1$, if $A \in IFS(X)$ is such that $\mu_A(x) = p, \nu_A(x) = q$ for all $x \in X$, then A is called a constant intuitionistic fuzzy set in X, otherwise it is called a non-constant intuitionistic fuzzy set in X.

In particular, when $p = 0, q = 1$, then the constant IFS A is denoted by $\tilde{0}$ and when $p = 1, q = 0$, then the constant IFS A is denoted by $\tilde{1}$. By an intuitionistic fuzzy point (IFP) $x_{(p,q)}$ of X, $x \in X$, $p, q \in [0, 1]$ such that $p + q \leq 1$ we mean $x_{(p,q)} \in IFS(X)$ is defined by

$$x_{(p,q)}(y) = \begin{cases} (p, q), & \text{if } y = x \\ (0, 1), & \text{otherwise} \end{cases}.$$

If $A \in IFS(X)$ and $x \in X$ such that $\mu_A(x) \geq p$ and $\nu_A(x) \leq q$, then $x_{(p,q)} \subseteq A$. We write it as $x_{(p,q)} \in A$. The intuitionistic fuzzy characteristic function of X with respect to a subset Y is denoted by χ_Y and is defined as:

$$\mu_{\chi_Y}(y) = \begin{cases} 1, & \text{if } y \in Y \\ 0, & \text{if otherwise} \end{cases}; \quad \nu_{\chi_Y}(y) = \begin{cases} 0, & \text{if } y \in Y \\ 1, & \text{otherwise.} \end{cases}$$

If $x = \theta$ and $p = 1, q = 0$, then $x_{(p,q)} = \theta_{(1,0)}$ (or $Y = \{\theta\}$) is called the intuitionistic fuzzy zero point of X and is denoted by $\chi_{\{\theta\}}$.

Definition 2.1. ([5,9]) Let $A \in IFS(R)$. Then A is called an intuitionistic fuzzy subring of R if for all $x, y \in R$, the following statements are satisfied:

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$;

(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;

(iv) $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

Definition 2.2. ([5,9]) Let $A \in IFS(R)$. Then A is called an intuitionistic fuzzy ideal (IFI) of R if for all $x, y \in R$, the following statements are satisfied:

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$;

(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;

(iv) $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Definition 2.3. ([8, 10–12]) Let $A \in IFS(M)$. Then A is called an intuitionistic fuzzy module (IFM) of M if for all $x, y \in M, r \in R$, the following statements are satisfied:

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(rx) \geq \mu_A(x)$;

(iii) $\mu_A(\theta) = 1$;

(iv) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;

(v) $\nu_A(rx) \leq \nu_A(x)$;

(vi) $\nu_A(\theta) = 0$.

Clearly, $\chi_{\{\theta\}}$, χ_M are IFMs of M and these are called trivial IFMs of M. Any IFM of M other than these is called non-trivial proper IFM of M. Let $IFM(M)$ denote the set of all intuitionistic fuzzy R-modules of M and $IFI(R)$ denote the set of all intuitionistic fuzzy ideals of R. We note that when $R = M$, then $A \in IFM(M)$ if and only if $\mu_A(\theta) = 1, \nu_A(\theta) = 0$ and $A \in IFI(R)$.

Let $A \in IFS(M)$ and $p, q \in [0,1]$ with $p+q \leq 1$. Then the set $A_{(p,q)} = \{x \in M \mid \mu_A(x) \geq p$ and $\nu_A(x) \leq q\}$ is called the (p,q)-cut subset of M with respect to A. Also $A \in IFM(M)$ if and only if $A_{(p,q)}$ is a submodule of M for all $p, q \in [0,1]$ with $p+q \leq 1$. If $x_{(p,q)}, y_{(s,t)} \in IFP(M)$, $r_{(u,v)} \in IFP(R)$, then $x_{(p,q)} + y_{(s,t)} = (x+y)_{(p\wedge s, q\vee t)}$ and $r_{(u,v)}x_{(p,q)} = (rx)_{(u\wedge p, v\vee q)}$. Moreover, $\langle A \rangle = \bigcap\{B \mid B \supseteq A, B \in IFM(M)\}$ be the intuitionistic fuzzy submodule of M generated by A.

Definition 2.4. ([10, 11]) Let M be an R-module and $A, B \in IFS(M)$, $C \in IFS(R)$, $r \in R$, we define $A + B,\ rA,\ C \cdot A$ as follows:

$$(\mu_{A+B}(x), \nu_{A+B}(x)) = \begin{cases} (\sup\{\mu_A(a) \wedge \mu_B(b)\}, \inf\{\nu_A(a) \vee \nu_B(b)\}), & \text{if } x = a+b, a, b \in M \\ (0,1), & \text{if } x \neq a+b, a, b \in M. \end{cases}$$

$$(\mu_{rA}(x), \nu_{rA}(x)) = \begin{cases} (\sup\{\mu_A(a)\}, \inf\{\nu_A(a)\}), & \text{if } x = ra, a \in M \\ (0,1), & \text{otherwise} \end{cases}$$

$$(\mu_{C\cdot A}(x), \nu_{C\cdot A}(x)) = \begin{cases} (\sup\{\mu_C(r) \wedge \mu_A(y)\}, \inf\{\nu_C(r) \vee \nu_A(y)\}), & \text{if } x = ry, r \in R, y \in M \\ (0,1), & \text{otherwise .} \end{cases}$$

Definition 2.5. ([4, 17]) For a non-constant $C \in IFI(R), C$ is called an intuitionistic fuzzy prime ideal of R if for any $x_{(p,q)}, y_{(s,t)} \in IFP(R)$, whenever $x_{(p,q)}y_{(s,t)} \subseteq C$ implies that either $x_{(p,q)} \subseteq C$ or $y_{(s,t)} \subseteq C$.

Definition 2.6. ([4, 17]) For a non-constant $C \in IFI(R), C$ is called an intuitionistic fuzzy semiprime ideal of R if for any $x_{(p,q)} \in IFP(R)$, whenever $x^2_{(p,q)} \subseteq C$ implies that $x_{(p,q)} \subseteq C$.

Definition 2.7. ([17]) For $A, B \in IFS(M)$ and $C \in IFS(R)$, define the residual quotient $(A : B)$ and $(A : C)$ as follows:

$$(A : B) = \bigcup\{D \mid D \in IFS(R) \text{ such that } D \cdot B \subseteq A\}$$

and

$$(A : C) = \bigcup\{E \mid E \in IFS(M) \text{ such that } C \cdot E \subseteq A\}.$$

Clearly, $(A : B) \in IFS(R)$ and $(A : C) \in IFS(M)$. By [17], Theorem 3.6, if $A, B \in IFM(M)$ and $C \in IFI(R)$, then $(A : B) \in IFI(R)$ and $(A : C) \in IFM(M)$.

Theorem 2.8. ([17]) *For $A, B \in IFS(M)$ and $C \in IFS(R)$. Then we have:*

(i) $(A : B) \cdot B \subseteq A$;

(ii) $C \cdot (A : C) \subseteq A$;

(iii) $C \cdot B \subseteq A \Leftrightarrow C \subseteq (A : B) \Leftrightarrow B \subseteq (A : C)$.

Definition 2.9. ([18]) A non-constant intuitionistic fuzzy submodule A of an R-module is said to be an intuitionistic fuzzy prime submodule if for any $C \in IFI(R)$ and $D \in IFM(M)$ such that $C \cdot D \subseteq A$, then either $D \subseteq A$ or $C \subseteq (A : \chi_M)$.

In terms of intuitionistic fuzzy points, a non-constant intuitionistic fuzzy submodule A of an R-module M is called an intuitionistic fuzzy prime submodule if for $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ such that $r_{(s,t)}x_{(p,q)} \in A$ implies that either $x_{(p,q)} \in A$ or $r_{(s,t)} \in (A : \chi_M)$.

Theorem 2.10. ([18]) *Let A be an intuitionistic fuzzy prime submodule of M. Then $(A : \chi_M)$ is an intuitionistic fuzzy prime ideal of R.*

Theorem 2.11. ([18]) *Let A be an intuitionistic fuzzy prime submodule of M. Then $A_{(p,q)}$ is a prime submodule of M for all $p, q \in [0, 1]$ such that $p + q \leq 1$.*

Definition 2.12. ([14]) An intuitionistic fuzzy submodule A of an R-module M is called an intuitionistic fuzzy primary submodule if for $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ such that $r_{(s,t)}x_{(p,q)} \in A$ implies that either $x_{(p,q)} \in A$ or $r^n_{(s,t)} \in (A : \chi_M)$ for some positive integer n.

Note that every intuitionistic fuzzy prime submodule is an intuitionistic fuzzy primary submodule.

Definition 2.13. ([13]) An R-module M is called an intuitionistic fuzzy multiplication module if and only if for each intuitionistic fuzzy submodule A of M, there exists an intuitionistic fuzzy ideal C of R with $C(0_R) = (1, 0)$ such that $A = C\chi_M$. One can easily show that if $A = C\chi_M$, then $A = (A : \chi_M)\chi_M$.

3 Intuitionistic fuzzy semiprime submodules

In this section we shall introduce the concept of intuitionistic fuzzy semi-prime submodules of an R-module M and investigate its connection between the intuitionistic fuzzy prime submodules, intuitionistic fuzzy primary submodules and intuitionistic fuzzy irreducible submodules. We shall also characterise intuitionistic fuzzy semi-prime submodules, when M is an intuitionistic fuzzy multiplication modules. Further, we shall also study the residual quotient of intuitionistic fuzzy semi-prime submodules which would be of great use in studying the decomposition of intuitionistic fuzzy ideal in terms of intuitionistic fuzzy semiprime submodules.

Definition 3.1. A non-constant intuitionistic fuzzy submodule A of an R-module M is called semiprime if for $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ (where $r \in R$, $x \in M$, $s, t, p, q \in [0, 1]$ with $s + t \leq 1, p + q \leq 1$) such that $r^2_{(s,t)}x_{(p,q)} \in A$ implies that $r_{(s,t)}x_{(p,q)} \in A$.

Theorem 3.2. *Every intuitionistic fuzzy prime submodule of an R-module M is intuitionistic fuzzy semiprime submodule of M.*

Proof. Let A be a non-constant intuitionistic fuzzy prime submodule of M. Let $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ such that $r^2_{(s,t)}x_{(p,q)} \in A$. This implies that either $x_{(p,q)} \in A$ or $r^2_{(s,t)} \in (A : \chi_M)$ (as A is an intuitionistic fuzzy prime submodule of M). If $x_{(p,q)} \in A$, then $r_{(s,t)}x_{(p,q)} =$

$(rx)_{(s\wedge p, t\vee q)} \in A$. On the other hand, if $r^2_{(s,t)} \in (A : \chi_M)$, then $r_{(s,t)} \in (A : \chi_M)$ (as $(A : \chi_M)$ is an intuitionistic fuzzy prime ideal of R). This further implies that $r_{(s,t)}\chi_M \subseteq A$ and therefore $r_{(s,t)}x_{(p,q)} \subseteq r_{(s,t)}\chi_M \subseteq A$. So $r_{(s,t)}x_{(p,q)} \in A$. Hence, A is an intuitionistic fuzzy semiprime submodule of M. □

Remark 3.3. The converse of the above theorem is not true in general, see the following example.

Example 3.4. Consider $M = \mathbb{Z}$ as a $\mathbb{Z}$-module and let A be an intuitionistic fuzzy set of M defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in \{0\} \\ 0.5, & \text{if } x \in 6\mathbb{Z} - \{0\} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in \{0\} \\ 0.3, & \text{if } x \in 6\mathbb{Z} - \{0\} \\ 1, & \text{otherwise} \end{cases}.$$

Clearly, A is an intuitionistic fuzzy submodule of M. Since $A_{(0,1)} = \mathbb{Z}$, $A_{(0.5,0.3)} = 6\mathbb{Z}$ and $A_{(\alpha,\beta)} = \{0\}$ for all $\alpha > 0.5, \beta < 0.3$. Clearly A is an intuitionistic fuzzy semiprime submodule of M. But A is not an intuitionistic fuzzy prime submodule of M, for $5_{(0.3,0.4)} \in IFP(\mathbb{Z})$, $6_{(0.6,0.4)} \in IFP(M)$, we have $5_{(0.3,0.4)}6_{(0.6,0.4)} = 30_{(0.3,0.4)} \subseteq A$, but $6_{(0.6,0.4)} \nsubseteq A$ and $5_{(0.3,0.4)} \nsubseteq (A : \chi_M)$, i.e., $5_{(0.3,0.4)}\chi_M \nsubseteq A$, i.e., $5_{(0.3,0.4)} \nsubseteq A$.

Theorem 3.5. *Let A be a proper intuitionistic fuzzy submodule of an R-module M. Then the following statements are equivalent:*

(i) A is an intuitionistic fuzzy semiprime submodule of M;

(ii) $(A : B)$ is an intuitionistic fuzzy semiprime ideal for all intuitionistic fuzzy submodules B that contain A properly;

(iii) $(A : \langle x_{(p,q)}\rangle)$ is an intuitionistic fuzzy semiprime ideal for all $x_{(p,q)} \notin A$.

Proof. $\underline{(i) \Rightarrow (ii)}$ It is clear that $(A : B)$ is an intuitionistic fuzzy ideal for all intuitionistic fuzzy submodules B that contain A properly. Now let $r_{(s,t)} \in \sqrt{(A : B)}$ so there exists a positive integer n such that $r^n_{(s,t)} \in (A : B)$. This implies that $r^n_{(s,t)}B \subseteq A$ and this further implies that $r^n_{(s,t)}x_{(p,q)} \subseteq A$, where $x_{(p,q)} \subseteq B$. As A is an intuitionistic fuzzy semiprime submodule of M we have $r_{(s,t)}x_{(p,q)} \in A$ and hence $r_{(s,t)} \in (A : \langle x_{(p,q)}\rangle) \subseteq (A : B)$. From this we observe that $\sqrt{(A : B)} \subseteq (A : B)$. We also know that $(A : B) \subseteq \sqrt{(A : B)}$ always. This implies that $(A : B) = \sqrt{(A : B)}$, therefore $(A : B)$ is an intuitionistic fuzzy semiprime ideal of R for all intuitionistic fuzzy submodules B of M that contains A properly.

$\underline{(ii) \Rightarrow (iii)}$ Let $x_{(p,q)} \in IFP(M)$ such that $x_{(p,q)} \notin A$. It is observed that the intuitionistic fuzzy ideal $(A : \langle x_{(p,q)}\rangle)$ is proper in R. Suppose that $r_{(s,t)} \in \sqrt{(A : \langle x_{(p,q)}\rangle)}$, so there exists a positive integer n such that $r^n_{(s,t)} \in (A : \langle x_{(p,q)}\rangle)$. But $A \subseteq A + \langle x_{(p,q)}\rangle$. It is clear that $r^n_{(s,t)} \subseteq (A : A + \langle x_{(p,q)}\rangle)$. By (ii) we have $r_{(s,t)} \subseteq (A : A + \langle x_{(p,q)}\rangle)$ and hence $r_{(s,t)} \in (A : \langle x_{(p,q)}\rangle)$ and this implies that $\sqrt{(A : \langle x_{(p,q)}\rangle)} \subseteq (A : \langle x_{(p,q)}\rangle)$, since $(A : \langle x_{(p,q)}\rangle) \subseteq (\sqrt{A : \langle x_{(p,q)}\rangle})$ always. Thus $(A : \langle x_{(p,q)}\rangle) = \sqrt{(A : \langle x_{(p,q)}\rangle)}$ and hence the $(A : \langle x_{(p,q)}\rangle)$ is semiprime ideal in R.

$\underline{(iii) \Rightarrow (i)}$ Let $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ such that $r^2_{(s,t)}x_{(p,q)} \in A$. If $x_{(p,q)} \in A$ then $r_{(s,t)}x_{(p,q)} = (rx)_{(s\wedge p, t\vee q)} \in A$. Suppose that $x_{(p,q)} \notin A$. Since $r^2_{(s,t)} \in (A : \langle x_{(p,q)}\rangle)$ and by (iii) we get $r_{(s,t)} \in (A : \langle x_{(p,q)}\rangle)$. This means that $r_{(s,t)}x_{(p,q)} \in A$ and, therefore, A is an intuitionistic fuzzy semiprime submodule of M. □

Corollary 3.6. *If A is an intuitionistic fuzzy semiprime submodule of M, then $(A : \chi_M)$ is an intuitionistic fuzzy semiprime ideal of R.*

Remark 3.7. The converse of the above corollary is not true in general, see the following example.

Example 3.8. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$-module and $N = 4\mathbb{Z} \oplus \langle 0\rangle$ is a proper submodule of M. Define $A \in IFS(M)$ as

$$\mu_A((a,b)) = \begin{cases} 1, & \text{if } (a,b) \in N \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_A((a,b)) = \begin{cases} 0, & \text{if } (a,b) \in N \\ 1, & \text{otherwise} \end{cases} .$$

Clearly, $A = \chi_N$ is an intuitionistic fuzzy submodule of M and $A_{(\alpha,\beta)} = N, \forall \alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$. Also $(A_{(\alpha,\beta)} : M) = (N : M) = \langle 0\rangle$ but $(A_{(\alpha,\beta)} : M) = (A : \chi_M)_{(\alpha,\beta)}$. This implies that $(A : \chi_M)_{(\alpha,\beta)} = \langle 0\rangle$ for all $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$, therefore,

$$(A : \chi_M)(r) = \begin{cases} (1,0), & \text{if } r \in \langle 0\rangle \\ (0,1), & \text{otherwise} \end{cases} .$$

It is easy to verify that $(A : \chi_M) = \chi_0$ is an intuitionistic fuzzy semiprime ideal in $\mathbb{Z}$. But A is not an intuitionistic fuzzy semiprime submodule of M, since $2^2_{(0.5,0.3)}(1,0)_{(0.5,0.3)} = (4,0)_{(0.5,0.3)} \subseteq A$ but $2_{(0.5,0.3)}(1,0)_{(0.5,0.3)} = (2,0)_{(0.5,0.3)} \nsubseteq A$.

Proposition 3.9. *If A is an intuitionistic fuzzy primary submodule of M, then $(A : \chi_M)$ is an intuitionistic fuzzy semiprime ideal of R if and only if A is an intuitionistic fuzzy prime submodule of M.*

Proof. Let A be an intuitionistic fuzzy primary submodule of M and let $(A : \chi_M)$ be an intuitionistic fuzzy semiprime ideal in R and let $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ such that $r_{(s,t)}x_{(p,q)} \in A$. Suppose that $x_{(p,q)} \notin A$, since A is an intuitionistic fuzzy primary submodule of M, then $r_{(s,t)} \in \sqrt{(A : \chi_M)}$ but $(A : \chi_M)$ is an intuitionistic fuzzy semiprime ideal of R, so $\sqrt{(A : \chi_M)} = (A : \chi_M)$, then $r_{(s,t)} \in (A : \chi_M)$. This means that A is an intuitionistic fuzzy prime submodule of M.

The second direction is immediate from Corollary 3.6. □

Corollary 3.10. *If A is an intuitionistic fuzzy primary submodule of M, then $(A : \chi_M)$ is an intuitionistic fuzzy semiprime ideal of R if and only if A is an intuitionistic fuzzy semiprime submodule of M.*

Proof. Immediately follows from Theorem 3.2, Corollary 3.6 and Proposition 3.9. □

Theorem 3.11. *If A is an intuitionistic fuzzy submodule of an intuitionistic fuzzy multiplication module M, then A is an intuitionistic fuzzy semiprime submodule of M if and only if $(A : \chi_M)$ is an intuitionistic fuzzy semiprime ideal of R.*

Proof. Suppose that $(A : \chi_M)$ is an intuitionistic fuzzy semiprime ideal of R. Then A is a proper intuitionistic fuzzy submodule of M. Now suppose that $r^2_{(s,t)}x_{(p,q)} \in A$ for some $r_{(s,t)} \in IFP(R)$ and $x_{(p,q)} \in IFP(M)$. As M is an intuitionistic fuzzy multiplication module, then $(\langle x_{(p,q)}\rangle : \chi_M)\chi_M = \langle x_{(p,q)}\rangle$, where $\langle x_{(p,q)}\rangle$ is the intuitionistic fuzzy submodule of M generated by $x_{(p,q)}$ and hence $x_{(p,q)} = \Sigma_{i=1}^n r_{(s_i,t_i)}m_{(p_i,q_i)}$, where $r_{(s_i,t_i)} \in (\langle x_{(p,q)}\rangle : \chi_M)$, $m_{(p_i,q_i)} \in IFP(M)$, $i = 1, 2, \ldots, n$, but $r^2_{(s_i,t_i)}r^2_{(s,t)} \in (\langle x_{(p,q)}\rangle : \chi_M) \subseteq (A : \chi_M)$, $i = 1, 2, \ldots, n$. Also because $(A : \chi_M)$ is an intuitionistic fuzzy semiprime ideal of R, then $r_{(s_i,t_i)}r_{(s,t)} \in (A : \chi_M)$, for all $i = 1, 2, \ldots, n$ and so $r_{(s,t)}x_{(p,q)} = \Sigma_{i=1}^n (r_{(s,t)}r_{(s_i,t_i)})m_{(p_i,q_i)} \in (A : \chi_M)\chi_M = A$. Hence A is an intuitionistic fuzzy semiprime submodule of M.

The second direction immediately follows from Corollary 3.6. $\square$

Theorem 3.12. *Let A be an intuitionistic fuzzy submodule of an R-module M. If A is an intersection of intuitionistic fuzzy prime submodules of M, then A is an intuitionistic fuzzy semiprime submodule of M.*

Proof. Let $A = \cap_{i\in J}A_i$, where each A_i $(i \in J)$ are intuitionistic fuzzy prime submodules of M. Let $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ such that $r^2_{(s,t)}x_{(p,q)} \in A = \cap_{i\in J}A_i$ implies that $r^2_{(s,t)}x_{(p,q)} \in A_i, \forall i \in J$. As each $A_i(i \in J)$ are intuitionistic fuzzy prime submodules of M, therefore, $x_{(p,q)} \in A_i$ or $r^2_{(s,t)} \in (A_i : \chi_M), \forall i \in J$.

If $x_{(p,q)} \in A_i$, then $r_{(s,t)}x_{(p,q)} = (rs)_{(s\wedge p,t\vee q)} \in A_i, \forall i \in J$.

If $r^2_{(s,t)} \in (A_i : \chi_M)$. As each $(A_i : \chi_M)$ is an intuitionistic fuzzy prime ideal of R, therefore, we get $r_{(s,t)} \in (A_i : \chi_M)$, which implies that $r_{(s,t)}\chi_M \subseteq A_i$ which further implies that $r_{(s,t)}x_{(p,q)} \subseteq r_{(s,t)}\chi_M \subseteq A_i$ gives that $r_{(s,t)}x_{(p,q)} \in A_i, \forall i \in J$. Thus we have $r_{(s,t)}x_{(p,q)} \in \cap_{i\in J}A_i = A$. Hence A is an intuitionistic fuzzy semiprime submodule of M. $\square$

Example 3.13. Let G be any finite Abelian group of order $n = p_1p_2\cdots p_k$, where p_i are distinct primes. Then by the structure theorem of finitely generated group we have

$$G \cong \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_k}.$$

Take $M = G$, then M is a $\mathbb{Z}$-module. Let $M = \langle x_1, x_2, \ldots, x_k\rangle$ such that $o(x_i) = p_i$, for $1 \leq i \leq k$. Let $M_0 = \langle 0\rangle$, $M_1 = \langle x_1\rangle$, $M_2 = \langle x_1, x_2\rangle, \ldots, M_k = \langle x_1, x_2, \ldots, x_k\rangle = M$ be the chain of maximal submodules of M such that $M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k$.

Let A be any intuitionistic fuzzy submodule of M defined by:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in M_0 \\ \alpha_1 & \text{if } x \in M_1\backslash M_0 \\ \alpha_2 & \text{if } x \in M_2\backslash M_1 \\ \cdots \\ \alpha_k & \text{if } x \in M_k\backslash M_{k-1} \end{cases} ; \qquad \nu_A(x) = \begin{cases} 0 & \text{if } x \in M_0 \\ \beta_1 & \text{if } x \in M_1\backslash M_0 \\ \beta_2 & \text{if } x \in M_2\backslash M_1 \\ \cdots \\ \beta_k & \text{if } x \in M_k\backslash M_{k-1} \end{cases},$$

where $1 = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_k$ and $0 = \beta_0 \leq \beta_1 \leq \cdots \leq \beta_k$ and the pair (α_i, β_i) are called double pins and the set $\wedge(A) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$ is called the set of double pinned flags for the IFM A of M (see Example 3.8 in [16]).

Define IFSs A_i on M as follows:

$$\mu_{A_i}(x) = \begin{cases} 1, & \text{if } x \in \langle p_i \rangle \\ \alpha_{i+1}, & \text{otherwise} \end{cases} ; \quad \nu_{A_i}(x) = \begin{cases} 0, & \text{if } x \in \langle p_i \rangle \\ \beta_{i+1}, & \text{otherwise} \end{cases} ,$$

where $\alpha_i, \beta_i \in (0,1)$ such that $\alpha_i + \beta_i \leq 1$, for $1 \leq i \leq k$ and $\alpha_{k+1} = \alpha_1$, $\beta_{k+1} = \beta_1$. Clearly, A_i are intuitionistic fuzzy prime submodules of M. It can be easily checked that $A = \cap_{i=1}^{n} A_i$ is an intuitionistic fuzzy prime decomposition of A and so by Theorem 2.11 A is an intuitionistic fuzzy semiprime submodule of M.

In order that the converse of Theorem 3.2 also holds, we shall first define the notion of intuitionistic fuzzy irreducible submodule of an R-module M.

Definition 3.14. Let A be an intuitionistic fuzzy submodule of an R-module M. Then A is called an intuitionistic fuzzy irreducible submodule if for any two intuitionistic fuzzy submodules A_1 and A_2 we have $A = A_1 \cap A_2$, then it implies that either $A = A_1$ or $A = A_2$, otherwise A is called reducible.

Theorem 3.15. *If A is an intuitionistic fuzzy irreducible submodule of an R-module M, then A is an intuitionistic fuzzy semiprime submodule of M if and only if A is an intuitionistic fuzzy prime submodule of M.*

Proof. Suppose that A is an intuitionistic fuzzy irreducible semiprime submodule of M. If possible, let A be not an intuitionistic fuzzy prime submodule of M. Then there exist $r_{(s,t)} \in IFP(R), x_{(p,q)} \in IFP(M)$ such that $r_{(s,t)}x_{(p,q)} \subseteq A$ but $x_{(p,q)} \notin A$ and $r_{(s,t)} \notin (A : \chi_M)$. Since $r_{(s,t)} \notin (A : \chi_M)$, then there exists $m_{(j,k)} \in IFP(M)$ such that $r_{(s,t)}m_{(j,k)} \notin A$.

We claim that $A = A_1 \cap A_2$, where $A_1 = A + \langle r_{(s,t)}m_{(j,k)} \rangle$, $A_2 = A + \langle x_{(p,q)} \rangle$. It is observed that $A \subseteq A_1 \cap A_2$. For the other inclusion, let $y_{(u,v)} \subseteq A_1 \cap A_2$, where $y_{(u,v)} \in IFP(M)$. This means that $y_{(u,v)} \subseteq A + \langle r_{(s,t)}m_{(j,k)} \rangle$ and $y_{(u,v)} \subseteq A + \langle x_{(p,q)} \rangle$. Therefore there exist $q_{(\alpha,\beta)}, z_{(\gamma,\delta)} \in IFP(M), a_{(c,d)}, b_{(e,f)} \in IFP(R)$ such that $y_{(u,v)} = q_{(\alpha,\beta)} + a_{(c,d)}r_{(s,t)}m_{(j,k)}$ and $y_{(u,v)} = z_{(\gamma,\delta)} + b_{(e,f)}x_{(p,q)}$. This implies that $z_{(\gamma,\delta)} - q_{(\alpha,\beta)} + b_{(e,f)}x_{(p,q)} = a_{(c,d)}r_{(s,t)}m_{(j,k)}$ and hence $r_{(s,t)}z_{(\gamma,\delta)} - r_{(s,t)}q_{(\alpha,\beta)} + r_{(s,t)}b_{(e,f)}x_{(p,q)} = a_{(c,d)}r^2_{(s,t)}m_{(j,k)}$ and here $a_{(c,d)}r^2_{(s,t)}m_{(j,k)} \subseteq A$. As A is an intuitionistic fuzzy semiprime submodule of M, this implies $q_{(\alpha,\beta)} + a_{(c,d)}r_{(s,t)}m_{(j,k)} \subseteq A$. This means that $y_{(u,v)} \subseteq A$ and so $A_1 \cap A_2 \subseteq A$. Thus $A = A_1 \cap A_2$, but this is a contradiction, as A is an irreducible intuitionistic fuzzy submodule of M. Hence A is an intuitionistic fuzzy prime submodule of M.

The second direction immediately follows from Theorem (3.2). □

Theorem 3.16. *Let A be an intuitionistic fuzzy semiprime submodule of an R-module M and B be an intuitionistic fuzzy submodule of M such that $B \nsubseteq A$. Then the intuitionistic fuzzy submodule $A \cap B$ be an intuitionistic fuzzy semiprime submodule of M.*

Proof. Since $B \nsubseteq A$. Therefore, $A \cap B$ is a proper intuitionistic fuzzy submodule of M properly contained in B. Now let $x_{(p,q)} \subseteq B$ and $r_{(s,t)} \in IFP(R)$ be such that $r^2_{(s,t)}x_{(p,q)} \subseteq A \cap B$. This gives $r^2_{(s,t)}x_{(p,q)} \subseteq A$ and $r^2_{(s,t)}x_{(p,q)} \subseteq B$. Now as A is an intuitionistic fuzzy semiprime submodule of M, this implies that $r_{(s,t)}x_{(p,q)} \in A$. Also, because $x_{(p,q)} \subseteq B$ implies that $r_{(s,t)}x_{(p,q)} \in B$, this means that $r_{(s,t)}x_{(p,q)} \in A \cap B$, which proves that $A \cap B$ is an intuitionistic fuzzy semiprime submodule of M. □

4 Conclusion

In this paper, we have discussed the concept of intuitionistic fuzzy semiprime submodule of an R-module M. We have investigated many properties of intuitionistic fuzzy semiprime submodules. It is shown that every intuitionistic fuzzy prime submodule A of an R-module M is an intuitionistic fuzzy semiprime submodule but the converse needs not be true. We have also proved that the converse holds when A is an intuitionistic fuzzy irreducible submodule of M. Further, we have shown that if A is an intuitionistic fuzzy semiprime submodule of M, then the residual quotient $(A : \chi_M)$ is an intuitionistic fuzzy semiprime ideal of R. Again, the converse of this result needs not be true, however we have shown that the converse of this result holds when either A is an intuitionistic fuzzy primary submodule of M or M is an intuitionistic fuzzy multiplication module. We have explored some related results, for example Theorem 3.12 and Theorem 3.16. These results would be of great use for studying the concepts like intuitionistic fuzzy semiprime radical and intuitionistic fuzzy semiprime decomposition theorem.

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