NIFS 10 (2004), 1, 18-26

Some Properties of Intuitionistic Fuzzy Metric Spaces

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Abstract: In this paper, we discuss some properties of intuitionistic fuzzy metric spaces introduced by Jin Han Park in the article titled by "Intuitionistic fuzzy metric spaces", we prove that the topology induced by any (complete) intuitionistic fuzzy metric space is (completely)metrizable and show that every separable intuitionistic fuzzy metric space admits a precompact intuitionistic fuzzy metric and that a intuitionistic fuzzy metric space is compact if and only if it is precompact and complete.
Keywords: intuitionistic fuzzy sets; inutitionistic fuzzy metric space; precompact; metrizable.

1. Introduction

Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. But in fuzzy sets theory, there is no means to incorporate the hesitation or uncertainty in the membership degrees. In 1983, Antanassov introduces the concept of intuitionistic fuzzy sets, which constitute a extension of fuzzy sets theory: intuitionistic fuzzy sets give both a membership degree and a non-membership degree. The only constraint on these two degrees is that the sum must smaller than or equal to 1. One of the main problems in the theory of fuzzy topological spaces is to obtain an appropriate and consistent notion of a fuzzy metric space. Many authors have investigated this question and several different notions of a fuzzy metric space have been defined and studied. In [2], Jin Han Park define the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani. He also define a Hausdorff topology on the intuitionistic fuzzy metric space and show that every metric induces an intuitionistic fuzzy metric space and proved some known results of metric spaces including Baire's theorem and Uniform limit theorem for intuitionistic fuzzy metric spaces. In the present paper, we will discuss some more properties for intuitionistic metric spaces.

2. Some definitions

Definition1 [1]: A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is continuous t-norm if $*$ is satisfying the following condition:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a*1=a$ for all $a\in[0,1]$;

(d) $a*b \le c*d$ whenever $a \le c$ and $b \le d$, and $a,b,c,d \in [0,1]$.

Definition2 [1]: A binary operation $\Diamond : [0,1]\times[0,1] \to [0,1]$ is continuous t-conorm if $\Diamond$ is satisfying the following condition:

(a) $\Diamond$ is commutative and associative;

(b) $\Diamond$ is continuous;

(c) $a \Diamond 0 = a$ for all $a \in [0,1]$;

(d) $a \Diamond b \le c \Diamond d$ whenever $a \le c$ and $b \le d$, and $a,b,c,d \in [0,1]$.

Definition3 [3]: An intuitionistic fuzzy set A in a (non-fuzzy) set E is defined as an object of the following form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$$

where the functions:

$$\mu_A : E \to [0,1]$$

and

$$\nu_A : E \to [0,1]$$

define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

Definition4 [2]: A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions: for all $x,y,z \in X, s,t > 0$,

(a) $M(x,y,t) + N(x,y,t) \le 1$;

(b) $M(x,y,t) > 0$;

(c) $M(x,y,t) = 1$ if and only if $x = y$;

(d) $M(x,y,t) = M(y,x,t)$;

(e) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$;

(f) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous;

(g) $N(x,y,t)>0$;

(h) $N(x,y,t)=0$ if and only if $x=y$;

(i) $N(x,y,t)=N(y,x,t)$;

(j) $N(x,y,t) \lozenge N(y,z,s) \geq N(x,z,t+s)$;

(k) $N(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous.

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t), N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Let (X,d) be a metric space, denote $a*b=ab$ and $a\lozenge b=\min\{1,a+b\}$ for all $a,b\in[0,1]$ and let M_d and N_d be fuzzy sets on $X^2\times(0,\infty)$ defined as follows:

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}, N_d(x,y,t)=\frac{d(x,y)}{t+d(x,y)},$$

then the space ($X,M_d,N_d,*,\lozenge$) is an intuitionistic fuzzy metric space by the definition of intuitionitstic fuzzy metric spaces, and call this intuitionistic fuzzy metric (M_d,N_d) induced by a metric d the standard intuitionistic fuzzy metric [2].

Definition5[2]: Let ($X,M,N,*,\lozenge$) be an intuitionistic fuzzy metric space, define $\tau_{(M,N)}=\{A\subset X:\text{ for each } x\in A, \text{there exist } t>0 \text{ and } r\in(0,1) \text{such that } B(x,y,t)\subset A\}$.

Then $\tau_{(M,N)}$ is a topology on X [2].

Defintion6: Let (X,τ) be a topological space, if there is an intuitionistic fuzzy metric (M,N) on X such that $\tau=\tau_{(M,N)}$, then we call the topological space (X,τ) admits a compatible intuitionistic fuzzy metric.

Definition7: A intuitionistic fuzzy metric space ($X,M,N,*,\lozenge$) is called precompact if for each r,t with $0<r<1$ and $t>0$, there is a finite subset A of X such that $X=\bigcup_{a\in A}B(a,r,t)$, in this case, we say that (M,N) is a precompact intuitionistic fuzzy metric on X.

3. Some properties of intuitionistic fuzzy metric spaces

Lemma1 [2]: In intuitionistic fuzzy metric space X, $M(x,y,\cdot)$ is non-decreasing and $N(x,y,\cdot)$ is non-increasing for all $x,y \in X$.

Lemma2 [4]: A T_1 topological space (X,τ) is metrizable if and only if it admits a compatible uniformity with a countable base.

Lemma3 [2]. Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space. Then $\tau_{(M,N)}$ is a Hausdorff topology and for each $x \in X, \left\{B\left(x,\frac{1}{n},\frac{1}{n}\right): n \in N\right\}$ is a neighborhood base at x for the topology $\tau_{(M,N)}$.

Theorem1: Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space. Then $\left(X,\tau_{(M,N)}\right)$ is a metrizable topological space.

Proof: For each $n \in N$ define

$$U_n = \left\{(x,y) \in X \times X : M\left(x,y,\frac{1}{n}\right) > 1-\frac{1}{n} \text{ and } N\left(x,y,\frac{1}{n}\right) < \frac{1}{n}\right\}.$$

Because for each $n \in N$,

$$\{(x,x) \in X \times X\} \subseteq U_n, U_{n+1} \subseteq U_n \text{ and } U_n = U_n^{-1},$$

and by the continuity of $*$ and $\Diamond$, there is an $m \in N$ such that

$$m > 2n \text{ and } \left(1-\frac{1}{m}\right) * \left(1-\frac{1}{m}\right) > \left(1-\frac{1}{n}\right)$$

$$m > 2n \text{ and } \frac{1}{m} \Diamond \frac{1}{m} < \frac{1}{n}.$$

By the non-decreasing of $M(x,y,\cdot)$ and non-increasing of $N(x,y,\cdot)$, we have

$$M\left(x,z,\frac{1}{n}\right) \geq M\left(x,z,\frac{2}{m}\right) \text{ and } N\left(x,z,\frac{1}{n}\right) \leq N\left(x,z,\frac{2}{m}\right),$$

where $(x,y) \in U_m$ and $(y,z) \in U_m$. So

$$M\left(x,z,\frac{1}{n}\right) \geq M\left(x,y,\frac{1}{m}\right) * M\left(y,z,\frac{1}{m}\right) \geq \left(1-\frac{1}{m}\right) * \left(1-\frac{1}{m}\right) > 1-\frac{1}{n}$$

$$N\left(x,z,\frac{1}{n}\right)\leq N\left(x,y,\frac{1}{m}\right)\Diamond N\left(y,z,\frac{1}{m}\right)\leq\frac{1}{m}\Diamond\frac{1}{m}<\frac{1}{n}$$

therefore $(x,z)\in U_n$, So $\{U_n : n\in N\}$ is a base for a uniformity □ on X. Since for each $x\in X$ and each $n\in N$

$$U_n(x)=\left\{y\in X : M\left(x,y,\frac{1}{n}\right)>1-\frac{1}{n}\text{ and }N\left(x,y,\frac{1}{n}\right)<\frac{1}{n}\right\}=B\left(x,\frac{1}{n},\frac{1}{n}\right).$$

We induce, from lemma2, that the topology induced by u coincides with $\tau_{(M,N)}$. By lemma2, $(X,\tau_{(M,N)})$ is a metrizable topological space.

Theorem2: Let $(X,M,N,*,\Diamond)$ be a complete intuitionistic fuzzy metric space, then $(X,\tau_{(M,N)})$ is completely metrizable.

Proof: It follows from the proof theorem1 that $\{U_n : n\in N\}$ is a base for a uniformity u on X compatible with $\tau_{(M,N)}$, when

$$U_n=\left\{(x,y)\in X\times X : M\left(x,y,\frac{1}{n}\right)>1-\frac{1}{n}\text{ and }N\left(x,y,\frac{1}{n}\right)<\frac{1}{n}\right\}\text{ for every } n\in N.$$

Then there exists a metric d on X whose induced continuity coincides with u, we want to show that the metric d is complete.

Given a Cauchy sequence $(x_n)_{n\in N}$ in (X,d), fix r,t with $0<r<1$ and $t>0$, choose a $k\in N$ such that $\frac{1}{k}\leq\min\{t,r\}$, then there is $n_0\in N$ such that

$$(x_n,x_m)\in U_k\text{ for every } m,n\geq n_0.$$

Sequently, for each $m,n\geq n_0$,

$$M(x_n,x_m,t)\geq M\left(x_n,x_m,\frac{1}{k}\right)>1-\frac{1}{k}\geq 1-r$$

$$N(x_n,x_m,t)\leq N\left(x_n,x_m,\frac{1}{k}\right)<\frac{1}{k}\leq r.$$

So $(x_n)_{n\in N}$ is a Cauchy sequence in the complete fuzzy metric space $(X,M,N,*,\Diamond)$, and it is convergent with respect to $\tau_{(M,N)}$. Hence d is a complete metric on X, we conclude that $\left(X,\tau_{(M,N)}\right)$ is completely metrizable.

Theorem3: A intuitionistic fuzzy metric space is precompact if and only if every sequence has a Cauchy subsequence.

Proof: Suppose that $(X,M,N,*,\Diamond)$ is a precompact fuzzy metric space. Let $(x_n)_{n\in N}$ be a sequence in X. For each $m\in N$, there is a finite subset A_m of X such that

$$X=\bigcup_{a\in A_m} B\left(a,\frac{1}{m},\frac{1}{m}\right).$$

Hence for $m=1$, there exists an $a_1\in A_1$ and a subsequence $\left(x_{1(n)}\right)_{n\in N}$ of $(x_n)_{n\in N}$ such that $x_{1(n)}\in B(a_1,1,1)$ for every $n\in N$. Similarly, there exist an $a_2\in A_2$ and a subsequence $\left(x_{2(n)}\right)_{n\in N}$ of $\left(x_{1(n)}\right)_{n\in N}$ such that $x_{2(n)}\in B\left(a_2,\frac{1}{2},\frac{1}{2}\right)$ for every $n\in N$. Following this process, for $m\in N, m>1$, there is an $a_m\in A_m$ and a subsequence $\left(x_{m(n)}\right)_{n\in N}$ of $\left(x_{(m-1)(n)}\right)_{n\in N}$ such that $x_{m(n)}\in B\left(a_m,\frac{1}{m},\frac{1}{m}\right)$ for every $n\in N$. Now, consider the subsequence $\left(x_{n(n)}\right)_{n\in N}$ of $(x_n)_{n\in N}$. Given r,t with $0<r<1$ and $t>0$, there is an $n_0\in N$ such that

$$\left(1-\frac{1}{n_0}\right)*\left(1-\frac{1}{n_0}\right)>1-r \text{ and } \frac{2}{n_0}<t$$

$$\frac{1}{n_0}\Diamond\frac{1}{n_0}<r \text{ and } \frac{2}{n_0}<t.$$

Then, for every $k,m\geq n_0$, we have

$$\begin{aligned}M\left(x_{k(k)},x_{m(m)},t\right)&\geq M\left(x_{k(k)},x_{m(m)},\frac{2}{n_0}\right)\\&\geq M\left(x_{k(k)},a_{n_0},\frac{1}{n_0}\right)*M\left(a_{n_0},x_{m(m)},\frac{1}{n_0}\right)\\&\geq\left(1-\frac{1}{n_0}\right)*\left(1-\frac{1}{n_0}\right)>1-r\end{aligned}$$

$$N\left(x_{k(k)},x_{m(m)},t\right)\leq N\left(x_{k(k)},x_{m(m)},\frac{2}{n_0}\right)$$
$$\leq M\left(x_{k(k)},a_{n_0},\frac{1}{n_0}\right)\lozenge M\left(a_{n_0},x_{m(m)},\frac{1}{n_0}\right)$$
$$\leq\frac{1}{n_0}\lozenge\frac{1}{n_0}<r$$

Hence $\left(x_{n(n)}\right)_{n\in N}$ is a Cauchy sequence in $(X,M,N,*,\lozenge)$.

Conversely, suppose that $(X,M,N,*,\lozenge)$ is a nonprecompact intuitionistic fuzzy metric space. Then there exist r,t with $0<r<1$ and $t>0$ such that for each finite subset A of X, $X\neq\bigcup_{a\in A}B(a,r,t)$. Fix $x_1\in X$, there is $x_2\in X\setminus B(x_1,r,t)$. Moreover, there is $x_3\in X\setminus\left(\bigcup_{k=1}^{2}B(x_k,r,t)\right)$, following this process, we construct a sequence $(x_n)_{n\in N}$ of distinct points in X such that $x_{n+1}\notin\left(\bigcup_{k=1}^{n}B(x_k,r,t)\right)$ for every $n\in N$. Therefore $(x_n)_{n\in N}$ has no Cauchy subsequence. This complete the proof.

Theorem4: A intuitionistic fuzzy metric space $(X,M,N,*,\lozenge)$ is separable if and only if $\left(X,\tau_{(M,N)}\right)$ admits a compatible precompact intuitionistic fuzzy metric.

Proof: Suppose that $(X,M,N,*,\lozenge)$ is a separable intuitionistic fuzzy metric space. By theorem1 and $\left(X,\tau_{(M,N)}\right)$ is a separable metrizable space, so it admits a compatible precompact metric d. We shall show that the standard intuitionistic fuzzy metric (M_d,N_d) induced by d is precompact:

Let $(x_n)_{n\in N}$ be a sequence in X. By precompactness of d, $(x_n)_{n\in N}$ has a Cauchy subsequence $\left(x_{k(n)}\right)_{n\in N}$ in (X,d). Given r,t with $0<r<1$ and $t>0$, put $\varepsilon=\frac{t}{1-r}-t$, then there is $n_0\in N$ such that $d\left(x_{k(n)},x_{k(m)}\right)<\varepsilon$ for every $m,n\geq n_0$. Therefore

$$M_d\left(x_{k(n)},x_{k(m)},t\right)>\frac{t}{t+\varepsilon}=1-r$$
$$N_d\left(x_{k(n)},x_{k(m)},t\right)<\frac{\varepsilon}{t+\varepsilon}=r$$
for every $m,n\geq n_0$.

So $(x_{k(n)})_{n\in N}$ is a Cauchy sequence in the intuitionistic fuzzy metric space $(X, M_d, N_d, *, \Diamond)$, by theorem3 $(X, M_d, N_d, *, \Diamond)$ is precompact.

Conversely, suppose that $(X, \tau_{(M,N)})$ admits a compatible precompact intuitionistic fuzzy metric (P,Q). Then for each $n \in N$, there is a finite subset A_n of X such that $X = \bigcup_{a\in A_n} B\left(a, \frac{1}{n}, \frac{1}{n}\right)$. Put $A = \bigcup_{n=1}^{\infty} A_n$, let $x \in X$ and $B\left(x, \frac{1}{m}, \frac{1}{m}\right)$ be a basic neighborhood of x, then there exists $a \in A_m$ such that $x \in B\left(a, \frac{1}{m}, \frac{1}{m}\right)$. Thus A is dense in X, we conclude that $(X, P, Q, *, \Diamond)$ is separable, i.e. $(X, \tau_{(M,N)})$ is a separable topological space.

Theorem5: Let $(X, M, N, *, \Diamond)$ be a intuitionistic fuzzy metric space. If a Cauchy sequence clusters to a point $x \in X$, then the sequence converges to x.

Proof: Let $(x_n)_{n\in N}$ be a Cauchy sequence in $(X, M, N, *, \Diamond)$ having a cluster point $x \in X$. Then there is a subsequence $(x_{k(n)})_{n\in N}$ of $(x_n)_{n\in N}$ that converges to x with respect to $\tau_{(M,N)}$. Thus given r,t with $0 < r < 1$ and $t > 0$, there is an $n_0 \in N$ such that for each $n \ge n_0$,

$$\begin{matrix} M\left(x, x_{k(n)}, \frac{t}{2}\right) > 1 - s \\ N\left(x, x_{k(n)}, \frac{t}{2}\right) < s \end{matrix} \quad \text{where } s > 0 \text{ satisfies } \begin{matrix} (1-s)*(1-s) > 1-r \\ s\Diamond s < r \end{matrix}.$$

On the other hand, there is $n_1 \ge k(n_0)$ such that for each $m, n \ge n_1$,

$$M\left(x_n, x_m, \frac{t}{2}\right) > 1 - s \text{ and } N\left(x_n, x_m, \frac{t}{2}\right) < s.$$

Therefore for each $n \ge n_1$, we have

$$\begin{aligned} M(x, x_n, t) &\ge M\left(x, x_{k(n)}, \frac{t}{2}\right) * M\left(x_{k(n)}, x_n, \frac{t}{2}\right) \\ &\ge (1\text{-}s) * (1\text{-}s) > 1 - r \end{aligned}$$

$$N(x,x_n,t) \le N\left(x,x_{k(n)},\frac{t}{2}\right) \Diamond N\left(x_{k(n)},x_n,\frac{t}{2}\right)$$
$$\le s \Diamond s < r$$

we conclude that the Cauchy sequence $(x_n)_{n\in N}$ converges to x.

Theorem6: A intuitionistic fuzzy metric space is compact if and only if it is precompact and complete.

Proof: Suppose that $(X,M,N,*,\Diamond)$ is a compact intuitionistic fuzzy metric space. For each r,t with $0<r<1$ and $t>0$, the open cover $\{B(x,r,t):x\in X\}$ has a finite subcover. Hence $(X,M,N,*,\Diamond)$ is precompact. On the other hand, every Cauchy sequence $(x_n)_{n\in N}$ in $(X,M,N,*,\Diamond)$ has a cluster point $y\in X$. By theorem5 $(x_n)_{n\in N}$ converges to y, thus $(X,M,N,*,\Diamond)$ is complete.

Conversely, let $(x_n)_{n\in N}$ be a sequence in X, from theorem3 and the completeness of $(X,M,N,*,\Diamond)$, it follows that $(x_n)_{n\in N}$ has a cluster point. Since, by theorem1, $(X,\tau_{(M,N)})$ is metrizable and every sequentially compact metrizable space is compact, we conclude that $(X,M,N,*,\Diamond)$ is compact.

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