Fifth Int. Conf. on IFSs, Sofia, 22-23 Sept. 2001
NIFS 7 (2001), 3, 43-48

Deductive Properties of Intuitionistic Fuzzy Logic

Nikolai G. Nikolov
CLBME-Bulgarian Academy of Sciences
Bl. 105, Acad. G. Bontchev Str., Sofia 1113, Bulgaria
e-mail: shte@iinf.bas.bg

Abstract. A formalisation is proposed of the notions of logical consequence and deductive closure in case of intuitionistic fuzzy logic. A relationship is shown between deductively closed sets and intuitionistic fuzzy valuations that respect Modus Ponens.
Keywords: intuitionistic fuzzy sets, deductive closure

1 Intuitionistic fuzzy logic

Below we present the necessary notions from Intuitionistic fuzzy logic (IFL); for a more detailed description the reader is referred to [1].

To each proposition (in the classical sense) we can assign its truth value: truth - denoted by 1, or falsity - 0. In the case of fuzzy logic this truth value is a real number in the interval $[0,1]$ and may be called "truth degree" of a particular proposition. Here we add one more value - "falsity degree" - which will be in the interval $[0,1]$ as well. Thus two real numbers, $\mu(p)$ and $\nu(p)$, are assigned to the proposition p with the following constraint to hold:

$$\mu(p) + \nu(p) \leq 1.$$

Let this assignment be provided by a valuation function V defined over a set of propositions S in such a way that:

$$V(p) = \langle \mu(p), \nu(p) \rangle.$$

Hence the function $V : S \rightarrow [0,1] \times [0,1]$ gives the truth and falsity degrees of all propositions in S.

We assume that the valuation function V assigns to the logical truth T

$$V(T) = \langle 1, 0 \rangle,$$

and to the logical falsity F

$$V(F) = \langle 0, 1 \rangle.$$

Below we will focus on the truth and falsity degrees of propositions obtained as a result of applying logical operations (unary and binary) over atomic propositions whose truth and falsity values are set by a given valuation function.

The evaluation of the negation $\neg p$ of a proposition p is defined as:

$$V(\neg p) = \langle \nu(p), \mu(p) \rangle.$$

When

$$\nu(p) = 1 - \mu(p),$$

i.e.

$$V(p) = \langle \mu(p), 1 - \mu(p) \rangle,$$

for $\neg p$ we get:

$$V(\neg p) = \langle 1 - \mu(p), \mu(p) \rangle,$$

and so reduces to the case of ordinary fuzzy logic.

When the values $V(p)$ and $V(q)$ of the propositions p and q are known, the valuation function V can be further extended to cover conjunction and disjunction as follows:

$$V(p\&q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle,$$
$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle.$$

Depending on the way of defining the implication operation, different variants of intuitionistic fuzzy propositional calculus can be obtained.

There are over 10 different versions of implication defined for fuzzy sets, but the attempts to transfer them on to IFL failed. The two most acceptable definitions are the following:
(a) sg–implication:

$$\begin{aligned} V(p \supset q) &= \langle 1 - (1 - \mu(q)).sg(\mu(p) - \mu(q)), \\ &\quad \nu(q).sg(\mu(p) - \mu(q)).sg(\nu(q) - \nu(p)) \rangle, \end{aligned}$$

where:

$$sg(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

and **(b)** $(max - min)$–implication:

$$V(p \supset q) = \langle \max(\nu(p), \mu(q)), \min(\mu(p), \nu(q)) \rangle.$$

The $(max - min)$–implication has the advantage that

$$V(p \supset q) = V(\neg p \vee q) = V(\neg(p\&\neg q))$$

and it is this implication that we will use throughout this paper.

However, a major drawback of it is that there are valuations for which Modus Ponens does not hold.

A notion that we will also need is that of tautology, a generalisation - in the IF case - of formulae valid in every interpretation.

Definition. A formula A will be called a *standard tautology* if $V(A) = \langle 1, 0 \rangle$ for all valuation functions V.

Definition. A formula A will be called an *intuitionistic fuzzy tautology* if $V(A) = \langle a, b \rangle$ implies $a \geq b$ for all valuation functions V.

2 Inference and deductive closure

In this section we recall relevant concepts from propositional calculus and first-order logic. This section is based on [2].

For the sake of simplicity, below we work with a propositional language L.

Let S be a set of formulae over L. We wish to formally define what is it for a formula p to be inferred by the formulae in S.

A possible approach would be, provided we can make use of a proof theory, to iteratively build first the direct consequences of the formulae in S, then the set of the direct consequences of the so augmented set etc. Finally we could take for a deductive closure the fixpoint of the operation.

However, there is another (although non-constructive) approach which is independent of the particular inference rules of a proof theory.

To this end, we would like to be able to assign a new set of formulae $C(S)$ to S, to be called the *deductive closure* of S, meant to contain S along with all of its logical consequences.

It is argued in [2] that $C(S)$ should satisfy the conditions

1. $S \subseteq C(S)$
2. if $S_1 \subseteq S_2$, then $C(S_1) \subseteq C(S_2)$
3. $C(C(S) = C(S)$.

The following operation satisfies the above conditions and is accepted as a formal counterpart of the intuitive notion of logical consequence.

Definition. A formula p is called a *logical consequence* of a set of formulae S, $S \models p$, if for all valuations V such that $V(s)$ is true for all $s \in S$, $V(p)$ is also true.

Definition. A set of formulae T is called *deductively closed* if for all formulae p,

$$T \models p \text{ iff } p \in T.$$

Now, we are ready to define the notion of deductive closure.

Definition. Given a set of formulae S,

$$C(S) = \bigcap \{T | S \subseteq T \text{ and } T \text{ is deductively closed.}\}$$

As a particular case, we have that $C(\emptyset)$ is the set of all tautologies in L.

3 Intuitionistic fuzzy deductive closure

In an intuitionistic fuzzy setting, we will no longer speak of sets of formulae, but rather of intuitionistic fuzzy valuations (IFV). Each IFV can be regarded as an intuitionistic fuzzy set of formulae over an universe L.

The basic question if we want to generalise the C operation to the case of IFL is how to interpret the relation of logical consequence. We cannot generalise directly the crisp notion as now every formula is true and false to some degree under any valuation.

Instead of trying to generalize the $\models$ relation between formulae, consider the definition of a deductively closed set. What we want is to capture the intuition that whenever a formula p infers a formula q in any valuation, the valuation of q in an intuitionistic fuzzy deductively closed set should be no less true than that of p.

We formalize this in the following definition.

Definition. An IFV V is called *deductively closed* if for all formulae of the form $p \supset q$ for which $\mu_V p \supset q \geq \nu_V p \supset q$,

$$\mu_V(p) \leq \mu_V(q),$$

$$\nu_V(q) \leq \nu_V(p),$$

Proposition. The intersection of finitely or infinitely many deductively closed IFV is a deductively closed IFV.

Proof. The intersection of IFVs is their intersection taken as intuitionistic fuzzy sets, so a direct check of the corresponding definition suffices.

Definition. Given an IFV V, the following IFV $C(V)$ will be called intuitionistic fuzzy deductive closure (IFDC) of V:

$$C(V) = \bigcap \{T | S \subseteq T \text{ and } T \text{ is deductively closed}\},$$

where the intersection and set inclusions are intuitionistic fuzzy.

Proposition. Given an IFV V and its IFDC $C(V)$, the following hold:

1. $S \subseteq C(S)$
2. if $S_1 \subseteq S_2$, then $C(S_1) \subseteq C(S_2)$
3. $C(C(S) = C(S)$.

Proof. 1. $V \subseteq T$ for all T from the definition of $C(V)$, and therefore is included in their IF intersection.

2. Assume towards a contradiction that $S_1 \subseteq S_2$ but $C(S_1) \not\subseteq C(S_2)$. For the pointwise intuitionistic fuzzy set inclusion this would mean that there exists a formula p such that $\mu_{C(S_1)}(p) > \mu_{C(S_2)}(p)$ or $\nu_{C(S_1)}(p) < \nu_{C(S_2)}(p)$. Without loss of generality, consider the case when $\mu_{C(S_1)}(p) > \mu_{C(S_2)}(p)$.

Now

$$\mu_{C(S_1)}(p) = \min_{S_1 \subseteq T, T\text{-closed}} \mu_T(p) \leq \min_{S_2 \subseteq T, T\text{-closed}} \mu_T(p) = \mu_{C(S_2)}(p)$$

which proves that the IFDC operator is monotonic.

3. Direct check of the definition.

Proposition. IFDC is a generalisation of the notion of deductive closure from first-order logic.

Proof. Check the definitions against valuations yielding $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ only.

We shoud mention that the syntactic form of the IFDC operator bears a great resemblance to the form of the intuitionistic fuzzy operator of topological closure. While they are both defined as the infinite intersection of IFS having certain properties, this comes to demonstrate that there is a deeper relation between these notions.

4 Intuitionistic fuzzy deductive closure and Modus Ponens

Definition. An IFV V will be said to *respect Modus Ponens* if, for all formulae p, $p \supset q$, whenever

$$\mu_V(p) \geq \nu_V(p) \text{ and } \mu_V(p \supset q) \geq \nu_V(p \supset q),$$

then

$$\mu_V(q) \geq \nu_V(p).$$

If we use the notion of classical tautology in the above definition, then trivially all valuations respect Modus Ponens.

There is, however, an interesting property that links deductively closed intuitionistic fuzzy valuations with those that respect Modus Ponens, which holds even in the case of IFT, i. e., even when the used notion of tautology does not support Modus Ponens.

Proposition. Every deductively closed IF valuation respects Modus Ponens.

Proof. Let V be a deductively closed IF valuation, and p, $p \supset q$ be such that

$$\mu_V(p) \geq \nu_V(p) \text{ and } \mu_V(p \supset q) \geq \nu_V(p \supset q).$$

Then

$$\mu_V(p) \leq \mu_V(q),$$

and

$$\nu_V(q) \leq \nu_V(p).$$

So we have

$$\nu_V(q) \leq \nu_V(p) \leq \mu_V(p) \leq \mu_V(q),$$

thus $\mu_V(q) \geq \nu_V(q)$, which completes the proof.

5 Intuitionistic fuzzy logical consequence

Basing on the above work, below we define logical consequence of a formula from a set of formulae in the intuitionistic fuzzy case.

Definition. For an intuitionistic fuzzy set of formulae S and a formula p, we will say that S infers p with a degree of truth μ and a degree of falsity ν, in symbols, $S \models_{\mu,\nu} p$, if

$$\mu = \mu_{C(S)}(p),$$
$$\nu = \nu_{C(S)}(p).$$

As can be checked, this is a correct generalisation of the first-order notion of logical consequence.

6 Conclusion

A formalisation is proposed of the notion of deductive closure in case of intuitionistic fuzzy logic. A relationship is shown between deductively closed sets and intuitionistic fuzzy valuations that respect Modus Ponens. Using the deductive closure approach, a concept is proposed of intuitionistic fuzzy logical consequence. The proposed notion is a generalisation of first-order logic inference despite the fact that intuitionistic fuzzy tautologies do not comply with Modus Ponens.

References

[1] Atanassov, K., Intuitionistic Fuzzy Sets, Springer-Verlag, 1999.

[2] Rasiowa, H., and R. Sikorski, The Mathematics of Metamathematics, Moscow, Nauka, 1972 (in Russian).

[3] Ginsberg, M., Bilattices and Modal Operators, Journal of Logic and Computation, 1(1):41-69, July 1990.