Notes on Intuitionistic Fuzzy Sets
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Generalized (T, S_T) intuitionistic fuzzy anti-ideals on semigroups

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Abstract: In this paper, we introduce generalized intuitionistic fuzzy (T, S_T) anti-ideals on semigroups. We first define intuitionistic fuzzy left anti-ideals, right anti-ideals, anti-ideals, and bi-anti-ideals in this setting. Then we study their basic properties and show how classical anti-ideals can be obtained via characteristic functions. We also give characterizations in terms of level subsets and examine the behavior of these structures under homomorphisms and isomorphisms. The results obtained show that the proposed approach extends known anti-ideal concepts and gives a useful framework for studying semigroups with uncertainty.
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1 Introduction

Semigroup theory, which studies nonempty sets equipped with an associative binary operation, occupies an important place in abstract algebra. The structure of a semigroup is often studied

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through its special subsets, such as left ideals, right ideals, bi-ideals, and their generalizations. The connections between semigroups and formal language theory, automata, combinatorics, and coding theory give this area both theoretical and practical importance [16].

The concept of fuzzy sets, introduced by Zadeh [19] in 1965, provided a mathematical framework for dealing with situations in which classical set theory based on strict membership is insufficient. Soon after, Rosenfeld [17] applied fuzzy set theory to groups, initiating the study of fuzzy algebraic structures. Since then, many semigroup-theoretic notions such as fuzzy ideals, fuzzy bi-ideals, fuzzy generalized bi-ideals, and their ordered or hyperstructured versions have been widely investigated [11, 12, 15].

Another important direction of research concerns anti-ideals. The notion of an anti-ideal in a semigroup was introduced by Schwarz [18] in 1953 in the study of maximal ideal theory. These subsets, which are disjoint from their own left or right translates, were shown to have important structural properties. Later, Iseki [8] generalized this concept by introducing (m, n)-anti-ideals, where the disjointness condition is defined using powers of the underlying set. These structures can be viewed as dual objects to ideals and naturally arise in the study of complements of ideals in semigroups.

More recently, anti-ideal theory has been extended to the fuzzy setting. Al Tahan *et al.* [4] introduced fuzzy left (right) anti-ideals and fuzzy bi-anti-ideals of semigroups, studied their basic properties, and connected them with classical anti-ideals through level sets. Later, Al Tahan and Hoskova-Mayerova [2] investigated interior anti-ideals and their fuzzy counterparts, proving that regular and intra-regular semigroups do not admit such structures.

The intuitionistic fuzzy framework introduced by Atanassov [5] provides a richer setting by assigning to each element both a membership degree and a non-membership degree. This allows uncertainty and hesitation to be modeled more effectively than ordinary fuzzy sets. Ahn *et al.* [1] initiated the study of intuitionistic fuzzy ideals in semigroups, and later studies extended these ideas to interior ideals, bi-ideals, and several generalized ideal structures [13]. However, anti-ideals within the intuitionistic fuzzy framework based on a t-norm and its dual t-conorm have not yet been studied systematically.

In this paper, we introduce generalized intuitionistic fuzzy (T, S_T) anti-ideals of semigroups. This notion generalizes the classical anti-ideal introduced by Schwarz [18], and the fuzzy anti-ideal framework studied in [4]. In particular, ordinary fuzzy anti-ideals appear as a special case of our approach. We establish several basic properties of these structures, characterize certain classes of semigroups using them, and study their behavior under semigroup homomorphisms.

2 Preliminaries

Throughout this paper, $(L, \leq, 0, 1)$ denotes a complete lattice with the bottom element 0 and the top element 1. For $u, v \in L$ and every family $\{b_\alpha \mid \alpha \in \Lambda\}$, we use the following notation such as

$$u \vee v = \sup\{u, v\}, \quad u \wedge v = \inf\{u, v\},$$
$$\bigvee_{\alpha\in\Lambda} b_\alpha = \sup\{b_\alpha \mid \alpha \in \Lambda\}, \quad \bigwedge_{\alpha\in\Lambda} b_\alpha = \inf\{b_\alpha \mid \alpha \in \Lambda\}.$$

Definition 1. [7] *Let L be a complete lattice. Then L is called an infinitely $\vee$-distributive lattice if*

$$a \wedge (\bigvee_{\alpha\in\Lambda} b_\alpha) = \bigvee_{\alpha\in\Lambda}(a \wedge b_\alpha)$$

for every $a \in L$ and every family $\{b_\alpha \mid \alpha \in \Lambda\} \subseteq L$.

Unless otherwise stated, L is a complete infinitely $\vee$-distributive lattice. The closed interval $[0, 1]$ equipped with the usual order "$\leq$" , is a complete infinitely $\vee$-distributive lattice.

Definition 2. [14] *A triangular norm, or t-norm, on L is a binary operation that is monotone, commutative, and associative and has 1 as a neutral element.*

Basic examples of t-norms on L are T_M, T_D, are defined by

$$T_M(u, v) = u \wedge v,$$

$$T_D(u, v) = \begin{cases} v \wedge u, & \text{if } u = 1 \text{ or } v = 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 3. [14] *Let T be a t-norm on L. Then T is called infinitely $\vee$-distributive if*

$$aT(\bigvee_{i\in I} b_\alpha) = \bigvee_{i\in I}(aTb_i),$$

for every $a \in L$ and every family $\{b_\alpha \mid \alpha \in \Lambda\} \subseteq L$.

Proposition 1. [14] *Let T be any t-norm on L. Then the following hold:*

(i) $uTv \leq u \wedge v$ *for all* $u, v \in L$,

(ii) $uT0 = 0Tu = 0$ *for all* $u \in L$,

(iii) $(\bigwedge_{\alpha\in\Lambda} a_\alpha)T(\bigwedge_{\beta\in\mho} b_\beta) \leq \bigwedge_{\alpha\in\Lambda,\beta\in\mho} a_\alpha Tb_\beta$.

Definition 4. [14] *A binary operation*

$$S : [0, 1] \times [0, 1] \to [0, 1]$$

is said to be a t-conorm *provided that it is commutative, associative, and monotone, with the boundary condition*

$$S(u, 0) = u$$

for every $u \in [0, 1]$.

If T is a t-norm on $[0, 1]$, then clearly $S(u, v) = 1 - T(1 - u, 1 - v)$ is a t-conorm on $[0, 1]$. The t-conorm S is called the dual of the t-norm T and is denoted by $S = S_T$.
Typical examples of t-conorms are given as follows:

$$S_M(u, v) = \max\{u, v\},$$

$$S_L(u, v) = \min\{1, u + v\},$$

and the drastic t-conorm

$$S_D(u, v) = \begin{cases} u, & \text{if } v = 0, \\ v, & \text{if } u = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 5. [10, 14]*A t-conorm S is said to be* infinitely $\vee$-distributive *if, for every family $\{b_i\}_{i\in I} \subseteq [0,1]$ and every $a \in [0,1]$,*

$$S\left(a, \bigvee_{i\in I} b_i\right) = \bigvee_{i\in I} S(a, b_i).$$

Similarly, S is called infinitely $\wedge$-distributive *if*

$$S\left(a, \bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I} S(a, b_i).$$

Theorem 1. [14] *Let S be a t-conorm on lattice L. Then*

$$(\bigvee_{\alpha\in\Lambda} a_\alpha) S (\bigvee_{\beta\in\mho} b_\beta) \geq \bigvee_{\alpha\in\Lambda,\beta\in\mho} a_\alpha S b_\beta.$$

A semigroup is a nonempty set S equipped with an associative binary operation. Unless otherwise stated, S denotes a semigroup throughout this paper. The *product* of two subsets A and B of a semigroup S is defined as

$$AB = \{\, ab \mid a \in A,\ b \in B \,\}.$$

Definition 6. [3, 4, 8] *Let A be any nonempty subset of S. Then*

(i) *A is called* a left anti-ideal of *S if $SA \cap A = \varnothing$,*

(ii) *A is called* a right anti-ideal of *S if $AS \cap A = \varnothing$,*

(iii) *A is called* an anti-ideal of *S if it is both* a left anti-ideal and a right anti-ideal of *S,*

(iv) *A is called* a bi-anti-ideal of *S if $ASA \cap A = \varnothing$,*

The sets of all left anti-ideals, right anti-ideals, anti-ideals, and bi-anti-ideals of S are denoted by $LAI(S)$, $RAI(S)$, $AI(S)$, and $BAI(S)$, respectively.

Definition 7. [19] *Let X be a nonempty set. A fuzzy set $\mathcal{P}$ on X is characterized by a membership function $\alpha_{\mathcal{P}} : X \to [0,1]$ and can be represented as*

$$\mathcal{P} = \{(p, \alpha_{\mathcal{P}}(p)) : p \in X\}$$

Definition 8. [5] *Let X be a nonempty set. An intuitionistic fuzzy set $\mathcal{P}$ on X is defined by*

$$\mathcal{P} = \{(p, \alpha_{\mathcal{P}}(p), \beta_{\mathcal{P}}(p)) : p \in X\},$$

where $\alpha_{\mathcal{P}}, \beta_{\mathcal{P}} : X \to [0,1]$ are mappings known as the membership function and non-membership function, respectively, and

$$0 \leq \alpha_{\mathcal{P}}(p) + \beta_{\mathcal{P}}(p) \leq 1$$

for all $p \in X$.

The hesitation function of $\mathcal{P}$ is defined by

$$\pi_{\mathcal{P}}(p) = 1 - \alpha_{\mathcal{P}}(p) - \beta_{\mathcal{P}}(p), \qquad p \in X.$$

The value $\pi_{\mathcal{P}}(p)$ represents the degree of hesitation associated with p.

The family of all intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

Definition 9. [5] *Let A be a nonempty subset of S. The intuitionistic fuzzy characteristic function of A, denoted by C^A, is defined by $C^A = (\alpha_{C^A}, \beta_{C^A})$, where*

$$\alpha_{C^A}(p) = \begin{cases} 1, & p \in A, \\ 0, & p \notin A, \end{cases} \qquad \beta_{C^A}(p) = \begin{cases} 0, & p \in A, \\ 1, & p \notin A. \end{cases}$$

Definition 10. [5] *Let X be a nonempty set and $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P})$, $\mathcal{R} = (\alpha_\mathcal{R}, \beta_\mathcal{R})$ be intuitionistic fuzzy sets on X. Then, for $\mathcal{P}$ and $\mathcal{R}$, some relations and operations can be defined as follows:*

(i) $\mathcal{P} \subseteq \mathcal{R}$ *if and only if, for all* $p \in X$, $\alpha_\mathcal{P}(p) \leq \alpha_\mathcal{R}(p)$ *and* $\beta_\mathcal{P}(p) \geq \beta_\mathcal{R}(p)$.

(ii) $\mathcal{P} \cup \mathcal{R} = \{\langle p, ((\alpha_\mathcal{P} \cup \alpha_\mathcal{R})(p), (\beta_\mathcal{P} \cap \beta_\mathcal{R})(p))\rangle \mid p \in X\}$, *where* $(\alpha_\mathcal{P} \cup \alpha_\mathcal{R})(p) = \alpha_\mathcal{P}(p) \vee \alpha_\mathcal{R}(p)$ *and* $(\beta_\mathcal{P} \cap \beta_\mathcal{R})(p) = \beta_\mathcal{P}(p) \wedge \beta_\mathcal{R}(p)$.

(iii) $\mathcal{P} \cap \mathcal{R} = \{\langle p, ((\alpha_\mathcal{P} \cap \alpha_\mathcal{R})(p), (\beta_\mathcal{P} \cup \beta_\mathcal{R})(p))\rangle \mid p \in X\}$, *where* $(\alpha_\mathcal{P} \cap \alpha_\mathcal{R})(p) = \alpha_\mathcal{P}(p) \wedge \alpha_\mathcal{R}(p)$ *and* $(\beta_\mathcal{P} \cup \beta_\mathcal{R})(p) = \beta_\mathcal{P}(p) \vee \beta_\mathcal{R}(p)$.

(iv) $\mathcal{P}^C(p) = (\beta_\mathcal{P}(p), \alpha_\mathcal{P}(p))$, *for all* $p \in X$.

(v) $(\Box\mathcal{P})(p) = (\alpha_\mathcal{P}(p), 1 - \alpha_\mathcal{P}(p)) = (\alpha_\mathcal{P}(p), \overline{\alpha}_\mathcal{P}(p))$, *for all* $p \in X$.

(vi) $(\diamond\mathcal{P})(p) = (1 - \beta_\mathcal{P}(p), \beta_\mathcal{P}(p)) = (\overline{\beta}_\mathcal{P}(p), \beta_\mathcal{P}(p))$, *for all* $p \in X$.

Definition 11. [5] *Let $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P})$ be an intuitionistic fuzzy set on X and let $\lambda \in [0, 1]$. Then the sets*

$$U(\alpha_\mathcal{P}, \lambda) = \{p \in X : \alpha_\mathcal{P}(p) \geq \lambda\},$$
$$L(\beta_\mathcal{P}, \lambda) = \{p \in X : \beta_\mathcal{P}(p) \leq \lambda\}$$

are called an α-level λ-cut and a β-level λ-cut of $\mathcal{P}$, respectively.

Definition 12. [5] $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P}) \in IFS(X)$ *and* $t, s \in [0, 1]$. *The (t, s)-level set of P is*

$$A_{(t,s)} = \{p \in X : \alpha_\mathcal{P}(p) \geq t \text{ and } \beta_\mathcal{P}(p) \leq s\} = U(\alpha_\mathcal{P}, t) \cap L(\beta_\mathcal{P}, s).$$

Definition 13. [9] *For a t-norm T and a t-conorm S, we use the symbols Δ_T and Δ_S as the sets:*

$$\Delta_T = \{a \in [0, 1] | T(a, a) = a\},$$
$$\Delta_S = \{a \in [0, 1] | S(a, a) = a\},$$

respectively.

Definition 14. [9] *We say that the intuitionistic fuzzy set $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P})$ on X satisfies the imaginable property if*

$$\Im(\alpha_\mathcal{P}) \subseteq \Delta_T \text{ and } \Im(\beta_\mathcal{P}) \subseteq \Delta_S.$$

Definition 15. [9] *Let X and Y be nonempty sets, and let $f : X \to Y$ be a mapping. Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ be an intuitionistic fuzzy set on X.*

The image *of $\mathcal{P}$ under f, denoted by $f(\mathcal{P}) = (\alpha_{f(\mathcal{P})}, \beta_{f(\mathcal{P})})$, is an intuitionistic fuzzy set on Y defined by*

$$\alpha_{f(\mathcal{P})}(y) = \begin{cases} \bigvee\{\alpha_{\mathcal{P}}(x) \mid x \in X,\ f(x) = y\}, & \text{if } f^{-1}(y) \neq \varnothing, \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_{f(\mathcal{P})}(y) = \begin{cases} \bigwedge\{\beta_{\mathcal{P}}(x) \mid x \in X,\ f(x) = y\}, & \text{if } f^{-1}(y) \neq \varnothing, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 16. *Let X and Y be nonempty sets, and let $f : X \to Y$ be a mapping. Let $\mathcal{R} = (\alpha_{\mathcal{R}}, \beta_{\mathcal{R}})$ be an intuitionistic fuzzy set on Y.*

The inverse image *of $\mathcal{R}$ under f, denoted by $f^{-1}(\mathcal{R}) = (\alpha_{f^{-1}(\mathcal{R})}, \beta_{f^{-1}(\mathcal{R})})$, is an intuitionistic fuzzy set on X defined by*

$$\alpha_{f^{-1}(\mathcal{R})}(x) = \alpha_{\mathcal{R}}(f(x)),$$
$$\beta_{f^{-1}(\mathcal{R})}(x) = \beta_{\mathcal{R}}(f(x)),$$

for all $x \in X$.

3 Intuitionistic fuzzy (T, S_T) anti-ideals

In this section, we introduce the notions of intuitionistic fuzzy (T, S_T) left (right) anti-ideals and explore their fundamental algebraic properties. In addition, the connections between intuitionistic fuzzy anti-ideals and their corresponding characteristic functions and level subsets are analyzed, and their relationships with classical anti-ideals are discussed in detail. Unless stated otherwise, S denotes a semigroup under the operation $'\cdot'$.

Definition 17. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}) \in IFS(S)$.*

(i) *$\mathcal{P}$ is called* an intuitionistic fuzzy (T, S_T) left anti-ideal *of S if*

$$\alpha_{\mathcal{P}}(p_2) T \alpha_{\mathcal{P}}(p_1 p_2) = 0,$$
$$\beta_{\mathcal{P}}(p_2) S_T \beta_{\mathcal{P}}(p_1 p_2) = 1,$$

(ii) *$\mathcal{P}$ is called* an intuitionistic fuzzy (T, S_T) right anti-ideal *of S if*

$$\alpha_{\mathcal{P}}(p_1) T \alpha_{\mathcal{P}}(p_1 p_2) = 0,$$
$$\beta_{\mathcal{P}}(p_1) S_T \beta_{\mathcal{P}}(p_1 p_2) = 1,$$

for all $p_1, p_2 \in S$.

An intuitionistic fuzzy set $\mathcal{P}$ on S is called an intuitionistic fuzzy (T, S_T) anti-ideal *if it is both an intuitionistic fuzzy (T, S_T) left anti-ideal of S and an intuitionistic fuzzy (T, S_T) right anti-ideal of S.*

All intuitionistic fuzzy (T, S_T) left anti-ideals, intuitionistic fuzzy (T, S_T) right anti-ideals, and intuitionistic fuzzy (T, S_T) anti-ideals of S are denoted by $IFSLAI(S)$, $IFSRAI(S)$, and $IFSAI(S)$, respectively.

Example 1. *Let $T = \wedge$ and $S_T = \vee$. Let $S = \{0, a, b\}$, with multiplication defined by*

$\cdot$	0	a	b
0	0	0	0
a	0	0	a
b	0	0	b

$\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ by

x	0	a	b
$\alpha_{\mathcal{P}}(x)$	0	$\frac{1}{2}$	0
$\beta_{\mathcal{P}}(x)$	1	$\frac{1}{4}$	1

For $p_1, p_2 \in S$, if $p_2 \in \{0, b\}$, then $\alpha_{\mathcal{P}}(p_2) = 0$ and $\beta_{\mathcal{P}}(p_2) = 1$. If $p_2 = a$, then $p_1 a = 0$ for all $p_1 \in S$. Thus

$$\alpha_{\mathcal{P}}(p_2) T \alpha_{\mathcal{P}}(p_1 p_2) = 0, \qquad \beta_{\mathcal{P}}(p_2) S_T \beta_{\mathcal{P}}(p_1 p_2) = 1.$$

Therefore, $\mathcal{P}$ is an intuitionistic fuzzy (T, S_T) left anti-ideal of S.

However, $\mathcal{P}$ is not an intuitionistic fuzzy (T, S_T) right anti-ideal. Indeed, for $p_1 = a$ and $p_2 = b$, we have $ab = a$, and so

$$\alpha_{\mathcal{P}}(a) T \alpha_{\mathcal{P}}(ab) = \frac{1}{2} \neq 0, \qquad \beta_{\mathcal{P}}(a) S_T \beta_{\mathcal{P}}(ab) = \frac{1}{4} \neq 1.$$

Example 2. *Let $S = \{0, a, b\}$ and define a multiplication on S by*

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	a	b

This operation is associative, so S is a semigroup.

Define $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ by the same functions:

x	0	a	b
$\alpha_{\mathcal{P}}(x)$	0	$\frac{1}{2}$	0
$\beta_{\mathcal{P}}(x)$	1	$\frac{1}{4}$	1

Again, $\mathcal{P} \in IFS(S)$.

For $p_1, p_2 \in S$, if $p_1 \in \{0, b\}$, then $\alpha_{\mathcal{P}}(p_1) = 0$ and $\beta_{\mathcal{P}}(p_1) = 1$. If $p_1 = a$, then $a p_2 = 0$ for all $p_2 \in S$. Thus

$$\alpha_{\mathcal{P}}(p_1) T \alpha_{\mathcal{P}}(p_1 p_2) = 0, \qquad \beta_{\mathcal{P}}(p_1) S_T \beta_{\mathcal{P}}(p_1 p_2) = 1.$$

Therefore, $\mathcal{P}$ is an intuitionistic fuzzy (T, S_T) right anti-ideal of S.

However, $\mathcal{P}$ is not an intuitionistic fuzzy (T, S_T) left anti-ideal. Indeed, for $p_1 = b$ and $p_2 = a$, we have $ba = a$, and so

$$\alpha_{\mathcal{P}}(a)T\alpha_{\mathcal{P}}(ba) = \frac{1}{2} \neq 0, \qquad \beta_{\mathcal{P}}(a)S_T\beta_{\mathcal{P}}(ba) = \frac{1}{4} \neq 1.$$

Classical anti-ideals of S can be characterized by intuitionistic fuzzy (T, S_T) anti-ideals of S by the following proposition.

Proposition 2. *Let $A \subseteq S$ and $A \neq \varnothing$. Then the following statements hold:*

(i) *$A \in LAI(S)$ if and only if $C^A \in IFSLAI(S)$,*

(ii) *$A \in RAI(S)$ if and only if $C^A \in IFSRAI(S)$,*

(iii) *$A \in AI(S)$ if and only if $C^A \in IFSAI(S)$.*

Proof. (i) Let $A \in LAI(S)$ and $p_1 \in S$, $p_2 \in A$. Since $A \in LAI(S)$, we have $p_1p_2 \notin A$. Thus,

$$\begin{aligned}\alpha_{C^A}(p_1p_2)T\alpha_{C^A}(p_2) &= 0T1\\ &= 0,\end{aligned}$$

$$\begin{aligned}\beta_{C^A}(p_1p_2)S_T\beta_{C^A}(p_2) &= 1S_T0\\ &= 1.\end{aligned}$$

Clearly, if $p_2 \notin A$,

$$\alpha_{C^A}(p_1p_2)T\alpha_{C^A}(p_2) = 0,$$

$$\beta_{C^A}(p_1p_2)S_T\beta_{C^A}(p_2) = 1.$$

Therefore $C^A \in IFSLAI(S)$.
Conversely, suppose that $C^A \in IFSLAI(S)$. Suppose that $SA \cap A \neq \varnothing$. Then there exist $p_1 \in S, p_2 \in A$ with $a = p_1p_2 \in SA \cap A$.

$$\alpha_{C^A}(p_1p_2)T\alpha_{C^A}(p_2) = 1 \neq 0$$

Therefore $A \in LAI(S)$. The proofs of the remaining assertions are similar. □

Proposition 3. *Let T be a t-norm and S_T be its dual t-conorm, $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}) \in IFS(S)$. Then*

(i) *If $\mathcal{P} \in IFSLAI(S)$, then so is $\Box\mathcal{P}$,*

(ii) *If $\mathcal{P} \in IFSRAI(S)$, then so is $\Box\mathcal{P}$,*

(iii) *If $\mathcal{P} \in IFSAI(S)$, then so is $\Box\mathcal{P}$.*

Proof. For all $p_1, p_2 \in S$, we have $\alpha_{\mathcal{P}}(p_2)T\alpha_{\mathcal{P}}(p_1p_2) = 0$. Hence

$$\begin{aligned}\overline{\alpha}_{\mathcal{P}}(p_2)S_T\overline{\alpha}_{\mathcal{P}}(p_1p_2) = (1 - \alpha_{\mathcal{P}}(p_2))S_T(1 - \alpha_{\mathcal{P}}(p_1p_2)) &= 1 - (\alpha_{\mathcal{P}}(p_2)T\alpha_{\mathcal{P}}(p_1p_2))\\ &= 1.\end{aligned}$$

Therefore $\Box\mathcal{P} = (\alpha_{\mathcal{P}}, \overline{\alpha}_{\mathcal{P}}) \in IFSLAI(S)$.

Statements (ii) and (iii) are obtained similarly to (i). □

Proposition 4. *Let T and S_T be dual norms, $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P}) \in IFS(S)$. Then*

(i) *If $\mathcal{P} \in IFSLAI(S)$, then so is $\diamond\mathcal{P}$,*

(ii) *If $\mathcal{P} \in IFSRAI(S)$, then so is $\diamond\mathcal{P}$,*

(iii) *If $\mathcal{P} \in IFSAI(S)$, then so is $\diamond\mathcal{P}$.*

Proof. For all $p_1, p_2 \in S$, we have $\beta_\mathcal{P}(p_2) S_T \beta_\mathcal{P}(p_1 p_2) = 1$. And so

$$\begin{aligned}\overline{\beta}_\mathcal{P}(p_2) T \overline{\beta}_\mathcal{P}(p_1 p_2) = (1 - \beta_\mathcal{P}(p_2)) T (1 - \beta_\mathcal{P}(p_1 p_2)) &= 1 - (\beta_\mathcal{P}(p_2) S_T \beta_\mathcal{P}(p_1 p_2)) \\ &= 0.\end{aligned}$$

Therefore $\diamond\mathcal{P} = (\overline{\beta}_\mathcal{P}, \beta_\mathcal{P}) \in IFSLAI(S)$.

Statements (ii) and (iii) are obtained similarly to (i). □

The following corollary can be readily established using the above two propositions.

Corollary 1. *Let T and S_T be dual norms, $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P}) \in IFSLAI(S)$ ($IFSRAI(S), IFSAI(S)$) if and only if $\Box\mathcal{P}$ and $\diamond\mathcal{P}$ are intuitionistic fuzzy (T, S_T) left (right) anti-ideal of S.*

Remark 1. For the intuitionistic fuzzy set $\mathcal{P} \in IFSLAI(S)$ of Example 1 one has $\alpha_\mathcal{P}(a) = 1/2$ and $\beta_\mathcal{P}(a) = 1/4$, hence $\pi_\mathcal{P}(a) = 1 - 1/2 - 1/4 = 1/4 > 0$. Therefore $\mathcal{P} \neq \Box\mathcal{P} = (\alpha_\mathcal{P}, 1 - \alpha_\mathcal{P})$ and $\mathcal{P} \neq \diamond\mathcal{P} = (1 - \beta_\mathcal{P}, \beta_\mathcal{P})$, i.e., $\mathcal{P}$ is a genuinely intuitionistic fuzzy object that cannot be obtained from any fuzzy anti-ideal by the $\Box$ or $\diamond$ operator.

Theorem 2. *Let $\{\mathcal{P}_t \mid t \in T\} \subset IFS(S)$. Then, the following statements hold:*

(i) *If $\{\mathcal{P}_t \mid t \in T\} \subseteq IFSLAI(S)$, then $\bigcap_{t\in T} \mathcal{P}_t \in IFSLAI(S)$,*

(ii) *If $\{\mathcal{P}_t \mid t \in T\} \subseteq IFSRAI(S)$, then $\bigcap_{t\in T} \mathcal{P}_t \in IFSRAI(S)$,*

(iii) *If $\{\mathcal{P}_t \mid t \in T\} \subseteq IFSAI(S)$, then $\bigcap_{t\in T} \mathcal{P}_t \in IFSAI(S)$.*

Proof. (i) Let $\mathcal{P}_t \in IFSLAI(S)$ and $p_1, p_2 \in S$.

$$\begin{aligned}\alpha_{\bigcap_{t\in T}\mathcal{P}_t}(p_1 p_2) T \alpha_{\bigcap_{t\in T}\mathcal{P}_t}(p_2) &= \bigwedge_{t\in T} \alpha_{\mathcal{P}_t}(p_1 p_2) T \bigwedge_{t\in T} \alpha_{\mathcal{P}_t}(p_2) \\ &\leq \bigwedge_{t\in T} (\alpha_{\mathcal{P}_t}(p_1 p_2) T \alpha_{\mathcal{P}_t}(p_2)) \\ &= 0\end{aligned}$$

$$\begin{aligned}\beta_{\bigcap_{t\in T}\mathcal{P}_t}(p_1 p_2) S_T \beta_{\bigcap_{t\in T}\mathcal{P}_t}(p_2) &= \bigvee_{t\in T} \beta_{\mathcal{P}_t}(p_1 p_2) S_T \bigvee_{t\in T} \beta_{\mathcal{P}_t}(p_2) \\ &\geq \bigvee_{t\in T} (\beta_{\mathcal{P}_t}(p_1 p_2) S_T \beta_{\mathcal{P}_t}(p_2)) \\ &= 1\end{aligned}$$

Therefore, $\bigcap_{t\in T} \mathcal{P}_t \in IFSLAI(S)$.

Statements (ii) and (iii) are obtained similarly to (i). □

Remark 2. The union of intuitionistic fuzzy (T, S_T) anti-ideals may not be an intuitionistic fuzzy (T, S_T) anti-ideal in general. We can illustrate this with the following example.

Example 3. *Let $S = \{0, 1, 2\}$ be a semigroup according to the following operation, $T = \wedge$, $S_T = \vee$, and $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P})$, $\mathcal{R} = (\alpha_\mathcal{R}, \beta_\mathcal{R})$ be intuitionistic fuzzy sets on S with the following tables:*

$\cdot$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

$\mathcal{P}$	0	1	2
$\alpha_\mathcal{P}$	0	1	0
$\beta_\mathcal{P}$	1	0	1

$\mathcal{R}$	0	1	2
$\alpha_\mathcal{R}$	0	0	1
$\beta_\mathcal{R}$	1	1	0

It is easily seen that $\mathcal{P}, \mathcal{R} \in IFSLAI(S)$. Since

$$\begin{aligned}\alpha_{\mathcal{P}\cup\mathcal{R}}(2 \cdot 2) \wedge \alpha_{\mathcal{P}\cup\mathcal{R}}(2) &= (\alpha_\mathcal{P}(1) \vee \alpha_\mathcal{R}(1)) \wedge (\alpha_\mathcal{P}(2) \vee \alpha_\mathcal{R}(2)) \\ &= 1 \\ &\neq 0\end{aligned}$$

we have that $\mathcal{P} \cup \mathcal{R} \notin IFSLAI(S)$.

Definition 18. [14] *Let $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P})$ and $\mathcal{R} = (\alpha_\mathcal{R}, \beta_\mathcal{R}) \in IFS(S)$. The (T, S_T)intersection of $\mathcal{P}$ and $\mathcal{R}$ is defined, for every $x \in S$, by*

$$\begin{aligned}\alpha_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(x) &= \alpha_\mathcal{P}(x) T \alpha_\mathcal{R}(x), \\ \beta_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(x) &= \beta_\mathcal{P}(x) S_T \beta_\mathcal{R}(x).\end{aligned}$$

Equivalently,

$$\mathcal{P} \cap_{(T,S_T)} \mathcal{R} = (\alpha_\mathcal{P} T \alpha_\mathcal{R}, \beta_\mathcal{P} S_T \beta_\mathcal{R}).$$

Remark 3. Since $\mathcal{P}, \mathcal{R} \in IFS(S)$, we have $\alpha_\mathcal{P}(x) \leq 1 - \beta_\mathcal{P}(x)$ and $\alpha_\mathcal{R}(x) \leq 1 - \beta_\mathcal{R}(x)$ for all $x \in S$. By the monotonicity of T and the De Morgan duality

$$S_T(u, v) = 1 - T(1 - u, 1 - v),$$

we obtain

$$T(\alpha_\mathcal{P}(x), \alpha_\mathcal{R}(x)) \leq T(1 - \beta_\mathcal{P}(x), 1 - \beta_\mathcal{R}(x)) = 1 - S_T(\beta_\mathcal{P}(x), \beta_\mathcal{R}(x)),$$

for all $x \in S$. Hence,

$$\alpha_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(x) + \beta_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(x) \leq 1$$

for all $x \in S$. Thus, $\mathcal{P} \cap_{(T,S_T)} \mathcal{R} \in IFS(S)$.

Theorem 3. *Let $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P})$ and $\mathcal{R} = (\alpha_\mathcal{R}, \beta_\mathcal{R}) \in IFS(S)$. Then*

(i) *If $\mathcal{P}, \mathcal{R} \in IFSLAI(S)$, then $\mathcal{P} \cap_{(T,S_T)} \mathcal{R} \in IFSLAI(S)$;*

(ii) *If $\mathcal{P}, \mathcal{R} \in IFSRAI(S)$, then $\mathcal{P} \cap_{(T,S_T)} \mathcal{R} \in IFSRAI(S)$;*

(iii) *If $\mathcal{P}, \mathcal{R} \in IFSAI(S)$, then $\mathcal{P} \cap_{(T,S_T)} \mathcal{R} \in IFSAI(S)$.*

Proof. (i) Let $p_1, p_2 \in S$. Since $\mathcal{P}, \mathcal{R} \in IFSLAI(S)$, we have

$$\alpha_{\mathcal{P}}(p_1p_2)T\alpha_{\mathcal{P}}(p_2) = 0$$

and

$$\alpha_{\mathcal{R}}(p_1p_2)T\alpha_{\mathcal{R}}(p_2) = 0.$$

Therefore, by the associativity and commutativity of T,

$$\begin{aligned}\alpha_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_1p_2)T\alpha_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_2) &= \Big(\big(\alpha_{\mathcal{P}}(p_1p_2)T\alpha_{\mathcal{P}}(p_2)\big)T\big(\alpha_{\mathcal{R}}(p_1p_2)T\alpha_{\mathcal{R}}(p_2)\big)\Big)\\ &= 0T0 = 0.\end{aligned}$$

Similarly,

$$\beta_{\mathcal{P}}(p_1p_2)S_T\beta_{\mathcal{P}}(p_2) = 1$$

and

$$\beta_{\mathcal{R}}(p_1p_2)S_T\beta_{\mathcal{R}}(p_2) = 1.$$

Hence, by the associativity and commutativity of S_T,

$$\begin{aligned}\beta_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_1p_2)S_T\beta_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_2) &= \Big(\big(\beta_{\mathcal{P}}(p_1p_2)S_T\beta_{\mathcal{P}}(p_2)\big)S_T\big(\beta_{\mathcal{R}}(p_1p_2)S_T\beta_{\mathcal{R}}(p_2)\big)\Big)\\ &= 1S_T1 = 1.\end{aligned}$$

Therefore, $\mathcal{P} \cap_{(T,S_T)} \mathcal{R} \in IFSLAI(S)$.

Statements (ii) and (iii) are obtained similarly to (i). □

Intuitionistic fuzzy anti-ideals can be characterized by their level subsets, as stated in the following theorem.

Definition 19. *An intuitionistic fuzzy (T, S_T) set of S is called an imaginable intuitionistic fuzzy (T, S_T) set on S if it satisfies the imaginable property.*

Theorem 4. *$\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ is an imaginable intuitionistic fuzzy (T, S_T) set on S. Then, the following statements hold:*

(i) *$\mathcal{P} \in IFSLAI(S)$ if and only if for every $t \in \Delta_T$ and $s \in \Delta_{S_T}$ with $t + s \leq 1$ and $(t, s) \neq (0, 1)$, the nonempty level set $A_{(t,s)}$ is a left anti-ideal on S,*

(ii) *$\mathcal{P} \in IFSRAI(S)$ if and only if for every $t \in \Delta_T$ and $s \in \Delta_{S_T}$ with $t + s \leq 1$ and $(t, s) \neq (0, 1)$, the nonempty level set $A_{(t,s)}$ is a right anti-ideal on S,*

(iii) *$\mathcal{P} \in IFSAI(S)$ if and only if for every $t \in \Delta_T$ and $s \in \Delta_{S_T}$ with $t + s \leq 1$ and $(t, s) \neq (0, 1)$, the nonempty level set $A_{(t,s)}$ is an anti-ideal on S.*

Proof. (i) Let $t \in \Delta_T$, $s \in \Delta_{S_T}$, $t + s \leq 1$, $(t, s) \neq (0, 1)$, and suppose that $\mathrm{A}_{(t,s)} \neq \varnothing$ contains elements p_2, p_1p_2 with $p_2 \in \mathrm{A}_{(t,s)}$, $p_1 \in S$. Since $\mathcal{P} \in IFSLAI(S)$, $\alpha_{\mathcal{P}}(p_2)T\alpha_{\mathcal{P}}(p_1p_2) = 0$ and $\beta_{\mathcal{P}}(p_2)S_T\beta_{\mathcal{P}}(p_1p_2) = 1$. On the other hand, $\alpha_{\mathcal{P}}(p_2)$, $\alpha_{\mathcal{P}}(p_1p_2) \geq t$ gives, by monotonicity of T and $t \in \Delta_T$,

$$0 = \alpha_{\mathcal{P}}(p_2)T\alpha_{\mathcal{P}}(p_1p_2) \geq tTt = t,$$

so $t = 0$; and $\beta_{\mathcal{P}}(p_2)$, $\beta_{\mathcal{P}}(p_1p_2) \leq s$ gives, by monotonicity of S_T and $s \in \Delta_{S_T}$,

$$1 = \beta_{\mathcal{P}}(p_2)S_T\beta_{\mathcal{P}}(p_1p_2) \leq sS_Ts = s,$$

so $s = 1$. Hence $(t, s) = (0, 1)$, contradicting the assumption. Thus no such pair exists, i.e., $s \cdot A_{(t,s)} \cap A_{\{t,s\}} = \varnothing$ and $A_{\{t,s\}} \in LAI(S)$.

Conversely, let $p_1, p_2 \in s$ and put $t_0 := \alpha_{\mathcal{P}}(p_2)T\alpha_{\mathcal{P}}(p_1p_2)$. Since $\mathcal{P}$ is imaginable and Δ_T is closed under T, $t_0 \in \Delta_T$. Suppose $t_0 > 0$, and set $s_0 := 1 - t_0 \in \Delta_{S_T}$. Then $t_0 + s_0 = 1$ and $(t_0, s_0) \neq (0, 1)$. Moreover $t_0 \leq \alpha_{\mathcal{P}}(p_2) \wedge \alpha_{\mathcal{P}}(p_1p_2)$ and, since $\beta_{\mathcal{P}}(x) \leq 1 - \alpha_{\mathcal{P}}(x)$, we have $\beta_{\mathcal{P}}(p_2)$, $\beta_{\mathcal{P}}(p_1p_2) \leq 1 - t_0 = s_0$. Hence $p_2, p_1p_2 \in A_{(t_0,s_0)}$, so by (ii) $A_{(t_0,s_0)}$ is a left anti-ideal; but $p_1p_2 = p_1 \cdot p_2 \in s \cdot A_{(t_0,s_0)} \cap A_{(t_0,s_0)}$, a contradiction. Therefore $t_0 = 0$. Next put $s_0' := \beta_{\mathcal{P}}(p_2)S_T\beta_{\mathcal{P}}(p_1p_2) \in \Delta_{S_T}$. Suppose $s_0' < 1$. Then $(0, s_0')$ is admissible $t + s = s_0' \leq 1$ and $(0, s_0') \neq (0, 1))$, and since $s_0' \geq \beta_{\mathcal{P}}(p_2) \vee \beta_{\mathcal{P}}(p_1p_2)$, we have $p_2, p_1p_2 \in A_{\{0,s_0'\}}$. By (ii), $A_{\{0,s_0'\}}$ is a left anti-ideal, contradicting $p_1p_2 \in s \cdot A_{\{0,s_0'\}} \cap A_{\{0,s_0'\}}$. Hence $s_0' = 1$.

Thus $\alpha_{\mathcal{P}}(p_2)T\alpha_{\mathcal{P}}(p_1p_2) = 0$ and $\beta_{\mathcal{P}}(p_2)S_T\beta_{\mathcal{P}}(p_1p_2) = 1$ for all $p_1, p_2 \in S$, i.e., $\mathcal{P} \in IFSLAI(S)$.

The proofs of (ii) and (iii) are similar. □

Corollary 2. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ be an imaginable intuitionistic fuzzy (T, S_T) set of S. Then, the following statements hold:*

(i) $\mathcal{P} \in IFSLAI(S)$ if and only if the non- empty sets $U(\alpha_{\mathcal{P}}, \lambda)$ and $L(\beta_{\mathcal{P}}, \lambda)$ are left anti-ideals on S for every $\lambda \in (0, 1)$ and $\lambda \in \Delta_T \cap \Delta_{S_T}$,

(ii) $\mathcal{P} \in IFSRAI(S)$ if and only if the non- empty sets $U(\alpha_{\mathcal{P}}, \lambda)$ and $L(\beta_{\mathcal{P}}, \lambda)$ are right anti-ideals on S for every $\lambda \in (0, 1)$ and $\lambda \in \Delta_T \cap \Delta_{S_T}$,

(iii) $\mathcal{P} \in IFSAI(S)$ if and only if the non- empty sets $U(\alpha_{\mathcal{P}}, \lambda)$ and $L(\beta_{\mathcal{P}}, \lambda)$ are anti-ideals on S for every $\lambda \in (0, 1)$ and $\lambda \in \Delta_T \cap \Delta_{S_T}$.

Theorem 5. *Let $f : S \to S'$ be a semigroup homomorphism, T is infinitely $\vee$-distributive, and S_T is an infinitely $\wedge$-distributive. $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}) \in IFS(S)$. Then*

(i) If $\mathcal{P} \in IFSLAI(S)$, then $f(\mathcal{P}) \in IFSLAI(S')$,

(ii) If $\mathcal{P} \in IFSRAI(S)$, then $f(\mathcal{P}) \in IFSRAI(S')$,

(iii) If $\mathcal{P} \in IFSAI(S)$, then $f(\mathcal{P}) \in IFSAI(S')$.

Proof. (i) Let $\mathcal{P} \in IFSLAI(S)$ and $p_1', p_2' \in S'$. Then there exist $p_1, p_2 \in S$ with $f(p_1) = p_1'$, $f(p_2) = p_2'$.

$$\begin{aligned}\alpha_{f(\mathcal{P})}(p_1'p_2')T\alpha_{f(\mathcal{P})}(p_2') &= \bigvee_{f(p_1p_2)=p_1'p_2'} \alpha_{\mathcal{P}}(p_1p_2)T \bigvee_{f(p_2)=p_2'} \alpha_{\mathcal{P}}(p_2) \\ &= \bigvee_{f(p_1p_2)=p_1'p_2', f(p_2)=p_2'} \alpha_{\mathcal{P}}(p_1p_2)T\alpha_{\mathcal{P}}(p_2) \\ &= 0\end{aligned}$$

Similarly, we can prove that $\beta_{f(\mathcal{P})}(p_1'p_2')S_T\beta_{f(\mathcal{P})}(p_2') = 1$. Hence, $f(\mathcal{P}) \in IFSLAI(S')$.

The proof of (ii) and (iii) is obtained similarly to (i). □

Theorem 6. *Let $f : S \to S'$ be a semigroup homomorphism, and let $\mathcal{R} = (\alpha_{\mathcal{R}}, \beta_{\mathcal{R}}) \in IFS(S')$. Then, the following statements hold:*

(i) If $\mathcal{R} \in IFSLAI(S')$, then $f^{-1}(\mathcal{R}) \in IFSLAI(S)$,

(ii) If $\mathcal{R} \in IFSRAI(S')$, then $f^{-1}(\mathcal{R}) \in IFSRAI(S)$,

(iii) If $\mathcal{R} \in IFSAI(S')$, then $f^{-1}(\mathcal{R}) \in IFSAI(S)$.

Proof. (i) Let $\mathcal{R} \in IFSLAI(S')$ and $p_1, p_2 \in S$.

$$\begin{aligned}\alpha_{f^{-1}(\mathcal{R})}(p_1p_2)T\alpha_{f^{-1}(\mathcal{R})}(p_2) &= \alpha_{\mathcal{R}}(f(p_1p_2))T\alpha_{\mathcal{R}}(f(p_2)) \\ &= \alpha_{\mathcal{R}}(f(p_1)f(p_2))T\alpha_{\mathcal{R}}(f(p_2)) \\ &= 0.\end{aligned}$$

Also, $\beta_{f^{-1}(\mathcal{R})}(p_1p_2)S_T\beta_{f^{-1}(\mathcal{R})}(p_2) = \beta_{\mathcal{R}}(f(p_1p_2))S_T\beta_{\mathcal{R}}(f(p_2)) = 1$.
Thus, $f^{-1}(\mathcal{R}) \in IFSLAI(S)$.

The proofs of the remaining assertions are similar. □

Definition 20. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}) \in IFS(S)$. $\mathcal{P}$ is called* an intuitionistic fuzzy (T, S_T) bi-anti-ideal *of S if*

$$\begin{aligned}\alpha_{\mathcal{P}}(p_1p_2p_3)T\alpha_{\mathcal{P}}(p_1)T\alpha_{\mathcal{P}}(p_3) = 0,\\ \beta_{\mathcal{P}}(p_1p_2p_3)S_T\beta_{\mathcal{P}}(p_1)S_T\beta_{\mathcal{P}}(p_3) = 1,\end{aligned}$$

for all $p_1, p_2, p_3 \in S$.

The set of all intuitionistic fuzzy (T, S_T) bi-anti-ideals is denoted by $IFSBAI(S)$.

Example 4. *Let $(\mathbb{N}^*, +)$ be the semigroup, $T = \wedge$, $S_T = \vee$ and $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ defined by:*

$$\alpha_{\mathcal{P}}(p) = \begin{cases} \frac{1}{3}, & p \leq 2 \\ 0, & \textit{otherwise.} \end{cases}$$

$$\beta_{\mathcal{P}}(p) = \begin{cases} 0, & p \leq 2 \\ 1, & \textit{otherwise.} \end{cases}$$

Since $\alpha_{\mathcal{P}}(p_1 + p_2 + p_3) = 0, \beta_{\mathcal{P}}(p_1 + p_2 + p_3) = 1$ for all $p_1, p_2, p_3 \in \mathbb{N}^*$ and hence $\mathcal{P} \in IFSBAI(\mathbb{N}^*)$.

Proposition 5. *Let $A \subset S$ and $A \neq \varnothing$. $A \in BAI(S)$ if and only if $C^A \in IFSBAI(S)$.*

Proof. Let $A \in BAI(S)$ and $p_1, p_3 \in A$, $p_2 \in S$. $p_1p_2p_3 \notin A$ since $A \in BAI(S)$. Thus,

$$\alpha_{C^A}(p_1p_2p_3)T\alpha_{C^A}(p_1)T\alpha_{C^A}(p_3) = 0$$

$$\beta_{C^A}(p_1p_2p_3)S_T\beta_{C^A}(p_1)S_T\beta_{C^A}(p_3) = 1.$$

Clearly, if $p_1 \notin A$ or $p_3 \notin A$,

$$\alpha_{C^A}(p_1p_2p_3)T\alpha_{C^A}(p_1)T\alpha_{C^A}(p_3) = 0T1T1 = 0,$$

$$\beta_{C^A}(p_1p_2p_3)S_T\beta_{C^A}(p_1)S_T\beta_{C^A}(p_3) = 1S_T0S_T0 = 1.$$

Therefore $C^A \in IFSBAI(S)$.

Conversely $C^A \in IFSBAI(S)$. Suppose that $ASA \cap A \neq \varnothing$. Then there exist $p_2 \in S$, $p_1, p_3 \in A$ with $a = p_1p_2p_3 \in ASA \cap A$.

$$\alpha_{C^A}(p_1p_2p_3)T\alpha_{C^A}(p_1)T\alpha_{C^A}(p_3) = 1 \neq 0$$

Therefore $A \in BAI(S)$. □

Proposition 6. *Let T and S_T be dual norms, $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P}) \in IFS(S)$. If $\mathcal{P} \in IFSBAI(S)$, then so is $\Box\mathcal{P} = (\alpha_\mathcal{P}, \overline{\alpha}_\mathcal{P})$.*

Proof. The proof is similar to that of Proposition 3. □

Proposition 7. *Let T and S_T be dual norms, $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P}) \in IFS(S)$. If $\mathcal{P} \in IFSBAI(S)$, then so is $\diamond\mathcal{P} = (\overline{\beta_\mathcal{P}}, \beta_\mathcal{P})$.*

Proof. The proof is similar to that of Proposition 4. □

Theorem 7. *Let $\{\mathcal{P}_t \mid t \in T\} \subset IFS(S)$. If $\{\mathcal{P}_t \mid t \in T\} \subseteq IFSBAI(S)$, then $\bigcap_{t\in T}\mathcal{P}_t \in IFSBAI(S)$.*

Proof. Let $\mathcal{P}_t \in IFSBAI(S)$ and $p_1, p_2, p_3 \in S$.

$$\begin{aligned}
\alpha_{\bigcap_{t\in T}\mathcal{P}_t}(p_1p_2p_3)T\alpha_{\bigcap_{t\in T}\mathcal{P}_t}(p_1)T\alpha_{\bigcap_{t\in T}\mathcal{P}_t}(p_3) &= \bigwedge_{t\in T}\alpha_{\mathcal{P}_t}(p_1p_2p_3)T\bigwedge_{t\in T}\alpha_{\mathcal{P}_t}(p_1)T\bigwedge_{t\in T}\alpha_{\mathcal{P}_t}(p_3)\\
&\leq \bigwedge_{t\in T}(\alpha_{\mathcal{P}_t}(p_1p_2p_3)T\alpha_{\mathcal{P}_t}(p_1)T\alpha_{\mathcal{P}_t}(p_3))\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\beta_{\bigcap_{t\in T}\mathcal{P}_t}(p_1p_2p_3)S_T\beta_{\bigcap_{t\in T}\mathcal{P}_t}(p_1)S_T\beta_{\bigcap_{t\in T}\mathcal{P}_t}(p_3) &= \bigvee_{t\in T}\beta_{\mathcal{P}_t}(p_1p_2p_3)S_T\bigvee_{t\in T}\beta_{\mathcal{P}_t}(p_1)S_T\bigvee_{t\in T}\beta_{\mathcal{P}_t}(p_3)\\
&= \bigvee_{t\in T}\left(\beta_{\mathcal{P}_t}(p_1p_2p_3)S_T\beta_{\mathcal{P}_t}(p_1)S_T\beta_{\mathcal{P}_t}(p_3)\right)\\
&= 1
\end{aligned}$$

Therefore, $\bigcap_{t\in T}\mathcal{P}_t \in IFSBAI(S)$. □

Remark 4. The union intuitionistic fuzzy (T, S_T) bi-anti-ideals may not be intuitionistic fuzzy (T, S_T) bi-anti-ideal in general.

Example 5. *Let* $S = \{0, 1, 2, 3\}$, $x \cdot y = \min(x + y, 3)$, $T = \wedge$, *and* $S_T = \vee$. *Let* $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P})$, $\mathcal{R} = (\alpha_\mathcal{R}, \beta_\mathcal{R})$ *be intuitionistic fuzzy sets on* S *with the following tables:*

$\mathcal{P}$	0	1	2	3
$\alpha_\mathcal{P}$	0	1	0	0
$\beta_\mathcal{P}$	1	0	1	1

$\mathcal{R}$	0	1	2	3
$\alpha_\mathcal{R}$	0	0	1	0
$\beta_\mathcal{R}$	1	1	0	1

It is easily seen that $\mathcal{P}, \mathcal{R} \in IFSBAI(S)$. *Since*

$$\begin{aligned}\alpha_{\mathcal{P}\cup\mathcal{R}}(1 \cdot 0 \cdot 1)T\alpha_{\mathcal{P}\cup\mathcal{R}}(1)T\alpha_{\mathcal{P}\cup\mathcal{R}}(1) &= (\alpha_\mathcal{P}(2) \vee \alpha_\mathcal{R}(2))T(\alpha_\mathcal{P}(1) \vee \alpha_\mathcal{R}(1))T(\alpha_\mathcal{P}(1) \vee \alpha_\mathcal{R}(1)) \\ &= (0 \vee 1)T(1 \vee 0)T(1 \vee 0) \\ &= 1 \\ &\neq 0,\end{aligned}$$

we have that $\mathcal{P} \cup \mathcal{R} \notin IFSBAI(S)$.

Theorem 8. *Let* $\mathcal{P} = (\alpha_\mathcal{P}, \beta_\mathcal{P})$ *and* $\mathcal{R} = (\alpha_\mathcal{R}, \beta_\mathcal{R})$ *be intuitionistic fuzzy sets on* S. *If* $\mathcal{P}, \mathcal{R} \in IFSBAI(S)$, *then*

$$\mathcal{P} \cap_{(T,S_T)} \mathcal{R} \in IFSBAI(S).$$

Proof. Let $p_1, p_2, p_3 \in S$. Since $\mathcal{P}, \mathcal{R} \in IFSBAI(S)$, we have

$$\begin{aligned}\alpha_\mathcal{P}(p_1p_2p_3)T\alpha_\mathcal{P}(p_1)T\alpha_\mathcal{P}(p_3) = 0, \\ \alpha_\mathcal{R}(p_1p_2p_3)T\alpha_\mathcal{R}(p_1)T\alpha_\mathcal{R}(p_3) = 0.\end{aligned}$$

Therefore, by the associativity and commutativity of T,

$$\begin{aligned}&\alpha_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_1p_2p_3)T\alpha_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_1)T\alpha_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_3) \\ &= \big(\alpha_\mathcal{P}(p_1p_2p_3)T\alpha_\mathcal{R}(p_1p_2p_3)\big)T\big(\alpha_\mathcal{P}(p_1)T\alpha_\mathcal{R}(p_1)\big)T\big(\alpha_\mathcal{P}(p_3)T\alpha_\mathcal{R}(p_3)\big) \\ &= \big(\alpha_\mathcal{P}(p_1p_2p_3)T\alpha_\mathcal{P}(p_1)T\alpha_\mathcal{P}(p_3)\big)T\big(\alpha_\mathcal{R}(p_1p_2p_3)T\alpha_\mathcal{R}(p_1)T\alpha_\mathcal{R}(p_3)\big) \\ &= 0T0 = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\beta_\mathcal{P}(p_1p_2p_3)S_T\beta_\mathcal{P}(p_1)S_T\beta_\mathcal{P}(p_3) = 1, \\ \beta_\mathcal{R}(p_1p_2p_3)S_T\beta_\mathcal{R}(p_1)S_T\beta_\mathcal{R}(p_3) = 1.\end{aligned}$$

Hence, by the associativity and commutativity of S_T,

$$\begin{aligned}&\beta_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_1p_2p_3)S_T\beta_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_1)S_T\beta_{\mathcal{P}\cap_{(T,S_T)}\mathcal{R}}(p_3) \\ &= \big(\beta_\mathcal{P}(p_1p_2p_3)S_T\beta_\mathcal{P}(p_1)S_T\beta_\mathcal{P}(p_3)\big)S_T\big(\beta_\mathcal{R}(p_1p_2p_3)S_T\beta_\mathcal{R}(p_1)S_T\beta_\mathcal{R}(p_3)\big) \\ &= 1S_T1 = 1.\end{aligned}$$

Therefore, $\mathcal{P} \cap_{(T,S_T)} \mathcal{R} \in IFSBAI(S)$. □

Theorem 9. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ be an imaginable intuitionistic fuzzy (T, S_T) set on S. $\mathcal{P} \in IFSBAI(S)$ if and only if for every $t \in \Delta_T$ and $s \in \Delta_{S_T}$ satisfying $t + s \leq 1$ and $(t, s) \neq (0, 1)$ the level set $A_{(t,s)}$, whenever nonempty, is a bi-anti-ideal of S.*

Proof. Suppose first that $\mathcal{P} \in IFSBAI(S)$. Let $A_{(t,s)}$ be nonempty and assume that it is not a bi-anti-ideal. Then there exist $p_1, p_3 \in A_{(t,s)}$ and $p_2 \in S$ such that $p_1p_2p_3 \in A_{(t,s)}$. Hence,

$$0 = \alpha_{\mathcal{P}}(p_1p_2p_3)T\alpha_{\mathcal{P}}(p_1)T\alpha_{\mathcal{P}}(p_3) \geq tTtTt = t,$$
$$1 = \beta_{\mathcal{P}}(p_1p_2p_3)S_T\beta_{\mathcal{P}}(p_1)S_T\beta_{\mathcal{P}}(p_3) \leq sS_TsS_Ts = s.$$

Thus, $t = 0$ and $s = 1$, contradicting $(t, s) \neq (0, 1)$. Therefore, $A_{(t,s)}$ is a bi-anti-ideal of S.

Conversely, suppose that every nonempty admissible level set $A_{(t,s)}$ is a bi-anti-ideal. For $p_1, p_2, p_3 \in S$, put

$$t = \min\{\alpha_{\mathcal{P}}(p_1p_2p_3), \alpha_{\mathcal{P}}(p_1), \alpha_{\mathcal{P}}(p_3)\},$$
$$s = \max\{\beta_{\mathcal{P}}(p_1p_2p_3), \beta_{\mathcal{P}}(p_1), \beta_{\mathcal{P}}(p_3)\}.$$

Since $\mathcal{P}$ is imaginable, $t \in \Delta_T$ and $s \in \Delta_{S_T}$. Moreover, $t + s \leq 1$.

If $(t, s) \neq (0, 1)$, then

$$p_1, p_3, p_1p_2p_3 \in A_{(t,s)},$$

which contradicts the assumption that $A_{(t,s)}$ is a bi-anti-ideal. Hence, $(t, s) = (0, 1)$. Therefore,

$$\alpha_{\mathcal{P}}(p_1p_2p_3)T\alpha_{\mathcal{P}}(p_1)T\alpha_{\mathcal{P}}(p_3) = 0$$

and

$$\beta_{\mathcal{P}}(p_1p_2p_3)S_T\beta_{\mathcal{P}}(p_1)S_T\beta_{\mathcal{P}}(p_3) = 1.$$

Thus, $\mathcal{P} \in IFSBAI(S)$. □

Corollary 3. *Let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ be an imaginable intuitionistic fuzzy (T, S_T) set on S. $\mathcal{P} \in IFSBAI(S)$ if and only if the nonempty sets $U(\alpha_{\mathcal{P}}, \lambda)$ and $L(\beta_{\mathcal{P}}, \lambda)$ are bi-anti-ideals of S for every $\lambda \in (0, 1)$ and $\lambda \in \Delta_T \cap \Delta_{S_T}$.*

Theorem 10. *Suppose that $f : S \to S'$ is a semigroup isomorphism, and let $\mathcal{P} = (\alpha_{\mathcal{P}}, \beta_{\mathcal{P}}) \in IFS(S)$. If $\mathcal{P} \in IFSBAI(S)$, then $f(\mathcal{P}) \in IFSBAI(S')$.*

Proof. Let $\mathcal{P} \in IFSBAI(S)$ and $p'_1, p'_2, p'_3 \in S'$. Then there exist $p_1, p_2, p_3 \in S$ with $f(p_1) = p'_1, f(p_2) = p'_2, f(p_3) = p'_3$.

$$\begin{aligned}\alpha_{f(\mathcal{P})}(p'_1p'_2p'_3)T\alpha_{f(\mathcal{P})}(p'_1)T\alpha_{f(\mathcal{P})}(p'_3) &= \bigvee_{f(p_1p_2p_3)=p'_1p'_2p'_3} \alpha_{\mathcal{P}}(p_1p_2p_3)T \bigvee_{f(p_1)=p'_1} \alpha_{\mathcal{P}}(p_1)T \bigvee_{f(p_3)=p'_3} \alpha_{\mathcal{P}}(p_3) \\ &= \bigvee_{f(p_1p_2p_3)=p'_1p'_2p'_3, f(p_1)=p'_1, f(p_3)=p'_3} \alpha_{\mathcal{P}}(p_1p_2p_3)T\alpha_{\mathcal{P}}(p_1)T\alpha_{\mathcal{P}}(p_3) \\ &= 0\end{aligned}$$

Similarly, we can prove that $\beta_{f(\mathcal{P})}(p'_1p'_2p'_3)S_T\beta_{f(\mathcal{P})}(p'_1)S_T\beta_{f(\mathcal{P})}(p'_3) = 1$.
Hence, $f(\mathcal{P}) \in IFSBAI(S')$. □

Theorem 11. *Suppose that $f : S \to S'$ is a semigroup homomorphism. $\mathcal{R} = (\alpha_{\mathcal{R}}, \beta_{\mathcal{R}}) \in IFS(S')$. If $\mathcal{R} \in IFSBAI(S')$, then $f^{-1}(\mathcal{R}) \in IFSBAI(S)$.*

Proof. Let $\mathcal{R} \in IFSBAI(S')$ and $p_1, p_2, p_3 \in S$.

$$\begin{aligned}
&\alpha_{f^{-1}(\mathcal{R})}(p_1p_2p_3)T\alpha_{f^{-1}(\mathcal{R})}(p_1)T\alpha_{f^{-1}(\mathcal{R})}(p_3) = \alpha_{\mathcal{R}}(f(p_1p_2p_3))T\alpha_{\mathcal{R}}(f(p_1))T\alpha_{\mathcal{R}}(f(p_3)) \\
&= \alpha_{\mathcal{R}}(f(p_1)f(p_2)f(p_3))T\alpha_{\mathcal{R}}(f(p_1))T\alpha_{\mathcal{R}}(f(p_3)) \\
&= 0.
\end{aligned}$$

Similarly, it can be seen that $\beta_{f^{-1}(\mathcal{R})}(p_1p_2p_3)S_T\beta_{f^{-1}(\mathcal{R})}(p_1)S_T\beta_{f^{-1}(\mathcal{R})}(p_3) = 1$. Thus, $f^{-1}(\mathcal{R}) \in IFSBAI(S)$. □

Theorem 12. *Every intuitionistic fuzzy (T, S_T) left or right anti-ideal of S is an intuitionistic fuzzy (T, S_T) bi-anti-ideal.*

Proof. Let $\mathcal{P} \in IFSLAI(S)$. For $p_1, p_2, p_3 \in S$,

$$0 \leq \alpha_{\mathcal{P}}(p_1p_2p_3)T\alpha_{\mathcal{P}}(p_1)T\alpha_{\mathcal{P}}(p_3) \leq \alpha_{\mathcal{P}}((p_1p_2)p_3)T\alpha_{\mathcal{P}}(p_3) = 0$$

Similarly, it can be seen that $1 \geq \beta_{\mathcal{P}}((p_1p_2)p_3)S_T\beta_{\mathcal{P}}(p_1)S_T\beta_{\mathcal{P}}(p_3) = 1$. Thus, $\mathcal{P} \in IFSBAI(S)$. The proof for a right anti-ideal is analogous. □

4 Conclusion

The definitions of intuitionistic fuzzy (T, S_T) left anti-ideals, right anti-ideals, anti-ideals, and bi-anti-ideals were given by using a t-norm together with its dual t-conorm. In this way, anti-ideal type structures were described by both membership and non-membership functions. The main properties of these structures were investigated. In particular, the connection between classical anti-ideals and intuitionistic fuzzy anti-ideals was obtained by means of characteristic functions. Moreover, characterizations based on level subsets were given, and the behavior of these structures under homomorphisms, inverse images, and isomorphisms was examined. It was also shown that every intuitionistic fuzzy (T, S_T) left or right anti-ideal is an intuitionistic fuzzy (T, S_T) bi-anti-ideal.

These results show that the proposed approach gives a natural extension of anti-ideal theory in semigroups. The framework may be useful for further studies on generalized anti-ideals, ordered semigroups, and other algebraic structures involving uncertainty.

References

[1] Ahn, T. C., Hur, K., Jang, K. W., & Roh, S. B. (2005). Intuitionistic fuzzy ideals of a semigroup. *Honam Mathematical Journal*, 27(4), 525–541.

[2] Al Tahan, M., & Hoskova-Mayerova, S. (2025). Characterizing a semigroup by its fuzzy interior antiideals and fuzzy m-interior antiideals. *Afrika Matematika*, 36, Article ID 63.

[3] Al Tahan, M., Hoskova-Mayerova, S., Al-Kaseasbeh, S. (2025). Generalization of bi-antiideals in semigroups. *European Journal of Pure and Applied Mathematics*, 18(1), Article ID 5646.

[4] Al Tahan, M., Hoskova-Mayerova, S., Davvaz, B., & Harikrishnan, P. (2025). Antiideal theory in semigroups and its fuzzification. *Soft Computing*, 29, 5465–5470.

[5] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

[6] Badhurays, L. B., Bashammakh, S. A., & Alshehri, N. O. (2017). Triangular norms based on intuitionistic fuzzy BCK-submodules. *Journal of Computational Analysis and Applications*, 23(5), 910–924.

[7] Birkhoff, G. (1948). *Lattice Theory*. American Mathematical Society, New York.

[8] Iseki, K. (1962). On (m, n)-antiideals in semigroups. *Proceedings of the Japan Academy*, 38(7), 316–317.

[9] Jun, Y. B., & Hong, S. M. (2001). On imaginable T-fuzzy subalgebras and imaginable T-fuzzy closed ideals in BCH-algebras. *International Journal of Mathematics and Mathematical Sciences*, 27(5), 269–287.

[10] Karaçal, F., & Khadjiev, D. (2005). $\vee$-Distributive and infinitely $\vee$-distributive t-norms on complete lattice. *Fuzzy Sets and Systems*, 151(2), 341–352.

[11] Kehayopulu, N. (2017). Hypersemigroups and fuzzy hypersemigroups. *European Journal of Pure and Applied Mathematics*, 10(5), 929–945.

[12] Kehayopulu, N., & Tsingelis, M. (2007). Fuzzy ideals in ordered semigroups. *Quasigroups and Related Systems*, 15 (2), 279–289.

[13] Kim, K. H. (2006). On intuitionistic Q-fuzzy ideals of semigroups. *Scientiae Mathematicae Japonicae*, 63(2), 119–126.

[14] Klement, E. P., Mesiar, R., & Pap, E. (2000). *Triangular Norms*. Kluwer Academic Publishers, Dordrecht.

[15] Kuroki, N. (1981). On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy Sets and Systems*, 5(2), 203–215.

[16] Mordeson, J. N., Malik, D. S., & Kuroki, N. (2003). *Fuzzy Semigroups*. Springer Science & Business Media, Berlin.

[17] Rosenfeld, A. (1971). Fuzzy groups. *Journal of Mathematical Analysis and Applications*, 35(3), 512–517.

[18] Schwarz, Š. (1953). On maximal ideals in the theory of semigroups, I. *Czechoslovak Mathematical Journal*, 3(2), 139–153.

[19] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8(3), 338–353.