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Convergence in measure of intuitionistic fuzzy observables

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Abstract: The aim of this paper is to define a convergence in measure $\mathbf{m}$, where $\mathbf{m}$ is an intuitionistic fuzzy state. We prove a version of weak law of large numbers for a sequence of independent intuitionistic fuzzy observables, too.
Keywords: Intuitionistic fuzzy observable, Intuitionistic fuzzy state, Joint intuitionistic fuzzy observable, Independence, Convergence in measure, Weak law of large numbers.
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1 Introduction

The different types of convergence of intuitionistic fuzzy observables, like convergence in distribution and almost everywhere convergence, were studied in papers [4–6, 8, 9, 13]. Many important theorems have been proved: the Central limit theorem, the Strong law of large numbers and the Fisher—Tippett—Gnedenko theorem. They are a basis for statistical and financial applications. Other versions of Strong law of large numbers were studied by P. Nowak and O. Hryniewicz in [12].

In this paper, we introduce the notion of convergence in measure $\mathbf{m}$ for intuitionistic fuzzy observables and we formulate a version of Weak law of large numbers for a sequence of independent intuitionistic fuzzy observables in this case. Our main idea is in a representation of a given sequence $(y_n)_n$ of intuitionistic fuzzy observables $y_n : \mathcal{B}(R) \to \mathcal{F}$ by a probability space $(\Omega, \mathcal{S}, P)$ with a sequence $(\eta_n)_n$ of random variables $\eta_n : \Omega \to R$. We work with a sequence of several intuitionistic fuzzy observables induced by Borel measurable function. Recall that a version of Weak law of large numbers for intuitionistic fuzzy observables using an intuitionistic fuzzy probability was proved in paper [11].

Remark that in a whole text we use a notation "IF" in short as the term "intuitionistic fuzzy".

2 IF-events, IF-states and IF-observables

In [1–3], K. T. Atanassov introduced the notion of intuitionistic fuzzy sets. We start with a definition of the basic notions:

Definition 2.1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.*

Definition 2.2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0, 1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$, $\mu_A : \Omega \longrightarrow [0, 1]$ will be called the membership function, $\nu_A : \Omega \longrightarrow [0, 1]$ be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Łukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by:

$$\mathbf{A} \oplus \mathbf{B} = ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)),$$
$$\mathbf{A} \odot \mathbf{B} = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))$$

and the partial ordering is then given by

$$\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

In paper we use max-min connectives defined by:

$$\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B),$$
$$\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$

and the De Morgan rules:

$$(a \vee b)^* = a^* \wedge b^*,$$
$$(a \wedge b)^* = a^* \vee b^*,$$

where $a^* = 1 - a$.

Example 2.3. *Fuzzy set $f : \Omega \longrightarrow [0, 1]$ can be regarded as IF-set, if we put*

$$\mathbf{A} = (f, 1_\Omega - f).$$

If $f = \chi_A$, then the corresponding IF-set has the form

$$\mathbf{A} = (\chi_A, 1_\Omega - \chi_A) = (\chi_A, \chi_{A'}).$$

In this case $\mathbf{A} \oplus \mathbf{B}$ corresponds to the union of sets, $\mathbf{A} \odot \mathbf{B}$ to the intersection of sets and $\leq$ to the set inclusion.

In the IF-probability theory ([15, 18]) instead of the notion of probability we use the notion of state.

Definition 2.4. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1, \mathbf{m}((0_\Omega, 1_\Omega)) = 0$*;*

(ii) If $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$*, then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$*;*

(iii) If $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.* $\mu_{A_n} \nearrow \mu_A$*,* $\nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$*.*

Probably the most useful result in the IF-state theory is the following representation theorem ([15]):

Theorem 2.5. *To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that*

$$\mathbf{m}(\mathbf{A}) = (1-\alpha)\int_\Omega \mu_A dP + \alpha\Big(1 - \int_\Omega \nu_A dP\Big)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$.

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form

$$[a, b) = \{x \in R : a \leq x < b\}.$$

Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets.

Definition 2.6. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega)$*,* $x(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) If $A \cap B = \emptyset$*, then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$*;*

(iii) If $A_n \nearrow A$*, then* $x(A_n) \nearrow x(A)$*.*

If we denote $x(A) = \big(x^\flat(A), 1_\Omega - x^\sharp(A)\big)$ for each $A \in \mathcal{B}(R)$, then $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \to [0,1]; f\ is\ \mathcal{S} - measurable\}$.

***Remark* 2.7.** *Sometimes we need to work with n-dimensional IF-observable $x : \mathcal{B}(R^n) \to \mathcal{F}$ defined as a mapping with the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$*,* $x(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) if $A \cap B = \emptyset$*,* $A, B \in \mathcal{B}(R^n)$*, then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$*;*

(iii) if $A_n \nearrow A$*, then* $x(A_n) \nearrow x(A)$ *for each* $A, A_n \in \mathcal{B}(R^n)$*.*

If $n = 1$ we simply say that x is an IF-observable.

If $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ is an IF-observable, and $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ is an IF-state, then the *IF-distribution function* of x is the function $\mathbf{F} : R \longrightarrow [0,1]$ defined by the formula

$$\mathbf{F}(t) = \mathbf{m}\big(x((-\infty, t))\big)$$

for each $t \in R$.

Similarly as in the classical case the following two theorems can be proved ([18]).

Theorem 2.8. *Let $\mathbf{F} : R \longrightarrow [0,1]$ be the IF-distribution function of an IF-observable $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$. Then $\mathbf{F}$ is non-decreasing on R, left continuous in each point $t \in R$ and*

$$\lim_{n \to -\infty} \mathbf{F}(t) = 0, \ \lim_{n \to \infty} \mathbf{F}(t) = 1.$$

Theorem 2.9. *Let $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ be an IF-state. Define the mapping $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ by the formula*

$$\mathbf{m}_x(C) = \mathbf{m}(x(C)).$$

Then $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0,1]$ is a probability measure.

Theorem 2.8 enables us to define IF-expectation and IF-dispersion of an IF-observable (see [4]).

Definition 2.10. *Let $\mathbf{F} : R \longrightarrow [0,1]$ be the IF-distribution function of an IF-observable $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$. If there exists $\int_R t \, d\mathbf{F}(t)$, then we define the IF-expectation of x by the formula*

$$\mathbf{E}(x) = \int_R t \, d\mathbf{F}(t).$$

Moreover if there exists $\int_R t^2 \, d\mathbf{F}(t)$, then we define the IF-dispersion $\mathbf{D}^2(x)$ by the formula

$$\mathbf{D}^2(x) = \int_R t^2 \, d\mathbf{F}(t) - \big(\mathbf{E}(x)\big)^2 = \int_R (t - \mathbf{E}(x))^2 \, d\mathbf{F}(t).$$

3 Product operation, joint IF-observable and independence

In [10] we introduced the notion of product operation on the family of IF-events $\mathcal{F}$ and showed an example of this operation.

Definition 3.1. *We say that a binary operation $\cdot$ on $\mathcal{F}$ is product if it satisfying the following conditions:*

(i) $(1_\Omega, 0_\Omega) \cdot (a_1, a_2) = (a_1, a_2)$ *for each* $(a_1, a_2) \in \mathcal{F}$*;*

(ii) The operation $\cdot$ is commutative and associative;

(iii) *If* $(a_1, a_2) \odot (b_1, b_2) = (0_\Omega, 1_\Omega)$ *and* $(a_1, a_2), (b_1, b_2) \in \mathcal{F}$*, then*

$$(c_1, c_2) \cdot \big((a_1, a_2) \oplus (b_1, b_2)\big) = \big((c_1, c_2) \cdot (a_1, a_2)\big) \oplus \big((c_1, c_2) \cdot (b_1, b_2)\big)$$

and

$$\big((c_1, c_2) \cdot (a_1, a_2)\big) \odot \big((c_1, c_2) \cdot (b_1, b_2)\big) = (0_\Omega, 1_\Omega)$$

for each $(c_1, c_2) \in \mathcal{F}$*;*

(iv) *If* $(a_{1n}, a_{2n}) \searrow (0_\Omega, 1_\Omega)$*,* $(b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$ *and* $(a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F}$*, then* $(a_{1n}, a_{2n}) \cdot (b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$*.*

The following theorem provides an example of product operation for IF-events.

Theorem 3.2 ([10, Theorem 1])**.** *The operation* $\cdot$ *defined by*

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$$

for each $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$ *is product operation on* $\mathcal{F}$*.*

In [16] B. Riečan defined the notion of a joint IF-observable and proved its existence.

Definition 3.3. *Let* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *be two IF-observables. The joint IF-observable of the IF-observables* x, y *is a mapping* $h : \mathcal{B}(R^2) \to \mathcal{F}$ *satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$*,* $h(\emptyset) = (0_\Omega, 1_\Omega)$*;*

(ii) *if* $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$*, then*

$$h(A \cup B) = h(A) \oplus h(B) \text{ and } h(A) \odot h(B) = (0_\Omega, 1_\Omega);$$

(iii) *if* $A, A_1, \ldots A_n \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$*, then* $h(A_n) \nearrow h(A)$*;*

(iv) $h(C \times D) = x(C) \cdot y(D)$ *for each* $C, D \in \mathcal{B}(R)$*.*

Theorem 3.4 ([16, Theorem 3.3])**.** *For each two IF-observables* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *there exists their joint IF-observable.*

***Remark* 3.5.** *The joint IF-observable of IF-observables* x, y *from Definition 3.3 is a two-dimensional IF-observable.*

Definition 3.6. *Let* $\mathbf{m}$ *be an IF-state. The IF-observables* $x_1, x_2, \ldots, x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ *are independent if for the* n*-dimensional IF-observable* $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ *there holds*

$$\mathbf{m}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \mathbf{m}\big(x_1(A_1)\big) \cdot \mathbf{m}\big(x_2(A_2)\big) \cdot \ldots \cdot \mathbf{m}\big(x_n(A_n)\big)$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$*.*

4 Kolmogorov construction and function of several IF-observables

In this section we show a notation and a construction of Kolmogorov probability space, see [7]. Consider the space R^N and the family of cylinders with finitely dimensional basis B. Such a cylinder has the form

$$A = \{(t_n)_1^\infty \in R^N : (t_1, \ldots, t_n) \in B\}.$$

If we introduce the notion of a projection (n-th coordinate random vector) $\pi_J : R^N \to R^n$,

$$\pi_J\big((t_n)_1^\infty\big) = (t_1, \ldots, t_n),$$

then the cylinder A can be expressed in the form

$$A = \pi_J^{-1}(B).$$

The volume of the cylinder A is product of the content $P_J(B)$ of its basis B and its height 1, hence

$$P(A) = P_J(B).$$

Proposition 4.1 ([7, Proposition 4.7]). *Let $\mathcal{C}$ be the family of all cylinders in R^N, i.e.*

$$\mathcal{C} = \{\pi_J^{-1}(A) \mid \emptyset \neq J \subset N, J \text{ finite}, A \in \mathcal{B}(R^{|J|})\}.$$

Then there exists exactly one probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$P(\pi_J^{-1}(A)) = P_J(A)$$

for each cylinders $\pi_J^{-1}(A)$. Particularly

$$\begin{aligned} P(\{(t_n)_1^\infty : t_i \in A_i, i = 1, 2, \ldots, n\}) &= \mathbf{m}(h_n(A_1 \times \ldots \times A_n)) \\ &= \mathbf{m}(x_1(A_1) \cdot \ldots \cdot x_n(A_n)). \end{aligned}$$

***Remark* 4.1.** *From Proposition 4.1 we have that*

$$P \circ \pi_J^{-1} = \mathbf{m} \circ h_n,$$

where h_n is a joint IF-observable of IF-observables $x_1, \ldots, x_n$.

Proposition 4.2 ([7, Proposition 4.8]). *Define the coordinate function $\xi_n : R^N \to R$ by the formula*

$$\xi_n((t_i)_1^\infty) = t_n.$$

Then ξ_n is a random variable with respect to $\sigma(\mathcal{C})$ such that

$$P_{\xi_n} = \mathbf{m} \circ x_n = \mathbf{m}_{x_n}.$$

***Remark* 4.2.** *By the preceding procedure to each sequence $(x_n)_n$ we can construct a sequence $(\xi_n)_n$ of a random variables.*

The following theorem will be used in the proofs in *Section 5*.

Theorem 4.3 ([4, Theorem 5])**.** *Let $(x_n)_n$ be a sequence of independent IF-observables in $(\mathcal{F}, \mathbf{m})$ with the same IF-distribution function. Define for each $n \in N$ the mapping $\xi_n : R^N \longrightarrow R$ by the formula*

$$\xi_n((t_i)_i) = t_n.$$

Then $(\xi_n)_n$ is a sequence of independent random variables in a space $(R^N, \sigma(\mathcal{C}), P)$. If there exists $\mathbf{E}(x_n)$ then $E(\xi_n) = \mathbf{E}(x_n)$. If there exists $\mathbf{D}^2(x_n)$ then $D^2(\xi_n) = \mathbf{D}^2(x_n)$.

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is a function of several IF-observables. In this regard, we provide the following definition, see [6].

Definition 4.4. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be IF-observables, h_n their joint IF-observable and $g_n : R^n \to R$ a Borel measurable function. Then we define the IF-observable $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ by the formula*

$$g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big).$$

for each $A \in \mathcal{B}(R)$.

Example 4.5. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be independent IF-observables and $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be their joint IF-observable. Then*

1. *the IF-observable $y_n = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} x_i - a\Big)$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} u_i - a\Big)$;

2. *the IF-observable $y_n = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} u_i$;

3. *the IF-observable $y_n = \frac{1}{n}\sum\limits_{i=1}^{n} (x_i - \mathbf{E}(x_i))$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} (u_i - \mathbf{E}(x_i))$;

4. *the IF-observable $y_n = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big)$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1},$$

where $g_n(u_1, \ldots, u_n) = \frac{1}{a_n}\big(\max(u_1, \ldots, u_n) - b_n\big)$.

5 Convergence in measure

In paper [8] we studied a convergence in distribution and an almost everywhere convergence with connection with IF-state $\mathbf{m}$ and we proved Central limit theorem and Strong law of large numbers. In this section we formulate the convergence in measure for IF-observables. First we start with definition in classical probability space.

Let $(\Omega, \mathcal{S}, P)$ be a probability space, $(\eta_n)_n$ be a sequence of random variables in this space. The sequence $(\eta_n)_n$ converges to 0 in measure P if

$$\lim_{n\to\infty} P\big(\eta_n^{-1}((-\varepsilon, \varepsilon))\big) = 1$$

for each real number $\varepsilon > 0$.

Similarly we can define the convergence in measure of a sequence $(y_n)_n$ of IF-observables on an IF-space $(\mathcal{F}, \mathbf{m})$.

Definition 5.1. *Let $(y_n)_n$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathbf{m})$. We say that $(y_n)_n$ converges in measure $\mathbf{m}$ to 0, if for each $\varepsilon > 0$, $\varepsilon \in R$*

$$\lim_{n\to\infty} \mathbf{m}\big(y_n((-\varepsilon, \varepsilon))\big) = 1.$$

In this form we shall now to present a generalization of Weak law of large numbers.

Theorem 5.2. (Weak law of large numbers) *Let $(\mathcal{F}, \mathbf{m})$ be an IF-space, $(x_n)_n$ be a sequence of independent IF-observables with the same distribution and such that $\mathbf{E}(x_n) = a, (n = 1, 2, \ldots)$ and $y_n = \frac{1}{n}\sum\limits_{i=1}^{n} x_i - a$. Then for each $\varepsilon > 0$, $\varepsilon \in R$*

$$\lim_{n\to\infty} \mathbf{m}\big(y_n((-\varepsilon, \varepsilon))\big) = 1,$$

i.e. the sequence $(y_n)_n$ converges in measure to 0.

Proof. For each $n = 1, 2, 3, \ldots$ let the Borel function $g_n : R^n \longrightarrow R$ be given by

$$g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i - a.$$

Let further the IF-observable $y_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be given by stipulation

$$y_n = h_n \circ g_n^{-1} = g_n(x_1, \ldots, x_n).$$

Consider the measure space $(R^N, \sigma(\mathcal{C}), P)$ and random variables

$$\xi_n((t_i)_i) = t_n, (n = 1, 2, ...).$$

Then by Theorem 4.3 the random variables ξ_n are independent. Moreover $E(\xi_n) = \mathbf{E}(x_n) = a$. Therefore by the classical Weak law of large numbers we have for each $\varepsilon > 0$, $\varepsilon \in R$

$$\lim_{n\to\infty} P\Big(\Big\{(u_i)_1^{\infty}\,;\, \Big|\frac{1}{n}\sum_{j=1}^{n} \xi_j((u_i)_1^{\infty}) - a\Big| < \varepsilon\Big\}\Big) = 1.$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{m}\big(y_n((-\varepsilon,\varepsilon))\big) &= \lim_{n\to\infty} \mathbf{m}\Big(h_n\big(g_n^{-1}((-\varepsilon,\varepsilon))\big)\Big) = \lim_{n\to\infty} P\Big(\pi_n\big(g_n^{-1}((-\varepsilon,\varepsilon))\big)\Big) \\
&= \lim_{n\to\infty} P\Big(\Big\{(u_i)_1^\infty; g_n\Big(\xi_1((u_i)_1^\infty),\ldots,\xi_n((u_i)_1^\infty)\Big) \in (-\varepsilon,\varepsilon)\Big\}\Big) \\
&= \lim_{n\to\infty} P\Big(\Big\{(u_i)_1^\infty;\ \Big|\frac{1}{n}\sum_{j=1}^{n}\xi_j((u_i)_1^\infty) - a\Big| < \varepsilon\Big\}\Big) = 1. \qquad \square
\end{aligned}$$

6 Conclusion

In this paper we introduced the notion of convergence in measure $\mathbf{m}$ for intuitionistic fuzzy observables and we formulated the weak law of large numbers in this case. Since the intuitionistic fuzzy probability $\mathcal{P}$ can be decomposed to two intuitionistic fuzzy states $\mathbf{m}$ (see [14, 17]), then we can use the results from Section 5 to define a convergence in measure $\mathcal{P}$ and prove the version of weak law of large numbers in connection with IF-probability.

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