Notes on Intuitionistic Fuzzy Sets
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Some modal operators with intuitionistic fuzzy sets

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Abstract: Intuitionistic fuzzy modal operators has been studied by many researchers. The characteristics of modal operators have been examined and their applications in different fields have been studied. Atanassov introduced new modal operators with intuitionistic fuzzy sets and he examined some properties of these generalized modal operators. In this paper, we defined new modal operators over the intuitionistic fuzzy sets which undefined over intuitionistic fuzzy sets by Atanasov and we obtained some characteristics of them.
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1 Introduction

Fuzzy Set Theory was introduced by Zadeh as an extension of crisp sets [12]. Several extension of fuzzy sets were defined. L-fuzzy sets, interval-valued fuzzy sets, rough sets and intuitionistic fuzzy sets are the importance extensions. The concept of Intuitionistic fuzzy sets was introduced by Atanassov [1], form an extension of fuzzy sets by expanding the truth value set to the lattice $[0,1] \times [0,1]$ is defined as following.

Definition 1. *Let* $L = [0,1]$ *then*
$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\}$
is a lattice with $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$".
For $(x_1, y_1), (x_2, y_2) \in L^*$, *the operators* $\wedge$ *and* $\vee$ *on* $(L^*, \leq)$ *are defined as following;*
$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$
$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$
For each $J \subseteq L^*$
$\sup J = (sup\{x : (x, y \in [0,1]), ((x, y) \in J)\}, inf\{y :(x, y \in [0,1])((x, y) \in J)\})$ *and*
$\inf J = (inf\{x : (x, y \in [0,1])((x, y) \in J)\}, sup\{y : (x, y \in [0,1])((x, y) \in J)\})$.

Definition 2 ([1])**.** *An intuitionistic fuzzy set (shortly IFS) on a set* X *is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ *is called the "degree of membership of* x *in* A *",* $\nu_A(x)$, $(\nu_A : X \to [0,1])$ *is called the "degree of non-membership of* x *in* A*", and where* μ_A *and* ν_A *satisfy the following condition:*

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{for all } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.
The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 3 ([1])**.** *An IFS* A *is said to be contained in an IFS* B *(notation* $A \sqsubseteq B$*) if and only if, for all* $x \in X : \mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4 ([1])**.** *Let* $A \in IFS$ *and let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *then the above set is called the complement of A*

$$\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \,|x \in X\}$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \,|x \in X\}$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \,|x \in X\}$$

Intuitionistic fuzzy modal operators has been mentioned by Atanassov, firstly in 1999.

Definition 5 ([1])**.** *Let* X *be universal and* $A \in IFS(X)$ *then*

1. $\Box(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$
2. $\Diamond(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$

New modal operators were defined and some properties were examined by several authors. Atanassov defined some intuitionistic fuzzy modal operators over intuitionistic fuzzy sets in [3]. In this study, we will define other modal operators with intuitionistic fuzzy sets and we will examine some characteristics of them.

First, let's define the operators we will deal with. Çuvalcıoğlu has introduced $E_{\alpha,\beta}$ operator in 2007. Then the operator $Z^{\omega}_{\alpha,\beta}$ and $Z^{\omega,\theta}_{\alpha,\beta}$ were defined as an extension of $E_{\alpha,\beta}$ by same author.

Definition 6 ([6]). *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \, | x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$. We define the following operator:*

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta)\rangle \, | x \in X\}$$

Definition 7 ([7]). *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \, | x \in X\} \in IFS(X)$, $\alpha, \beta, \omega \in [0,1]$ then*

$$Z^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta)\rangle \, | x \in X\}$$

Definition 8 ([7]). *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0,1]$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta)\rangle \, | x \in X\}$$

The concept of intuitionistic fuzzy uni-type operators was defined by Çuvalcıoğlu. Some of them are as follows.

Definition 9 ([8]). *Let X be a universal, $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$ then*

1. $\boxplus^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha(\beta\nu_A(x) + \omega - \omega\beta)\rangle \, | x \in X\}$
2. $\boxtimes^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega\alpha), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle \, | x \in X\}$

Definition 10 ([8]). *Let X be a set, $A \in IFS(X)$ and $\alpha, \beta \in [0,1]$ then*

1. $B_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle \, | x \in X\}$
2. $\boxminus_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\beta)\nu_A(x)), \alpha((1-\alpha)\mu_A(x) + \nu_A(x))\rangle \, | x \in X\}$

In 2014, $\otimes_{\alpha,\beta,\gamma,\delta}$ was introduced by Atanassov and his friends.

Definition 11 ([4]). *Let X be a universal and $A \in IFS(X)$.*

$$\otimes_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, \alpha\mu_A(x) + \gamma\nu_A(x), \beta\mu_A(x) + \delta\nu_A(x)\rangle\},$$

where $\alpha, \beta, \gamma, \delta \in [0,1]$ and $\alpha + \beta \leq 1, \gamma + \delta \leq 1$.

The one type modal operators $L^{\omega}_{\alpha,\beta}$ and $K^{\omega}_{\alpha,\beta}$ were studied by Yılmaz and Bal in 2014. After then, the second type modal operators $T_{\alpha,\beta}$ and $S_{\alpha,\beta}$ were defined by Yılmaz and Çuvalcıoğlu.

Definition 12 ([11]). *Let X be a set and $A \in IFS(X), \alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$*

1. $L^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\rangle\, x \in X\}$

2. $K^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha(1-\beta)\mu_A(x) + \alpha\beta(1-\omega), \alpha\nu_A(x) + \omega(1-\alpha)\rangle\, x \in X\}$

Definition 13 **([10]).** *Let X be a set and $A \in IFS(X), \alpha, \beta, \alpha+\beta \in [0,1]$.*

1. $T_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x) + \alpha), \alpha(\nu_A(x) + (1-\beta)\mu_A(x))\rangle | x \in X\}$ *where* $\alpha + \beta \in [0,1]$.

2. $S_{\alpha,\beta}(A) = \{\langle x, \alpha(\mu_A(x) + (1-\beta)\nu_A(x)), \beta(\nu_A(x) + (1-\alpha)\mu_A(x) + \alpha)\rangle | x \in X\}$ *where* $\alpha + \beta \in [0,1]$.

2 Main results

Definition 14. *Let X be a set and $A \in IFS(X)$. The generalized modal operators are defined as follows:*

1. $E_B(A) = \left\{\left\langle \begin{array}{c} x, \nu_B(x)(\mu_B(x)\mu_A(x) + 1 - \mu_B(x)), \\ \mu_B(x)(\nu_B(x)\nu_A(x) + 1 - \nu_B(x)) \end{array} \right\rangle | x \in X \right\}$

2. $Z^{\omega}_B(A) = \left\{\left\langle \begin{array}{c} x, \nu_B(x)(\mu_B(x)\mu_A(x) + \omega - \omega\mu_B(x)), \\ \mu_B(x)(\nu_B(x)\nu_A(x) + \omega - \omega\nu_B(x)) \end{array} \right\rangle | x \in X \right\}$

3. $Z^{C}_B(A) = \left\{\left\langle \begin{array}{c} x, \nu_B(x)(\mu_B(x)\mu_A(x) + \mu_C(x) - \mu_C(x)\mu_B(x)), \\ \mu_B(x)(\nu_B(x)\nu_A(x) + \nu_C(x) - \nu_C(x)\nu_B(x)) \end{array} \right\rangle | x \in X \right\}$

4. $\boxplus^{\omega}_B(A) = \left\{\left\langle \begin{array}{c} x, \nu_B(x)(\mu_A(x) + (1 - \nu_B(x))\nu_A(x)), \\ \mu_B(x)(\nu_B(x)\nu_A(x) + \omega - \omega\nu_B(x)) \end{array} \right\rangle | x \in X \right\}$

5. $\boxtimes^{\omega}_B(A) = \left\{\left\langle \begin{array}{c} x, \nu_B(x)(\mu_B(x)\mu_A(x) + \omega - \omega\mu_B(x)), \\ \mu_B(x)((1 - \nu_B(x))\mu_A(x) + \nu_A(x)) \end{array} \right\rangle | x \in X \right\}$

6. $B_C(A) = \left\{\left\langle \begin{array}{c} x, \nu_C(x)(\mu_A(x) + (1 - \mu_C(x))\nu_A(x)), \\ \mu_C(x)((1 - \nu_C(x))\mu_A(x) + \nu_A(x)) \end{array} \right\rangle | x \in X \right\}$

7. $\boxminus_B(A) = \left\{\left\langle \begin{array}{c} x, \nu_B(x)(\mu_A(x) + (1 - \nu_B(x))\nu_A(x)), \\ \mu_B(x)((1 - \mu_B(x))\mu_A(x) + \nu_A(x)) \end{array} \right\rangle | x \in X \right\}$

8. $L^{\omega}_B(A) = \left\{\left\langle \begin{array}{c} x, \mu_B(x)\mu_A(x) + \omega(1 - \mu_B(x)), \\ \mu_B(x)(1 - \nu_B(x))\nu_A(x) + \mu_B(x)\nu_B(x)(1-\omega) \end{array} \right\rangle | x \in X \right\}$

9. $K^{\omega}_B(A) = \left\{\left\langle \begin{array}{c} x, \mu_B(x)(1 - \nu_B(x))\mu_A + \mu_B(x)\nu_B(x)(1-\omega), \\ \mu_B(x)\mu_A(x) + \omega(1 - \mu_B(x)) \end{array} \right\rangle | x \in X \right\}$

10. $\otimes_{B,C}(A) = \left\{\left\langle \begin{array}{c} x, \mu_B(x)\mu_A(x) + \mu_C(x)\nu_A(x), \\ \nu_B(x)\mu_A(x) + \nu_C(x)\nu_A(x) \end{array} \right\rangle | x \in X \right\}$

11. $T_B(A) = \left\{ \left\langle \begin{array}{c} x, \nu_B(x)(\mu_A(x) + (1-\mu_B(x))\nu_A(x) + \mu_B(x)), \\ \mu_B(x)(\nu_A(x) + (1-\nu_B(x))\mu_A(x)) \end{array} \right\rangle | x \in X \right\}$

12. $S_B(A) = \left\{ \left\langle \begin{array}{c} x, \mu_B(x)(\mu_A(x) + (1-\nu_B(x))\nu_A(x)), \\ \nu_B(x)(\nu_A(x) + (1-\mu_B(x))\mu_A(x) + \mu_B(x)) \end{array} \right\rangle | x \in X \right\}$

Theorem 1. *Let X be a set and $A, B, C \in IFS(X)$.*

1. $\overline{E_B(\bar{A})} = E_{\overline{B}}(A)$
2. $E_{\overline{B}}(\overline{A}) = \overline{E_B(A)}$
3. $\overline{Z_B^{\omega}(\bar{A})} = Z_{\overline{B}}^{\omega}(A)$
4. $Z_{\overline{B}}^{\omega}(\bar{A}) = \overline{Z_B^{\omega}(A)}$
5. $\overline{Z_B^{C}(\bar{A})} = Z_{\overline{B}}^{\overline{C}}(A)$
6. $\overline{B_C(\bar{A})} = B_{\overline{C}}(A)$
7. $B_{\overline{C}}(\bar{A}) = \overline{B_C(A)}$
8. $\overline{B_C(\bar{A})} = B_{\overline{B}}(A)$
9. $\overline{\boxminus_B(\bar{A})} = \boxminus_{\overline{B}}(A)$
10. $\boxminus_{\overline{B}}(\bar{A}) = \overline{\boxminus_B(A)}$
11. $\otimes_{\overline{B},\overline{C}}(\overline{A}) = \overline{\otimes_{B,C}(\overline{A})}$
12. $\otimes_{B,\overline{C}}(A) = \overline{\otimes_{\overline{B},C}(A)}$
13. $\otimes_{\overline{B},C}(A) = \overline{\otimes_{B,\overline{C}}(A)}$

Proof. (1)

$$\begin{aligned} \overline{E_B(\overline{A})} &= \left\{ \left\langle \begin{array}{c} x, \nu_B(x)(\mu_B(x)\nu_A(x) + 1 - \mu_B(x)), \\ \mu_B(x)(\nu_B(x)\mu_A(x) + 1 - \nu_B(x)) \end{array} \right\rangle | x \in X \right\} \\ &= \left\{ \left\langle \begin{array}{c} x, \mu_B(x)(\nu_B(x)\mu_A(x) + 1 - \nu_B(x)), \\ \nu_B(x)(\mu_B(x)\nu_A(x) + 1 - \mu_B(x)) \end{array} \right\rangle | x \in X \right\} \\ &= E_{\overline{B}}(A) \end{aligned}$$

(5)

$$\begin{aligned} Z_{\overline{B}}^{\overline{C}}(A) &= \left\{ \left\langle \begin{array}{c} x, \mu_B(x)(\nu_B(x)\mu_A(x) + \nu_C(x) - \nu_C(x)\nu_B(x)), \\ \nu_B(x)(\mu_B(x)\nu_A(x) + \mu_C(x) - \mu_C(x)\mu_B(x)) \end{array} \right\rangle | x \in X \right\} \\ &= \left\{ \left\langle \begin{array}{c} x, \mu_B(x)(\nu_B(x)\mu_A(x) + \nu_C(x) - \nu_C(x)\nu_B(x)), \\ \nu_B(x)(\mu_B(x)\nu_A(x) + \mu_C(x) - \mu_C(x)\mu_B(x)) \end{array} \right\rangle | x \in X \right\} \\ &= \overline{Z_B^{C}(\overline{A})} \end{aligned}$$

(10)

$$\begin{aligned}
\boxminus_{\overline{B}}\left(\overline{A}\right) &= \left\{\left\langle \begin{array}{c} x, \mu_B\left(x\right)\left(\nu_A\left(x\right)+\left(1-\mu_B\left(x\right)\right)\mu_A\left(x\right)\right), \\ \nu_B\left(x\right)\left(\left(1-\nu_B\left(x\right)\right)\nu_A\left(x\right)+\mu_A\left(x\right)\right) \end{array} \right\rangle | x \in X \right\} \\
&= \left\{\left\langle \begin{array}{c} x, \mu_B\left(x\right)\left(\left(1-\mu_B\left(x\right)\right)\mu_A\left(x\right)+\nu_A\left(x\right)\right), \\ \nu_B\left(x\right)\left(\mu_A\left(x\right)+\left(1-\nu_B\left(x\right)\right)\nu_A\left(x\right)\right) \end{array} \right\rangle | x \in X \right\} \\
&= \overline{\boxminus_B\left(A\right)}
\end{aligned}$$

(11)

$$\begin{aligned}
\otimes_{\overline{B},\overline{C}}\left(\overline{A}\right) &= \left\{\left\langle \begin{array}{c} x, \nu_B\left(x\right)\nu_A\left(x\right)+\nu_C\left(x\right)\mu_A\left(x\right), \\ \mu_B\left(x\right)\nu_A\left(x\right)+\mu_C\left(x\right)\mu_A\left(x\right) \end{array} \right\rangle | x \in X \right\} \\
&= \left\{\left\langle \begin{array}{c} x, \nu_B\left(x\right)\nu_A\left(x\right)+\nu_C\left(x\right)\mu_A\left(x\right), \\ \mu_B\left(x\right)\nu_A\left(x\right)+\mu_C\left(x\right)\mu_A\left(x\right) \end{array} \right\rangle | x \in X \right\} \\
&= \overline{\otimes_{B,C}\left(\overline{A}\right)}
\end{aligned}$$

The other properties can be proved similarly. □

Theorem 2. *Let X be a set and $A, B, C \in IFS(X)$.*

1. $B \subseteq C \Rightarrow E_A\left(B\right) \subseteq E_A\left(C\right)$
2. $B \subseteq C \Rightarrow Z_A^{\omega}\left(B\right) \subseteq Z_A^{\omega}\left(C\right)$
3. $A \subseteq D \Rightarrow Z_B^{C}\left(A\right) \subseteq Z_B^{C}\left(D\right)$
4. $A \subseteq C \Rightarrow L_B^{\omega}\left(A\right) \subseteq L_B^{\omega}\left(C\right)$
5. $A \subseteq C \Rightarrow K_B^{\omega}\left(A\right) \subseteq K_B^{\omega}\left(C\right)$

Proof. (1)

$$\begin{aligned}
\mu_B\left(x\right) &\leq \mu_C\left(x\right) \Rightarrow \mu_A\left(x\right)\mu_B\left(x\right) \leq \mu_A\left(x\right)\mu_C\left(x\right) \\
&\Rightarrow \mu_A\left(x\right)\mu_B\left(x\right)+1-\mu_A\left(x\right) \leq \mu_A\left(x\right)\mu_C\left(x\right)+1-\mu_A\left(x\right) \\
&\Rightarrow \nu_A\left(x\right)\left[\mu_B\left(x\right)\mu_B\left(x\right)+1-\mu_A\left(x\right)\right] \leq \nu_A\left(x\right)\left[\mu_A\left(x\right)\mu_C\left(x\right)+1-\mu_A\left(x\right)\right] \\
&\Rightarrow \mu_{E_A(B)}\left(x\right) \leq \mu_{E_A(C)}\left(x\right)
\end{aligned}$$

and

$$\begin{aligned}
\nu_B\left(x\right) &\geq \nu_C\left(x\right) \Rightarrow \nu_A\left(x\right)\nu_B\left(x\right) \geq \nu_A\left(x\right)\nu_C\left(x\right) \\
&\Rightarrow \nu_A\left(x\right)\nu_B\left(x\right)+1-\nu_A\left(x\right) \geq \nu_A\left(x\right)\nu_C\left(x\right)+1-\nu_A\left(x\right) \\
&\Rightarrow \mu_A\left(x\right)\left(\nu_A\left(x\right)\nu_B\left(x\right)+1-\nu_A\left(x\right)\right) \geq \mu_A\left(x\right)\left(\nu_A\left(x\right)\nu_C\left(x\right)+1-\nu_A\left(x\right)\right) \\
&\Rightarrow \nu_{E_A(B)}\left(x\right) \leq \nu_{E_A(C)}\left(x\right)
\end{aligned}$$

(3)

$$\begin{aligned}
\mu_A(x) &\leq \mu_D(x) \Rightarrow \mu_B(x)\,\mu_A(x) \leq \mu_B(x)\,\mu_D(x) \\
&\Rightarrow \mu_B(x)\,\mu_A(x) + \mu_C(x) - \mu_C(x)\,\mu_B(x) \\
&\leq \mu_B(x)\,\mu_D(x) + \mu_C(x) - \mu_C(x)\,\mu_B(x) \\
&\Rightarrow \nu_B(x)\left(\mu_B(x)\,\mu_A(x) + \mu_C(x) - \mu_C(x)\,\mu_B(x)\right) \\
&\leq \nu_B(x)\left(\mu_B(x)\,\mu_A(x) + \mu_C(x) - \mu_C(x)\,\mu_B(x)\right) \\
&\Rightarrow \mu_{Z_B^C(A)}(x) \leq \mu_{Z_B^C(B)}(x)
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x) &\geq \nu_D(x) \Rightarrow \nu_B(x)\,\nu_A(x) \geq \nu_B(x)\,\nu_D(x) \\
&\Rightarrow \nu_B(x)\,\nu_A(x) + \nu_C(x) - \nu_C(x)\,\nu_B(x) \\
&\geq \nu_B(x)\,\nu_D(x) + \nu_C(x) - \nu_C(x)\,\nu_B(x) \\
&\Rightarrow \mu_B(x)\left(\nu_B(x)\,\nu_A(x) + \nu_C(x) - \nu_C(x)\,\nu_B(x)\right) \\
&\geq \mu_B(x)\left(\nu_B(x)\,\nu_D(x) + \nu_C(x) - \nu_C(x)\,\nu_B(x)\right) \\
&\Rightarrow \nu_{Z_B^C(A)}(x) \leq \nu_{Z_B^C(D)}(x)
\end{aligned}$$

(4)

$$\begin{aligned}
\mu_A(x) &\leq \mu_C(x) \Rightarrow \mu_B(x)\,\mu_A(x) \leq \mu_B(x)\,\mu_C(x) \\
&\Rightarrow \mu_B(x)\,\mu_A(x) + \omega\left(1 - \mu_B(x)\right) \leq \mu_B(x)\,\mu_C(x) + \omega\left(1 - \mu_B(x)\right) \\
&\Rightarrow \mu_{L_B^\omega(A)}(x) \leq \mu_{L_B^\omega(C)}(x)
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x) &\geq \nu_C(x) \\
&\Rightarrow \mu_B(x)\left(1 - \nu_B(x)\right)\nu_A(x) \geq \mu_B(x)\left(1 - \nu_B(x)\right)\nu_C(x) \\
&\Rightarrow \mu_B(x)\left(1 - \nu_B(x)\right)\nu_A(x) + \mu_B(x)\,\nu_B(x)\left(1 - \omega\right) \\
&\geq \mu_B(x)\left(1 - \nu_B(x)\right)\nu_C(x) + \mu_B(x)\,\nu_B(x)\left(1 - \omega\right) \\
&\Rightarrow \nu_{L_B^\omega(A)}(x) \leq \mu_{L_B^\omega(C)}(x)
\end{aligned}$$

□

Theorem 3. *Let X be a set and $A, B, C, D \in IFS(X)$.*

1. $E_B(A) \sqcap E_C(A) \sqsubseteq E_{B \cap C}(A)$
2. $T_B(A) \sqcap T_C(A) \sqsubseteq T_{B \cap C}(A)$
3. $S_B(A) \sqcap S_C(A) \sqsubseteq S_{B \cap C}(A)$
4. $\boxminus_B(A) \sqcap \boxminus_C(A) \sqsubseteq \boxminus_{B \cap C}(A)$
5. $B_B(A) \sqcap B_C(A) \sqsubseteq B_{B \cap C}(A)$

Proof. (1)

$$E_{B\cap C}(A)$$

$$=\left\{\left\langle \begin{array}{c} x, \max\left(\nu_B(x), \nu_C(x)\right)\left(\min\left(\mu_B(x), \mu_C(x)\right)\mu_A(x)+1-\min\left(\mu_B(x)\mu_C(x)\right)\right) \\ \min\left(\mu_B(x), \mu_C(x)\right)\left(\max\left(\nu_B(x), v_C(x)\right)\nu_A(x)+1-\max\left(\nu_B(x), \nu_C(x)\right)\right) \end{array}\right\rangle |x\in X\right\}$$

and

$$E_B(A)\cap E_B(C)=\left\{\left\langle \begin{array}{c} x, \min\left\{\begin{array}{c} \nu_B(x)\mu_B(x)\mu_A(x)+\nu_B(x)-\mu_B(x)\nu_B(x), \\ \nu_C(x)\mu_C(x)\mu_A(x)+\nu_C(x)-\nu_C(x)\mu_C(x) \end{array}\right\} \\ \max\left\{\begin{array}{c} \mu_B(x)\nu_B(x)\nu_A(x)+\mu_B(x)-\mu_B(x)\nu_B(x) \\ \mu_C(x)\nu_C(x)\nu_A(x)+\mu_C(x)-\mu_C(x)\nu_C(x) \end{array}\right\} \end{array}\right\rangle |x\in X\right\}$$

So, it can be seen easily that $E_B(A)\sqcap E_C(A)\sqsubseteq E_{B\cap C}(A)$.

(3)

$$S_B(A)\cap S_C(A)$$

$$=\left\{\left\langle \begin{array}{c} x, \min\left\{\begin{array}{c} \mu_B(x)\mu_A(x)+\mu_B(x)\mu_A(x)(1-\nu_B(x)), \\ \mu_C(x)\mu_A(x)+\mu_C(x)\mu_A(x)(1-\nu_C(x)) \end{array}\right\}, \\ \max\left\{\begin{array}{c} \nu_B(x)\nu_A(x)+\nu_B(x)\nu_A(x)(1-\mu_B(x))+\nu_B(x)\mu_B(x), \\ \nu_C(x)\nu_A(x)+\nu_C(x)\mu_A(x)(1-\mu_C(x))+\nu_C(x)\mu_C(x) \end{array}\right\} \end{array}\right\rangle |x\in X\right\}$$

$$S_{B\cap C}(A)$$

$$=\left\{\left\langle \begin{array}{c} x, \min\left(\mu_A(x), \mu_C(x)\right)\mu_A(x)+ \\ \min\left(\mu_B(x), \mu_C(x)\right)\left(1-\max\left(\nu_B(x), \nu_C(x)\right)\right)\nu_A(x), \\ \max\left(\nu_B(x), v_C(x)\right)\nu_A(x)+ \\ \max\left(\nu_B(x), \nu_C(x)\right)\left(1-\min\left(\mu_B(x), \mu_C(x)\right)\right)\mu_A(x)+ \\ \max\left(\nu_B(x), \nu_C(x)\min\left(\mu_B(x), \mu_C(x)\right)\right) \end{array}\right\rangle |x\in X\right\}$$

We obtain that $S_B(A)\sqcap S_C(A)\sqsubseteq S_{B\cap C}(A)$.

(5)

$$B_B(A)\cap B_C(A)=\left\{\left\langle \begin{array}{c} x, \min\left\{\begin{array}{c} \nu_B(x)\mu_A(x)+\nu_B(x)\mu_A(x)(1-\mu_B(x)), \\ \nu_C(x)\mu_A(x)+\nu_C(x)\nu_A(x)(1-\mu_C(x)) \end{array}\right\}, \\ \max\left\{\begin{array}{c} \mu_B(x)\mu_A(x)(1-\nu_B(x))+\mu_B(x)\nu_A(x), \\ \mu_C(x)\mu_A(x)(1-\nu_C(x))+\mu_C(x)\mu_A(x) \end{array}\right\} \end{array}\right\rangle |x\in X\right\}$$

and

$$B_{B\cap C}(A)=\left\{\left\langle \begin{array}{c} x, \max\left(\nu_B(x), \nu_C(x)\right)\left(\mu_A(x)+\left(1-\min\left(\mu_B(x), \mu_C(x)\right)\nu_A(x)\right)\right), \\ \min\left(\mu_B(x), \mu_C(x)\right)\left(\left(1-\max\left(\nu_B(x), \nu_C(x)\right)\mu_A(x)+\nu_A(x)\right)\right) \end{array}\right\rangle\right\}$$

Similarly, $B_B(A)\sqcap B_C(A)\sqsubseteq B_{B\cap C}(A)$. □

3 Conclusion

The idea of defining modal operators over intuitionistic fuzzy sets, which was originally introduced by Atanassov, was expanded for other operators. Some of the properties obtained there were also obtained for new modal operators. In addition, new properties were obtained for the intersection operation.

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