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Numerical solution of intuitionistic fuzzy differential equations by Euler and Taylor methods

B. Ben Amma, S. Melliani* and L. S. Chadli

LMACS, Laboratoire de Mathématiques Appliquées & Calcul Scientifique
Sultan Moulay Slimane University, PO Box 523, 23000 Beni Mellal, Morocco
e-mail: said.melliani@gmail.com
* Corresponding author

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Abstract: In this paper, numerical algorithms for solving intuitionistic fuzzy differential equations are considered. A scheme based on the Euler and Taylor methods of order p are discussed in detail and this is followed by a complete error analysis. Some numerical illustrations are given to show the efficiency of the algorithms.
Keywords: Euler and p-order Taylor methods, Intuitionistic fuzzy Cauchy problem, Intuitionistic fuzzy number.
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1 Introduction

Fuzzy Differential Equation (FDE) models have wide range of applications in many branches of engineering and in the field of medicine. The concept of fuzzy derivative was first introduced by S. L. Change and L. A. Zadeh in [5]. The fuzzy differential equation and initial value problems were extensively studied by O. Kaleva in [7, 8] and by S. Seikkala in [14]. Recently, many research papers are focused on numerical solution of fuzzy initial value problems (FIVPS). Numerical Solution of fuzzy differential equations has been introduced by M. Ma, M. Friedman and A. Kandel in [6] through Euler method and by S. Abbasbandy and T. Allahviranloo in [1] by Taylor method.

The idea of intuitionistic fuzzy sets was first published by Atanassov [2,3] as a generalization of the notion of fuzzy sets. The existence and uniqueness of the solution of a differential equation with intuitionistic fuzzy data has been discussed in [11]. In this work, intuitionistic fuzzy Cauchy problem is solved numerically by Euler and Taylor of order p methods under generalised differentiability concept.

The structure of this paper is organized as follows. In Section 2, some basic results on intuitionistic fuzzy sets and metric spaces, which have been discussed in [10–12], are given. In Section 3, we define the problem that is an intuitionistic fuzzy differential equation one. Its numerical solutions are the main interest of this work. Solving numerically the intuitionistic fuzzy differential equations by Euler and Taylor methods of order p are discussed in Section 4 and 5. The proposed algorithms are illustrated by an example in Section 6 and the conclusion is in Section 7.

2 Preliminaries

An intuitionistic fuzzy set (IFS) $A \in X$ is defined as an object of the following form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in\},$$

where the functions $\mu_A, \nu_A(x) : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$, respectively, and for every $x \in X$

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Obviously, each ordinary fuzzy set may be written as

$$\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}.$$

Definition 1. *[3] The value of*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

is called the degree of non-determinacy (or uncertainty) of the element $x \in X$ to the intuitionistic fuzzy set A.

Remark 1. *[3] Clearly, in the case of ordinary fuzzy sets, $\pi_A(x) = 0$ for every $x \in X$.*

We denote by

$$IF_1 = \{\langle u, v\rangle \mid R \to [0,1]^2, \ \ \forall x \in \mathbb{R} \ \ 0 \leq u(x) + v(x) \leq 1\}$$

the collection of all intuitionistic fuzzy number by $\mathbb{F}_1$. An element $\langle u, v\rangle$ of $\mathbb{F}_1$ is called intuitionistic fuzzy number if it satisfies the following conditions

(i) is normal, i.e., there exists x_0, $x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous.

(iv) $supp(u) = cl\{x \in \mathbb{R} : v(x) < 1\}$ is bounded.

Remark 2. *[11] If $\langle u, v\rangle$ a fuzzy number, so we can see $[\langle u, v\rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v\rangle]^\alpha$ as $[1 - v]^\alpha$.*

A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u, v\rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v:

$$u(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \le x \le a_2 \\ \\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2 \le x \le a_3 \\ 0 & \text{otherwise} \end{cases},$$

$$v(x) = \begin{cases} \frac{a_2-x}{a_2-a_1'} & \text{if } a_1' \le x \le a_2 \\ \\ \frac{x-a_2}{a_3'-a_2} & \text{if } a_2 \le x \le a_3' \\ \\ 1 & \text{otherwise} \end{cases},$$

where $a_1' \le a_1 \le a_2 \le a_3 \le a_3'$ and $u(x), v(x) \le 0.5$ for $u(x) = v(x)$, $\forall x \in \mathbb{R}$.

This TIFN is denoted by $\langle u, v\rangle$=$\langle a_1, a_2, a_3; a_1', a_2, a_3'\rangle$ where,

$$[\langle u, v\rangle]_\alpha = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)],$$

$$[\langle u, v\rangle]^\alpha = [a_1' + \alpha(a_2 - a_1'), a_3' - \alpha(a_3' - a_2)].$$

Definition 2. *[10] The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$0_{(1,0)}(t) = \begin{cases} (1, 0) & t = 0 \\ (0, 1) & t \neq 0 \end{cases}.$$

Definition 3. *[10] Let $\langle u, v\rangle$, $\langle u', v'\rangle \in \mathbb{F}_1$ and $\lambda \in \mathbb{R}$, we define the following operations by :*

$$\Big(\langle u, v\rangle \oplus \langle u', v'\rangle \Big)(z) = \Big(\sup_{z=x+y} \min\left(u(x), u'(y)\right), \inf_{z=x+y} \max\left(v(x), v'(y)\right) \Big)$$

$$\lambda \langle u, v\rangle = \begin{cases} \langle \lambda u, \lambda v\rangle & \textit{if } \lambda \neq 0 \\ 0_{(1,0)} & \textit{if } \lambda = 0 \end{cases}.$$

According to Zadeh's extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space $\mathbb{F}_1$ as follows :

$$\Big[\langle u, v\rangle \oplus \langle z, w\rangle\Big]^\alpha = \Big[\langle u, v\rangle\Big]^\alpha + \Big[\langle z, w\rangle\Big]^\alpha, \quad \Big[\lambda \langle z, w\rangle\Big]^\alpha = \lambda\Big[\langle z, w\rangle\Big]^\alpha,$$
$$\Big[\langle u, v\rangle \oplus \langle z, w\rangle\Big]_\alpha = \Big[\langle u, v\rangle\Big]_\alpha + \Big[\langle z, w\rangle\Big]_\alpha, \quad \Big[\lambda \langle z, w\rangle\Big]_\alpha = \lambda\Big[\langle z, w\rangle\Big]_\alpha,$$

where $\langle u, v\rangle$, $\langle z, w\rangle \in \mathbb{F}_1$ and $\lambda \in \mathbb{R}$.

Definition 4. *[10] Let $\langle u, v\rangle$ an element of $I\!F_1$ and $\alpha \in [0,1]$, we define the following sets :*

$$[\langle u,v\rangle]_l^+(\alpha) = \inf\{x \in \mathbb{R} | u(x) \geq \alpha\}, \qquad [\langle u,v\rangle]_r^+(\alpha) = \sup\{x \in \mathbb{R} | u(x) \geq \alpha\},$$
$$[\langle u,v\rangle]_l^-(\alpha) = \inf\{x \in \mathbb{R} | v(x) \leq 1-\alpha\}, \qquad [\langle u,v\rangle]_r^-(\alpha) = \sup\{x \in \mathbb{R} | v(x) \leq 1-\alpha\}.$$

Remark 3. *[10]*

$$[\langle u,v\rangle]_\alpha = \left[[\langle u,v\rangle]_l^+(\alpha), [\langle u,v\rangle]_r^+(\alpha)\right], \qquad [\langle u,v\rangle]^\alpha = \left[[\langle u,v\rangle]_l^-(\alpha), [\langle u,v\rangle]_r^-(\alpha)\right]$$

On the space $I\!F_1$ we will consider the following L_p-metric,

Theorem 1. *[10] For $1 \leq p \leq \infty$*

$$\begin{aligned} d_p(\langle u,v\rangle, \langle z,w\rangle) = (\frac{1}{4})^{\frac{1}{p}} \Big\{ \int_0^1 \Big| [\langle u,v\rangle]_r^+(\alpha) - [\langle z,w\rangle]_r^+(\alpha) \Big|^p d\alpha \\ + \int_0^1 \Big| [\langle u,v\rangle]_l^+(\alpha) - [\langle z,w\rangle]_l^+(\alpha) \Big|^p d\alpha + \int_0^1 \Big| [\langle u,v\rangle]_r^-(\alpha) - [\langle z,w\rangle]_r^-(\alpha) \Big|^p d\alpha \\ \int_0^1 \Big| [\langle u,v\rangle]_l^-(\alpha) - [\langle z,w\rangle]_l^-(\alpha) \Big|^p d\alpha \Big\}^{\frac{1}{p}} \end{aligned}$$

and for $p = \infty$

$$\begin{aligned} d_\infty(\langle u,v\rangle, \langle z,w\rangle) = \frac{1}{4} \Big[\sup_{0<\alpha\leq 1} \Big| [\langle u,v\rangle]_r^+(\alpha) - [\langle z,w\rangle]_r^+(\alpha) \Big| \\ + \sup_{0<\alpha\leq 1} \Big| [\langle u,v\rangle]_l^+(\alpha) - [\langle z,w\rangle]_l^+(\alpha) \Big| + \sup_{0<\alpha\leq 1} \Big| [\langle u,v\rangle]_r^-(\alpha) - [\langle z,w\rangle]_r^-(\alpha) \Big| \\ + \sup_{0<\alpha\leq 1} \Big| [\langle u,v\rangle]_l^-(\alpha) - [\langle z,w\rangle]_l^-(\alpha) \Big| \Big] \end{aligned}$$

is a metric on $I\!F_1$.

Definition 5. *[12] Let $F : [a,b] \to I\!F_1$ be an intuitionistic fuzzy valued mapping and $t_0 \in [a,b]$. Then F is called intuitionistic fuzzy continuous in t_0 iff:*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall t \in [a,b] \ \textit{ such that } \ |t - t_0| < \delta) \Rightarrow d_\infty(F(t), F(t_0)) < \varepsilon.$$

Definition 6. *[11] F is called intuitionistic fuzzy continuous iff is intuitionistic fuzzy continuous in every point of $[a,b]$.*

Definition 7. *[12] Let $\langle u,v\rangle$ and $\langle u',v'\rangle \in I\!F_1$, the H-difference is the IFN $\langle z,w\rangle \in I\!F_1$, if it exists, such that*

$$\langle u,v\rangle \ominus \langle u',v'\rangle = \langle z,w\rangle \Longleftrightarrow \langle u,v\rangle = \langle u',v'\rangle \oplus \langle z,w\rangle.$$

Definition 8. *[12] A mapping $F : [a,b] \to I\!F_1$ is said to be Hukuhara derivable at t_0 if there exist $F'(t_0) \in I\!F_1$ such that both limits:*

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) \ominus F(t_0)}{\Delta t} \quad \textit{and} \quad \lim_{\Delta t \to 0^+} \frac{F(t_0) \ominus F(t_0 - \Delta t)}{\Delta t}$$

exist and they are equal to $F'(t_0) = \langle u'(t_0), v'(t_0)\rangle$, which is called the Hukuhara derivative of F at t_0. At the end points of $[a,b]$ we consider only the one-sided derivatives.

3 The intuitionistic fuzzy differential equation

In this section, we consider the initial value problem for the intuitionistic fuzzy differential equation

$$\begin{cases} x'(t) = f(t, x(t)) & t \in I \\ x(t_0) = \langle u_{t_0}, v_{t_0} \rangle \in \mathbb{F}_1 \end{cases}, \tag{1}$$

where $x \in \mathbb{F}_1$ is unknown $I = [t_0, T]$ and $f : I \times \mathbb{F}_1 \to \mathbb{F}_1$.

$x(t_0)$ is an intuitionistic fuzzy number.

Denote the $\alpha-$ level set

$$[x(t)]_\alpha = \Big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big], \quad [x(t)]^\alpha = \Big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big]$$

and

$$[x(t_0)]_\alpha = \Big[[x(t_0)]_l^+(\alpha), [x(t_0)]_r^+(\alpha)\Big] \quad [x(t_0)]^\alpha = \Big[x(t_0)]_l^-(\alpha), [x(t_0)]_r^-(\alpha)\Big]$$

$$[f(t, x(t))]_\alpha = \Big[f_1^+(t, x(t); \alpha), f_2^+(t, x(t); \alpha)\Big] \quad [f(t, x(t))]^\alpha = \Big[f_3^-(t, x(t); \alpha), f_4^-(t, x(t); \alpha)\Big]$$

where

$$\begin{aligned} f_1^+(t, x(t); \alpha) &= \min\Big\{f(t, u) | u \in \big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big]\Big\}, \\ f_2^+(t, x(t); \alpha) &= \max\Big\{f(t, u) | u \in \big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big]\Big\}, \\ f_3^-(t, x(t); \alpha) &= \min\Big\{f(t, u) | u \in \big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big]\Big\}, \\ f_4^-(t, x(t); \alpha) &= \max\Big\{f(t, u) | u \in \big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big]\Big\}. \end{aligned} \tag{2}$$

Denote

$$\begin{aligned} f_1^+(t, x(t); \alpha) &= G\big(t, [x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big), \\ f_2^+(t, x(t); \alpha) &= H\big(t, [x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big), \\ f_3^-(t, x(t); \alpha) &= L\big(t, [x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big), \\ f_4^-(t, x(t); \alpha) &= K\big(t, [x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big). \end{aligned} \tag{3}$$

The mapping $f(t, x)$ is an intuitionistic fuzzy process and the derivative $f^{(i)}(t, x)$, for $i = 1, \ldots, p$ is defined by

$$\begin{aligned} [f^{(i)}(t, x(t))]_\alpha &= \Big[f_1^{+(i)}(t, x(t); \alpha), f_2^{+(i)}(t, x(t); \alpha)\Big], \\ [f^{(i)}(t, x(t))]^\alpha &= \Big[f_3^{-(i)}(t, x(t); \alpha), f_4^{-(i)}(t, x(t); \alpha)\Big], \end{aligned}$$

provided that these equations determine the intuitionistic fuzzy number $f^{(i)}(t, x(t)) \in \mathbb{F}_1$, where

$$\begin{aligned} f_1^{+(i)}(t, x(t); \alpha) &= \min\Big\{f^{(i)}(t, u) | u \in \big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big]\Big\}, \\ f_2^{+(i)}(t, x(t); \alpha) &= \max\Big\{f^{(i)}(t, u) | u \in \big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\big]\Big\}, \\ f_3^{-(i)}(t, x(t); \alpha) &= \min\Big\{f^{(i)}(t, u) | u \in \big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big]\Big\}, \\ f_4^{-(i)}(t, x(t); \alpha) &= \max\Big\{f^{(i)}(t, u) | u \in \big[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\big]\Big\}. \end{aligned} \tag{4}$$

Denote by $C(I, \mathbb{F}_1)$ the set of all continuous mappings from I to $\mathbb{F}_1$.

Defining the metric

$$D(f, g) = \sup_{t \in I} d_\infty((f_{1,t}, f_{2,t}), (g_{1,t}, g_{2,t}))$$

with $f(t) = (f_{1,t}, f_{2,t})$ and $g(t) = (g_{1,t}, g_{2,t})$

Definition 9. *[11] $x : I \to \mathbb{F}_1$ is a solution of the initial value problem (1), if and only if it is continuous and satisfies the integral equation*

$$x(t) = x(t_0) \oplus \int_{t_0}^{t} f(s, x(s))ds.$$

Denote by $C(I \times \mathbb{F}_1, \mathbb{F}_1)$ the set of all continuous mappings from $I \times \mathbb{F}_1$ to $\mathbb{F}_1$.

Theorem 2. *[11] Assume that $f \in C(I \times \mathbb{F}_1, \mathbb{F}_1)$ and satisfies*

$$\begin{aligned}
\big|[f(s, x(s)]_r^+(\alpha) - [f(s, y(s)]_r^+(\alpha)]\big| &\leq k\big|[x(s)]_r^+(\alpha) - [y(s)]_r^+(\alpha)]\big|, \\
\big|[f(s, x(s)]_l^+(\alpha) - [f(s, y(s)]_l^+(\alpha)]\big| &\leq k\big|[x(s)]_l^+(\alpha) - [y(s)]_l^+(\alpha)]\big|, \\
\big|[f(s, x(s)]_r^-(\alpha) - [f(s, y(s)]_r^-(\alpha)]\big| &\leq k\big|[x(s)]_r^-(\alpha) - [y(s)]_r^-(\alpha)]\big|, \\
\big|[f(s, x(s)]_l^-(\alpha) - [f(s, y(s)]_l^-(\alpha)]\big| &\leq k\big|[x(s)]_l^-(\alpha) - [y(s)]_l^-(\alpha)]\big|
\end{aligned}$$

with $k|T - t_0| \leq 1$. Then the initial value problem (1) has a unique solution.

Proof. See [11]. □

4 Euler method

Let

$$[X(t_n)]_\alpha = \Big[[X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\Big], \qquad [X(t_n)]^\alpha = \Big[[X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\Big]$$

be the exact solutions of (1) and

$$[x(t_n)]_\alpha = \Big[[x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\Big], \qquad [x(t_n)]^\alpha = \Big[[x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\Big]$$

be approximated solutions at t_n, $0 \leq n \leq N$. The solutions are calculated by grid points at

$$t_0 < t_1 < t_2 < \ldots < t_N = T, \;\; h = \frac{T - t_0}{N}, \;\; t_n = t_0 + nh, \;\; n = 0, 1, \ldots, N \tag{5}$$

Euler's method is based on first order the approximation $[X(t)]_l^+(\alpha)$, $[X(t)]_r^+(\alpha)$, $X(t)]_l^-(\alpha)$ and $[X(t)]_r^-(\alpha)$, given by:

$$Z'(t, \alpha) = \frac{Z(t + h, \alpha) - Z(t, \alpha)}{h}$$

where $Z(t, \alpha)$ is $[X(t)]_l^+(\alpha)$ or $[X(t)]_r^+(\alpha)$ or $[X(t)]_l^-(\alpha)$ or $[X(t)]_r^-(\alpha)$.

We get

$$\begin{cases} [X(t_{n+1})]_l^+(\alpha) \approx [X(t_n)]_l^+(\alpha) + hG_n(\alpha) \\ [X(t_{n+1})]_r^+(\alpha) \approx [X(t_n)]_r^+(\alpha) + hH_n(\alpha) \\ [X(t_{n+1})]_l^-(\alpha) \approx [X(t_n)]_l^-(\alpha) + hL_n(\alpha) \\ [X(t_{n+1})]_r^-(\alpha) \approx [X(t_n)]_r^-(\alpha) + hK_n(\alpha) \end{cases}, \tag{6}$$

where

$$G_n(\alpha) = G\big(t_n, [X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\big), \quad H_n(\alpha) = H\big(t_n, [X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\big),$$
$$L_n(\alpha) = L\big(t_n, [X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\big), \quad K_n(\alpha) = K\big(t_n, [X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\big).$$

From Equation (6) is defined Euler's method as follows :

$$\begin{cases} [x(t_{n+1})]_l^+(\alpha) = [x(t_n)]_l^+(\alpha) + hG\big(t_n, [x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\big) \\ [x(t_{n+1})]_r^+(\alpha) = [x(t_n)]_r^+(\alpha) + hH\big(t_n, [x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\big) \\ [x(t_{n+1})]_l^-(\alpha) = [x(t_n)]_l^-(\alpha) + hL\big(t_n, [x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\big) \\ [x(t_{n+1})]_r^-(\alpha) = [x(t_n)]_r^-(\alpha) + hK\big(t_n, [x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\big) \end{cases}. \tag{7}$$

Our goal is to determine the convergence of the Euler method to exact solutions, i.e., we will show

$$d_\infty\big(x(t_n), X(t_n)\big) \longrightarrow 0 \ \ when \ \ \ h \longrightarrow 0.$$

Let $G(t, u^+, v^+)$, $H(t, u^+, v)^+$, $L(t, u^-, v)^-$ and $K(t, u^-, v^-)$ be the functions of (3), where u^+, v^+, u^- and v^- are the constants and $u^+ \leq v^+$ and $u^- \leq v^-$.

The domain of G and H is

$$M_1 = \{(t, u^+, v^+) \backslash \ t_0 \leq t \leq T, \ \ \infty < u^+ \leq v^+, \ \ -\infty < v^+ < +\infty\}$$

and the domain of L and K is

$$M_2 = \{(t, u^-, v^-) \backslash \ t_0 \leq t \leq T, \ \ \infty < u^- \leq v^-, \ \ -\infty < v^- < +\infty\},$$

where $M_1 \subseteq M_2$

The proof of the following theorem is similar to the convergence's theorem of Taylor method.

Theorem 3. *Let $G(t, u^+, v^+)$, $H(t, u^+, v^+)$ belong to $C^1(M_1)$ and $L(t, u^-, v^-)$, $K(t, u^-, v^-)$ belong to $C^1(M_2)$ and the partial derivatives of G, H and L, K be bounded over M_1 and M_2, respectively. Then, for arbitrarily fixed $0 \leq \alpha \leq 1$, the numerical solutions of (7) converge to the exact solutions $[X(t)]_l^+(\alpha)$, $[X(t)]_r^+(\alpha)$, $[X(t)]_l^-(\alpha)$ and $[X(t)]_r^-(\alpha)$ uniformly in t.*

5 Taylor method of order p

Let the exact solutions

$$[X(t)]_\alpha = \Big[[X(t)]_l^+(\alpha), [X(t)]_r^+(\alpha)\Big], \qquad [X(t)]^\alpha = \Big[[X(t)]_l^-(\alpha), [X(t)]_r^-(\alpha)\Big]$$

be approximated by

$$[x(t)]_\alpha = \Big[[x(t)]_l^+(\alpha), [x(t)]_r^+(\alpha)\Big], \qquad [x(t)]^\alpha = \Big[[x(t)]_l^-(\alpha), [x(t)]_r^-(\alpha)\Big].$$

The Taylor method of order p is based on the expansion

$$x(t+h;\alpha) = \sum_{i=0}^{p} \frac{h^i}{i!} x^{(i)}(t;\alpha), \tag{8}$$

where $x(t;\alpha)$ is $[X(t)]_l^+(\alpha)$ or $[X(t)]_r^+(\alpha)$ or $[X(t)]_l^-(\alpha)$ or $[X(t)]_r^-(\alpha)$.

We define:

$$\begin{array}{lcl} G[t,x;\alpha] & = & \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_1^{+(i)}(t,x;\alpha), \\ H[t,x;\alpha] & = & \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_2^{+(i)}(t,x;\alpha), \\ L[t,x;\alpha] & = & \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_3^{-(i)}(t,x;\alpha), \\ K[t,x;\alpha] & = & \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_4^{-(i)}(t,x;\alpha). \end{array} \tag{9}$$

The exact and approximate solutions at t_n, $0 \le n \le N$ are denoted by

$$[X(t_n)]_\alpha = \Big[[X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\Big], \qquad [X(t_n)]^\alpha = \Big[[X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\Big]$$

and

$$[x(t_n)]_\alpha = \Big[[x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\Big], \qquad [x(t_n)]^\alpha = \Big[[x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\Big],$$

respectively. The solutions are calculated at the grid points of (5).

Using the Taylor method of order p and substituting $[X(t)]_l^+$,$[X(t)]_r^+$,$[X(t)]_l^-$ and $[X(t)]_r^-$ into (8) and considering (9), we have :

$$\begin{cases} [X(t_{n+1})]_l^+(\alpha) \approx [X(t_n)]_l^+(\alpha) + hG[t, X(t_n);\alpha] \\ [X(t_{n+1})]_r^+(\alpha) \approx [X(t_n)]_r^+(\alpha) + hH[t, X(t_n);\alpha] \\ [X(t_{n+1})]_l^-(\alpha) \approx [X(t_n)]_l^-(\alpha) + hL[t, X(t_n);\alpha] \\ [X(t_{n+1})]_r^-(\alpha) \approx [X(t_n)]_r^-(\alpha) + hK[t, X(t_n);\alpha] \end{cases}. \tag{10}$$

Hence, we get

$$\begin{cases} [x(t_{n+1})]_l^+(\alpha) = [x(t_n)]_l^+(\alpha) + hG[t, x(t_n);\alpha] \\ [x(t_{n+1})]_r^+(\alpha) = [x(t_n)]_r^+(\alpha) + hH[t, x(t_n);\alpha] \\ [x(t_{n+1})]_l^-(\alpha) = [x(t_n)]_l^-(\alpha) + hL[t, x(t_n);\alpha] \\ [x(t_{n+1})]_r^-(\alpha) = [x(t_n)]_r^-(\alpha) + hK[t, x(t_n);\alpha] \end{cases}, \tag{11}$$

where

$$[x(t_0)]_\alpha = \Big[[x(t_0)]_l^+(\alpha), [x(t_0)]_r^+(\alpha)\Big], \qquad [x(t_0)]^\alpha = \Big[x(t_0)]_l^-(\alpha), [x(t_0)]_r^-(\alpha)\Big].$$

The following lemmas will be applied to show convergence of these approximations,

$$d_\infty\big(x(t_n), X(t_n)\big) \longrightarrow 0 \ \text{ when } \ h \longrightarrow 0.$$

Lemma 1. *Let the sequence of numbers* $\{W_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \le A|W_n| + B, \ \ 0 \le n \le N-1$$

for the given positive constants A *and* B. *Then*

$$|W_n| \le A^n|W_0| + B\frac{A^n - 1}{A-1}, \ \ 0 \le n \le N.$$

Proof. See [9]. □

Lemma 2. *Let the sequence of numbers* $\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \le |W_n| + A\max\{|W_n|, |V_n|\} + B$$

$$|V_{n+1}| \le |V_n| + A\max\{|W_n|, |V_n|\} + B$$

for the given positive constants A *and* B. *Then, denoting*

$$U_n = |W_n| + |V_n|, \ \ 0 \le n \le N,$$

we have

$$U_n \le \overline{A}^n|U_0| + \overline{B}\frac{\overline{A}^n - 1}{\overline{A}-1}, \ \ 0 \le n \le N,$$

where $\overline{A} = 1 + 2A$ *and* $\overline{B} = 2B$

Proof. See [9]. □

Let $G^*(t, u^+, v^+)$, $H^*(t, u^+, v)^+$, $L^*(t, u^-, v)^-$ and $K^*(t, u^-, v^-)$ be the functions of G, H, L and K, respectively, in (9), where u^+, v^+, u^- and v^- are the constants and $u^+ \le v^+$ and $u^- \le v^-$. In other words,

$$\begin{aligned} G^*(t, u^+, v^+) = &\ \textstyle\sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} \min\{f^{(i)}(t,\tau) | \tau \in [u^+, v^+]\}, \\ H^*(t, u^+, v^+) = &\ \textstyle\sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} \max\{f^{(i)}(t,\tau) | \tau \in [u^+, v^+]\}, \\ L^*(t, u^-, v^-) = &\ \textstyle\sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} \min\{f^{(i)}(t,\tau) | \tau \in [u^-, v^-]\}, \\ K^*(t, u^-, v^-) = &\ \textstyle\sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} \max\{f^{(i)}(t,\tau) | \tau \in [u^-, v^-]\}, \end{aligned} \tag{12}$$

or $G^*(t,u^+,v^+)$, $H^*(t,u^+,v^+)$, $L^*(t,u^-,v^-)$ and $K^*(t,u^-,v^-)$ are obtained by substituting $[x(t)]_\alpha = [u^+,v^+]$ and $[x(t)]^\alpha = [u^-,v^-]$ into (9).
The domain of G^* and H^* is

$$M_1 = \{(t,u^+,v^+) \backslash \ t_0 \le t \le T, \ \ \infty < u^+ \le v^+, \ \ -\infty < v^+ < +\infty\}$$

and the domain of L^* and K^* is

$$M_2 = \{(t,u^-,v^-) \backslash \ t_0 \le t \le T, \ \ \infty < u^- \le v^-, \ \ -\infty < v^- < +\infty\},$$

where $M_1 \subseteq M_2$.

Theorem 4. *Let $G^*(t,u^+,v^+)$, $H^*(t,u^+,v^+)$ belong to $C^{p-1}(M_1)$, and $L^*(t,u^-,v^-)$, $K^*(t,u^-,v^-)$ belong to $C^{p-1}(M_2)$, and the partial derivatives of G^*, H^* and L^*, K^* be bounded over M_1 and M_2, respectively. Then, for arbitrarily fixed $0 \le \alpha \le 1$, the numerical solutions of (11) converge to the exact solutions $[X(t)]_l^+(\alpha)$, $[X(t)]_r^+(\alpha)$, $[X(t)]_l^-(\alpha)$ and $[X(t)]_r^-(\alpha)$ uniformly in t.*

Proof. It is sufficient to show

$$d_\infty\big(x(t_N), X(t_N)\big) \longrightarrow 0 \ \text{ when } \ \ h \longrightarrow 0,$$

i.e.,

$$\begin{aligned}
\lim\nolimits_{h\to 0}[x(t_N)]_l^+(\alpha) &= \ [X(t_N)]_l^+(\alpha) \\
\lim\nolimits_{h\to 0}[x(t_N)]_r^+(\alpha) &= \ [X(t_N)]_r^+(\alpha) \\
\lim\nolimits_{h\to 0}[x(t_N)]_l^-(\alpha) &= \ [X(t_N)]_l^-(\alpha) \\
\lim\nolimits_{h\to 0}[x(t_N)]_r^-(\alpha) &= \ [X(t_N)]_r^-(\alpha),
\end{aligned} \tag{13}$$

where $t_N = T$. For $n = 0, 1, ..., N-1$, using the Taylor theorem, we get:

$$\begin{aligned}
&[X(t_{n+1})]_l^+(\alpha) = \\
&[X(t_n)]_l^+(\alpha) + hG^*\big(t_n, [X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\big) + \frac{h^{p+1}}{(p+1)!}[X^{(p+1)}(t_n)]_l^+(\alpha)(\zeta_{n,1}) \\
&[X(t_{n+1})]_r^+(\alpha) = \\
&[X(t_n)]_r^+(\alpha) + hH^*\big(t_n, [X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\big) + \frac{h^{p+1}}{(p+1)!}[X^{(p+1)}(t_n)]_r^+(\alpha)(\zeta_{n,2}) \\
&[X(t_{n+1})]_l^-(\alpha) = \\
&[X(t_n)]_l^-(\alpha) + hL^*\big(t_n, [X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\big) + \frac{h^{p+1}}{(p+1)!}[X^{(p+1)}(t_n)]_l^-(\alpha)(\zeta_{n,3}) \\
&[X(t_{n+1})]_r^-(\alpha) = \\
&[X(t_n)]_r^-(\alpha) + hK^*\big(t_n, [X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\big) + \frac{h^{p+1}}{(p+1)!}[X^{(p+1)}(t_n)]_r^-(\alpha)(\zeta_{n,4}),
\end{aligned} \tag{14}$$

where $\zeta_{n,1}, \zeta_{n,2}, \zeta_{n,3}, \zeta_{n,4} \in (t_n, t_{n+1})$. Denoting

$$\begin{aligned} W_n^+ = \; & [X(t_n)]_l^+(\alpha) - [x(t_n)]_l^+(\alpha) \quad , \qquad V_n^+ = [X(t_n)]_r^+(\alpha) - [x(t_n)]_r^+(\alpha) \\ W_n^- = \; & [X(t_n)]_l^-(\alpha) - [x(t_n)]_l^-(\alpha) \quad , \qquad V_n^- = [X(t_n)]_r^-(\alpha) - [x(t_n)]_r^-(\alpha) \end{aligned} \tag{15}$$

from (11) and (14) it follows that:

$$W_{n+1}^+ = W_n^+ + h\Big\{G^*\big(t_n, [X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\big) - G^*\big(t_n, [x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\big)\Big\} + \frac{h^{p+1}}{(p+1)!}[X^{(p+1)}(t_n)]_l^+(\alpha)(\zeta_{n,1})$$

$$V_{n+1}^+ = V_n^+ + h\Big\{H^*\big(t_n, [X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\big) - H^*\big(t_n, [x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\big)\Big\} + \frac{h^{p+1}}{(p+1)!}[X^{(p+1)}(t_n)]_r^+(\alpha)(\zeta_{n,2})$$

$$W_{n+1}^- = W_n^- + h\Big\{L^*\big(t_n, [X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\big) - L^*\big(t_n, [x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\big)\Big\} + \frac{h^{p+1}}{(p+1)!}[X^{(p+1)}(t_n)]_l^-(\alpha)(\zeta_{n,3})$$

$$V_{n+1}^- = V_n^- + h\Big\{K^*\big(t_n, [X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\big) - K^*\big(t_n, [x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\big)\Big\} + \frac{h^{p+1}}{(p+1)!}[X^{(p+1)}(t_n)]_r^-(\alpha)(\zeta_{n,4})$$

Hence,

$$\begin{aligned} |W_{n+1}^+| \leq \; & |W_n^+| + 2hD^+ \max\{|W_n^+|, |V_n^+|\} + \tfrac{h^{p+1}}{(p+1)!}C, \\ |V_{n+1}^+| \leq \; & |V_n^+| + 2hD^+ \max\{|W_n^+|, |V_n^+|\} + \tfrac{h^{p+1}}{(p+1)!}C, \\ |W_{n+1}^-| \leq \; & |W_n^-| + 2hD^- \max\{|W_n^-|, |V_n^-|\} + \tfrac{h^{p+1}}{(p+1)!}C, \\ |V_{n+1}^-| \leq \; & |V_n^-| + 2hD^- \max\{|W_n^-|, |V_n^-|\} + \tfrac{h^{p+1}}{(p+1)!}C, \end{aligned} \tag{16}$$

where

- ⋄ $C_1^+ = \max\Big|[X^{(p+1)}(t_n)]_l^+(\alpha)\Big|$, $C_2^+ = \max\Big|[X^{(p+1)}(t_n)]_r^+(\alpha)\Big|$,
- ⋄ $C_1^- = \max\Big|[X^{(p+1)}(t_n)]_l^-(\alpha)\Big|$, $C_2^- = \max\Big|[X^{(p+1)}(t_n)]_r^-(\alpha)\Big|$,
- ⋄ $C^+ = \max\{C_1^+, C_2^+\}$, $C^- = \max\{C_1^-, C_2^-\}$

for $t \in [t_0, T]$.

$C = \max\{C^+, C^-\}$ and $D^+ > 0$ is a bound for the partial derivatives of G^* and H^*, and $D^- > 0$ is a bound for the partial derivatives of L^* and K^*. Therefore, from Lemma 2, we obtain

$$\begin{aligned} |W_n^+| \leq & \quad (1+4hD^+)^n |U_0^+| + \tfrac{2h^{p+1}}{(p+1)!} C \tfrac{(1+4hD^+)^n - 1}{4hD^+}, \\ |V_n^+| \leq & \quad (1+4hD^+)^n |U_0^+| + \tfrac{2h^{p+1}}{(p+1)!} C \tfrac{(1+4hD^+)^n - 1}{4hD^+}, \\ |W_n^-| \leq & \quad (1+4hD^-)^n |U_0^-| + \tfrac{2h^{p+1}}{(p+1)!} C \tfrac{(1+4hD^-)^n - 1}{4hD^-}, \\ |V_n^-| \leq & \quad (1+4hD^-)^n |U_0^-| + \tfrac{2h^{p+1}}{(p+1)!} C \tfrac{(1+4hD^-)^n - 1}{4hD^-}, \end{aligned} \tag{17}$$

where $|U_0^+| = |W_0^+| + |V_0^-|$ et $|U_0^-| = |W_0^-| + |V_0^-|$.

In particular

$$\begin{aligned} |W_N^+| \leq & \quad (1+4hD^+)^N |U_0^+| + \tfrac{h^{p+1}}{(p+1)!} C \tfrac{(1+4hD^+)^{\frac{T}{h}} - 1}{2hD^+}, \\ |V_N^+| \leq & \quad (1+4hD^+)^N |U_0^+| + \tfrac{h^{p+1}}{(p+1)!} C \tfrac{(1+4hD^+)^{\frac{T}{h}} - 1}{2hD^+}, \\ |W_N^-| \leq & \quad (1+4hD^-)^N |U_0^-| + \tfrac{h^{p+1}}{(p+1)!} C \tfrac{(1+4hD^-)^{\frac{T}{h}} - 1}{2hD^-}, \\ |V_N^-| \leq & \quad (1+4hD^-)^N |U_0^-| + \tfrac{h^{p+1}}{(p+1)!} C \tfrac{(1+4hD^-)^{\frac{T}{h}} - 1}{2hD^-}. \end{aligned} \tag{18}$$

Since $|W_0^+| = |V_0^+| = |W_0^-| = |V_0^-| = 0$, we have

$$|W_N^+| \leq C \frac{e^{4TD^+} - 1}{2D^+(p+1)!} h^p, \quad |V_N^+| \leq C \frac{e^{4TD^+} - 1}{2D^+(p+1)!} h^p,$$

$$|W_N^-| \leq C \frac{e^{4TD^-} - 1}{2D^-(p+1)!} h^p, \quad |V_N^-| \leq C \frac{e^{4TD^-} - 1}{2D^-(p+1)!} h^p.$$

We have

$$d_\infty(x(t_N), X(t_N)) \quad = \quad \frac{1}{4} \Big[\sup_{0<\alpha\leq 1} |W_N^+| + \sup_{0<\alpha\leq 1} |V_N^+| + \sup_{0<\alpha\leq 1} |W_N^-| + \sup_{0<\alpha\leq 1} |V_N^-| \Big].$$

Thus, if $h \to 0$, we get $W_N^+ \to 0$, $V_N^+ \to 0$, $W_N^- \to 0$ and $V_N^- \to 0$, which completes the proof. $\square$

6 Example

Consider the intuitionistic fuzzy initial value problem

$$\begin{cases} x'(t) + x(t) & = \quad \sigma(t), \forall t \geq 0 \\ \qquad x_0 & = \quad (-1, 1, 0, -\frac{3}{2}, \frac{3}{2}) \end{cases} \tag{19}$$

and $\sigma(t) = 2\exp(-t)x_0$.

Applying the method of solution proposed in [11] we get the differential system

$$\begin{cases} [x'(t)]_l^+(\alpha) + [x(t)]_l^+(\alpha) = & 2(\alpha-1)\exp(-t) \quad , \quad [x(0)]_l^+ = (\alpha-1) \\ [x'(t)]_r^+(\alpha) + [x(t)]_r^+(\alpha) = & 2(1-\alpha)\exp(-t) \quad , \quad [x(0)]_r^+ = (1-\alpha) \\ [x'(t)]_l^-(\alpha) + [x(t)]_l^-(\alpha) = & -3\alpha\exp(-t) \quad , \quad [x(0)]_l^- = -\frac{3}{2}\alpha \\ [x'(t)]_r^-(\alpha) + [x(t)]_r^-(\alpha) = & 3\alpha\exp(-t) \quad , \quad [x(0)]_r^- = \frac{3}{2}\alpha \end{cases},$$

we find

$$\begin{cases} [x(t)]_l^+(\alpha) = & (\alpha-1)\exp(-t)(1+2t) \\ [x(t)]_r^+(\alpha) = & (1-\alpha)\exp(-t)(1+2t) \\ [x(t)]_l^-(\alpha) = & (3t+\frac{3}{2})(\alpha-1))\exp(-t) \\ [x(t)]_r^-(\alpha) = & (3t+\frac{3}{2})(1-\alpha))\exp(-t) \end{cases}.$$

Therefore, the exact solutions is given by

$$[X(t)]_\alpha = \Big[(\alpha-1)\exp(-t)(1+2t), (1-\alpha)\exp(-t)(1+2t)\Big],$$

$$[X(t)]^\alpha = \Big[(3t+\tfrac{3}{2})(\alpha-1)\exp(-t), (3t+\tfrac{3}{2})(1-\alpha)\exp(-t)\Big],$$

which at $t = 0.3$ are

$$[X(0.3)]_\alpha = \Big[(\alpha-1)\exp(-0.3)(1.6), (1-\alpha)\exp(-0.3)(1.6)\Big],$$

$$[X(0.3)]^\alpha = \Big[(0.9+\tfrac{3}{2})(\alpha-1)\exp(-0.3), (0.9+\tfrac{3}{2})(1-\alpha)\exp(-0.3)\Big].$$

Using the Euler method, we have

$$[x(t_{n+1})]_l^+(\alpha) = (1-h)[x(t_n)]_l^+(\alpha) + h[\sigma(t_n)]_l^+(\alpha)$$

$$[x(t_{n+1})]_r^+(\alpha) = (1-h)[x(t_n)]_r^+(\alpha) + h[\sigma(t_n)]_r^+(\alpha)$$

$$[x(t_{n+1})]_l^-(\alpha) = (1-h)[x(t_n)]_l^-(\alpha) + h[\sigma(t_n)]_l^-(\alpha)$$

$$[x(t_{n+1})]_r^-(\alpha) = (1-h)[x(t_n)]_r^-(\alpha) + h[\sigma(t_n)]_r^-(\alpha)$$

and the Taylor method of order p, we have

$$[x(t_{n+1})]_l^+(\alpha) = [x(t_n)]_l^+(\alpha) + \sum_{i=0}^{p-1} \frac{h^{i+1}}{(i+1)!}\big([\sigma^{(i)}(t_n)]_l^+(\alpha) - [x^{(i)}(t_n)]_l^+(\alpha)\big)$$

$$[x(t_{n+1})]_r^+(\alpha) = [x(t_n)]_r^+(\alpha) + \sum_{i=0}^{p-1} \frac{h^{i+1}}{(i+1)!}\big([\sigma^{(i)}(t_n)]_r^+(\alpha) - [x^{(i)}(t_n)]_r^+(\alpha)\big)$$

$$[x(t_{n+1})]_l^-(\alpha) = [x(t_n)]_l^-(\alpha) + \sum_{i=0}^{p-1} \frac{h^{i+1}}{(i+1)!}\big([\sigma^{(i)}(t_n)]_l^-(\alpha) - [x^{(i)}(t_n)]_l^-(\alpha)\big)$$

$$[x(t_{n+1})]_r^-(\alpha) = [x(t_n)]_r^-(\alpha) + \sum_{i=0}^{p-1} \frac{h^{i+1}}{(i+1)!}\big([\sigma^{(i)}(t_n)]_r^-(\alpha) - [x^{(i)}(t_n)]_r^-(\alpha)\big)$$

The exact and approximate solutions by Euler method are plotted at $t = 0.3$ and $h = 0.25$ in Fig. 1.

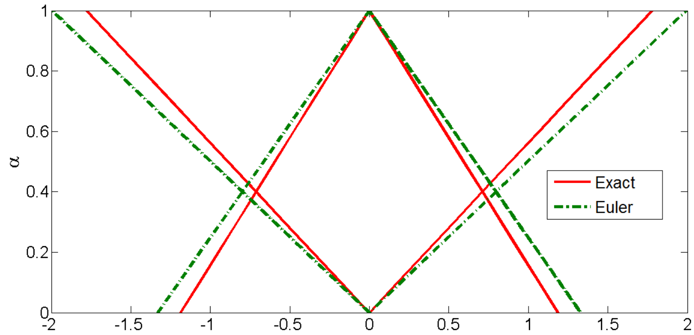

Figure 1. $h = 0.25$

The exact and approximate solutions for $p = 2$ and $p = 4$ are compared and plotted at $t = 0.3$ and $h = 0.25$ in Figs. 2 and 3.

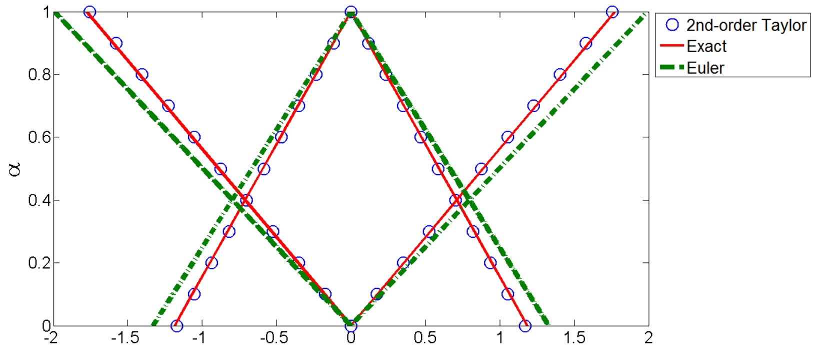

Figure 2. $h = 0.25$

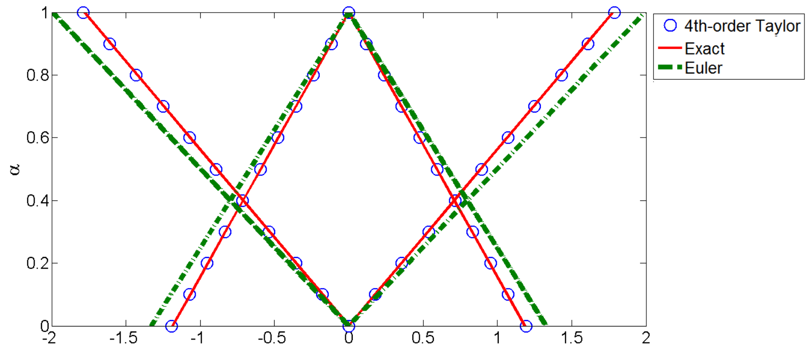

Figure 3. $h = 0.25$

The error between the Euler and the 2nd-order and 4th-order Taylor method is plotted in Fig. 4.

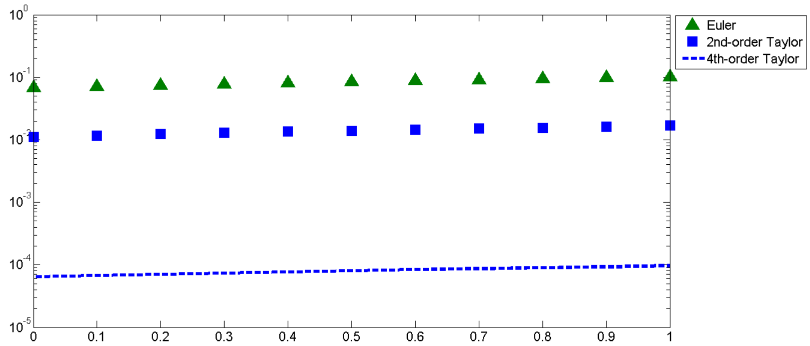

Figure 4. $h = 0.25$

7 Conclusion

It is shown that the convergence order of the Taylor method is $O(h^p)$, while the Euler method converges with the rate $O(h)$ only. Comparison of the solutions of the example shows that the Taylor method gives a better solution than the Euler method.

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