Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy probability and two theorems from extreme value theory

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Abstract: The aim of this contribution is to formulate a variation of Fisher–Tipett–Gnedenko theorem and a variation of Pickand–Balkema–de Haan theorem for intuitionistic fuzzy observables using an intuitionistic fuzzy probability. Such the intuitionistic fuzzy probability can be decomposed to two intuitionistic fuzzy states, we will try to apply the results from extreme value theory, which were proving in connection with the intuitionistic fuzzy state, see papers [4, 8].
Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy probability, Intuitionistic fuzzy observables, Joint intuitionistic fuzzy observables, Independence, Excess intuitionistic fuzzy distribution, Maximum domain of attraction for intutionistic fuzzy probability, Fisher–Tipett–Gnedenko theorem, Pickands–Balkema–de Haan theorem.
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1 Introduction

It is known that the classical theory of extreme values is used in statistics to analyze extreme events or values. It is used in various fields, most often in finance, for example, to model extreme returns from stock markets (see [5]). In papers [4, 8] we studied two theorems from extreme value theory in case of intuitionistic fuzzy sets. We proved a variation of Fisher–Tipett–Gnedenko theorem and a variation of Pickand–Balkema–de Haan theorem for intuitionistic fuzzy observables in connection with an intuitionistic fuzzy state.

The theory of intuitionistic fuzzy sets is now more than 40 years old. It was founded by K. T. Atanassov in 1983 (see [1]) and it is a generalization of Zadeh's theory of fuzzy sets [20, 21].

An *intuitionistic fuzzy set* $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$. Namely if $\mu_A : \Omega \longrightarrow [0,1]$ is a fuzzy set, then $\mathbf{A} = (\mu_A, 1 - \mu_A)$ is the corresponding intuitionistic fuzzy set.

In this paper we will work with intuitionistic fuzzy events. As an *intuitionistic fuzzy event* we understand an intuitionistic fuzzy set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable (see [2, 3, 16]). The family of all intuitionistic fuzzy events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$.

We will also work with the notion of intuitionistic fuzzy probability. In the paper [13, 14] B. Riečan formulated its axiomatic definition. By *intuitionistic fuzzy probability* he means a mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$, which is satisfying the following conditions:

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1,1]$, $\mathcal{P}((0_\Omega, 1_\Omega)) = [0,0]$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, then $\mathcal{P}(\mathbf{A} \oplus \mathbf{B}) = \mathcal{P}(\mathbf{A}) + \mathcal{P}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$, then $\mathcal{P}(\mathbf{A}_n) \nearrow \mathcal{P}(\mathbf{A})$.

There $[\alpha_n, \beta_n] \nearrow [\alpha, \beta]$ means that $\alpha_n \nearrow \alpha$, $\beta_n \nearrow \beta$, but $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n}) \nearrow \mathbf{A} = (\mu_A, \nu_A)$ means $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$.

The Lukasiewicz binary operations $\oplus, \odot$ on the family of intuitionistic fuzzy events $\mathcal{F}$ are given as follows

$$
\begin{aligned}
\mathbf{A} \oplus \mathbf{B} &= ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)), \\
\mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))
\end{aligned}
$$

for $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$ and the partial ordering is given by

$$
\mathbf{A} \leq \mathbf{B} \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.
$$

A very important fact is that the intuitionistic fuzzy probability can be expressed using two intuitionistic fuzzy states, see [15]. There B. Riečan proved that $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ is an intuitionistic fuzzy probability if and only if $\mathcal{P}^\flat : \mathcal{F} \to [0,1]$ and $\mathcal{P}^\sharp : \mathcal{F} \to [0,1]$ are intuitionistic fuzzy states, where $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$. Such $[\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ is an interval, then $\mathcal{P}^\flat(\mathbf{A}) \leq \mathcal{P}^\sharp(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{F}$.

Recall that by an *intuitionistic fuzzy state* we understand a mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ which is satisfying the following three conditions:

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e., $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$, see [18]. In this article, we will try to use the results from extreme value theory proven for intuitionistic fuzzy states in [4, 8] and apply them to intuitionistic fuzzy probability.

The article is organized as follows. In the first part, we explain the basic notions from intuitionistic fuzzy theory connected with intuitionistic fuzzy probability. In the second part, we recall the basic notions from extreme value theory valid for intuitionistic fuzzy observables and for intuitionistic fuzzy states. We will prove a variation of Fisher–Tipett–Gnedenko theorem and a variation of Pickand–Balkema–de Haan theorem for intuitionistic fuzzy observables using an intuitionistic fuzzy probability.

Throughout the text we will use the abbreviation "IF" for "intuitionistic fuzzy".

2 Basic notions

In this section we will explain some basic notions like IF-observable, IF-mean value, IF-dispersion and IF-distribution function in relation to the IF-probability. In the next section, we will work with independent IF-observables, so we will also explain independence there.

By the notion of *n-dimensional IF-observable* in IF-space $(\mathcal{F}, \mathcal{P})$ we mean each mapping $x : \mathcal{B}(R^n) \to \mathcal{F}$ satisfying the following conditions:

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\varnothing) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \varnothing$ and $A, B \in \mathcal{B}(R^n)$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$ and $A_n, A \in \mathcal{B}(R^n)$, $n \in N$, then $x(A_n) \nearrow x(A)$.

For $n = 1$ we say simply about IF-observable, see [18]. There $\mathcal{B}(R)$ is the σ-algebra of the family of all intervals in R of the form $[a, b) = \{x \in R : a \leq x < b\}$, (see [19]).

In [12] we were defined the moments for IF-observable in the sense of IF-probability.

Definition 2.1. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability, $\mathcal{P}^\flat, \mathcal{P}^\sharp$ be the corresponding IF-states and $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. We say that IF-observable x is integrable, if the integrals $\int_R t\, d\mathcal{P}_x^\flat(t)$, $\int_R t\, d\mathcal{P}_x^\sharp(t)$ exist. Then the IF-mean values are defined by*

$$\mathbf{E}^\flat(x) = \int_R t\, d\mathcal{P}_x^\flat(t) \quad , \quad \mathbf{E}^\sharp(x) = \int_R t\, d\mathcal{P}_x^\sharp(t).$$

We say that IF-observable x is square integrable, if the integrals $\int_R t^2\, d\mathcal{P}_x^\flat(t)$, $\int_R t^2\, d\mathcal{P}_x^\sharp(t)$ exist. Then the IF-dispersions are defined by

$$\begin{aligned} \mathbf{D}_\flat^2(x) &= \int_R (t - \mathbf{E}^\flat(x))^2\, d\mathcal{P}_x^\flat(t) = \int_R t^2\, d\mathcal{P}_x^\flat(t) - \big(\mathbf{E}^\flat(x)\big)^2, \\ \mathbf{D}_\sharp^2(x) &= \int_R (t - \mathbf{E}^\sharp(x))^2\, d\mathcal{P}_x^\sharp(t) = \int_R t^2\, d\mathcal{P}_x^\sharp(t) - \big(\mathbf{E}^\sharp(x)\big)^2. \end{aligned}$$

We can see that by Definition 2.1 the IF-mean value (or IF-dispersion) of IF-observable x induced by IF-probability $\mathcal{P}$ consists of two IF-mean values (or IF-dispersions) induced by IF-state, i.e., $\mathbf{E}(x) = [\mathbf{E}^\flat(x), \mathbf{E}^\sharp(x)]$ and $\mathbf{D}^2(x) = [\mathbf{D}_\flat^2(x), \mathbf{D}_\sharp^2(x)]$.

Now we recall the definition of an IF-distribution function in the sense of IF-probability, see [10].

Definition 2.2. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be an IF-probability and $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. Then a mapping $\mathbf{F} : R \to \mathcal{J}$ defined by formula*

$$\mathbf{F}(t) = \mathcal{P} \circ x((-\infty, t)) = \left[\mathcal{P}^\flat\big((x(-\infty, t))\big), \mathcal{P}^\sharp\big((x(-\infty, t))\big)\right] = [\mathbf{F}^\flat(t), \mathbf{F}^\sharp(t)]$$

for each $t \in R$ is called IF- distribution function related to the IF-probability $\mathcal{P}$, where $\mathbf{F}^\flat, \mathbf{F}^\sharp : R \to [0,1]$ are the corresponding IF-distribution functions related to the IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.

Remark 2.1. *Such $[\mathbf{F}^\flat(t), \mathbf{F}^\sharp(t)]$ is an interval, then $\mathbf{F}^\flat(t) \leq \mathbf{F}^\sharp(t)$ for each $t \in R$.*

In [18] B. Riečan defined *IF-distribution function related to the IF-state* as the function $\mathbf{F} : R \longrightarrow [0,1]$ given by the formula $\mathbf{F}(t) = \mathbf{m}\big(x((-\infty, t))\big)$ for each $t \in R$. There $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ is an IF-observable and $\mathbf{m} : \mathcal{F} \longrightarrow [0,1]$ is an IF-state.

In the next section, we will work with independent IF-observables, so we will explain what independence is in relation to IF-probability, see [17].

Definition 2.3. *Let $\mathcal{P}$ be an IF-probability, $\mathcal{P}^\flat, \mathcal{P}^\sharp$ be the corresponding IF-states. Then the IF-observables $x_1, x_2, \ldots, x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ are independent with respect to $\mathcal{P}$ if for the n-dimensional IF-observable $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ there holds*

$$\mathcal{P}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \Big[\mathcal{P}^\flat\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\flat\big(x_n(A_n)\big), \mathcal{P}^\sharp\big(x_1(A_1)\big) \cdot \ldots \cdot \mathcal{P}^\sharp\big(x_n(A_n)\big)\Big]$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, $n \in N$. The map h_n is called the joint IF-observable of $x_1, x_2, \ldots, x_n$.

By a *joint IF-observable* of two IF-observables x, y we understand a two-dimensional IF-observable $h : \mathcal{B}(R^2) \to \mathcal{F}$, which satisfies the following condition:

$$h(C \times D) = x(C) * y(D)$$

for each $C, D \in \mathcal{B}(R)$, see [16]. There $*$ is the product operation on the family of IF-events $\mathcal{F}$ given by $\mathbf{A} * \mathbf{B} = \big(\mu_A \cdot \mu_B, 1 - (1 - \nu_A) \cdot (1 - \nu_B)\big) = \big(\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B\big)$ for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$ and $\cdot$ is a multiplication, see [11].

There is the following relationship between the independence of IF-observables related to IF-probability and the independence of IF-observables related to IF-states.

Theorem 2.1. *Let $\mathcal{P}$ be an IF-probability, $\mathcal{P}^\flat, \mathcal{P}^\sharp$ be the corresponding IF-states. The IF-observables $x_1, x_2, \ldots, x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ are independent with respect to IF-probability $\mathcal{P}$ if and only if the IF-observables $x_1, x_2, \ldots, x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ are independent with respect to IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.*

Proof. Proof is straightforward. □

Recall that the *IF-observables* $x_1, x_2, \ldots, x_n : \mathcal{B}(R) \longrightarrow \mathcal{F}$ are independent with respect to IF-state $\mathbf{m}$ if for the n-dimensional IF-observable $h_n : \mathcal{B}(R^n) \longrightarrow \mathcal{F}$ there holds

$$\mathbf{m}\big(h_n(A_1 \times A_2 \times \ldots \times A_n)\big) = \mathbf{m}\big(x_1(A_1)\big) \cdot \mathbf{m}\big(x_2(A_2)\big) \cdot \ldots \cdot \mathbf{m}\big(x_n(A_n)\big)$$

for each $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$, see [9].

3 Two theorems from extreme value theory

In this section, we will discuss variations of Fisher–Tippett–Gnedenko theorem and Pickands–Balkema–de Haan theorem in relation to IF-observables. In the first subsection, we will present these two theorems in relation to IF-state, and in the second subsection, we will show the formulations of these theorems for IF-probability. In both subsections, we will work with functions of several IF-observables, so we will explain what we mean by this term, see [6].

Definition 3.1. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be IF-observables, h_n be their joint IF-observable and $g_n : R^n \to R$ be a Borel measurable function. Then we define the IF-observable $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ by the formula*

$$g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big)$$

for each $A \in \mathcal{B}(R)$.

Example 3.1. *Let $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ be the IF-observables and $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be their joint IF-observable. Then the IF-observable $y_n = g_n(x_1, \ldots, x_n) = \frac{1}{a_n}\big(\max(x_1, \ldots, x_n) - b_n\big)$ is defined by the equality $y_n = h_n \circ g_n^{-1}$, where $g_n(u_1, \ldots, u_n) = \frac{1}{a_n}\big(\max(u_1, \ldots, u_n) - b_n\big)$ for all real numbers $u_1, \ldots, u_n$.*

To conclude this section, let us repeat what we mean by convergence in distribution in relation to IF-probability, see [7].

Definition 3.2. *Let $(y_n)_n$ be a sequence of IF-observables in the IF-space $(\mathcal{F}, \mathcal{P})$, $\mathcal{P}$ be an IF-probability. We say that $(y_n)_n$ converges in distribution to a function $\Psi : R \longrightarrow [0,1]$, if for each $t \in R$*

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\infty, t))\big) = \big[\Psi(t), \Psi(t)\big] = \big\{\Psi(t)\big\}.$$

Of course there is a relation between a convergence in distribution with respect to the IF-probability $\mathcal{P}$ and a convergence in distribution with respect to the corresponding IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$, see [7].

Theorem 3.1. *A sequence $(y_n)_n$ of an IF-observables converges in distribution to a function $\Psi : R \longrightarrow [0,1]$ with respect to the IF-probability $\mathcal{P}$ if and only if it converges in distribution to a function Ψ with respect to the IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.*

Recall that we say that a sequence of IF-observables $(y_n)_n$ converges in distribution with respect to the IF-state to a function $\Psi : R \longrightarrow [0,1]$, if

$$\lim_{n\to\infty} \mathbf{m}\big(y_n((-\infty, t))\big) = \Psi(t)$$

for each $t \in R$, see [9].

3.1 Two theorems from extreme value theory in the sense of IF-state

In this subsection we will recall the variations of Fisher–Tippett–Gnedenko theorem and Pickands–Balkema–de Haan theorem for IF-states. The results can be found in the papers [4, 8]. We will use them in the next subsection.

Let $x_1, x_2, \ldots$ be independent, equally distributed IF-observables on $\mathcal{F}$. Denote by $\mathbf{M}_n$ the maximum of n IF-observables: $\mathbf{M}_1 = x_1$, $\mathbf{M}_n = \max(x_1, \ldots, x_n)$ for $n \geq 2$.

Theorem 3.2. (Variation of Fisher–Tippett–Gnedenko theorem for an IF-state). *Let $x_1, x_2, \ldots$ be a sequence of independent, equally distributed IF-observables such that $\mathbf{D}^2(x_n) = \sigma^2$, $\mathbf{E}(x_n) = \mu, (n = 1, 2, \ldots)$. If there exist two sequences of real constants $a_n > 0$ and b_n and a non-degenerate distribution function H, such that*

$$\lim_{n\to\infty} \mathbf{m}\bigg(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\bigg) = H(t),$$

then the distribution function H is one of the following three types of distributions:

1. *Gumbel*
$$H_{\mu,\sigma}(t) = \exp\left(-e^{-\left(\frac{t-\mu}{\sigma}\right)}\right), \quad t \in R,$$
2. *Fréchet*
$$H_{\mu,\sigma,\alpha}(t) = \begin{cases} 0, & \text{for } t \leq \mu, \\ \exp\left(-\left(\frac{t-\mu}{\sigma}\right)^{-\alpha}\right), & \text{for } t > \mu, \alpha > 0, \end{cases}$$
3. *Weibull*
$$H_{\mu,\sigma,\alpha}(t) = \begin{cases} \exp\left(-\left(-\frac{t-\mu}{\sigma}\right)^{\alpha}\right), & \text{for } t \leq \mu, \alpha > 0, \\ 1, & \text{for } t > \mu. \end{cases}$$

There a parameter $\mu \in R$ is the location parameter and a parameter $\sigma > 0$ is the scale parameter.

Let x be an IF-observable on $\mathcal{F}$ and $\mathbf{F}$ be an IF-distribution function of x connected with IF-state $\mathbf{m}$. We denote by

$$t_{\mathbf{F}} = \sup\{t \in R : \mathbf{F}(t) < 1\}$$

the right endpoint of IF-distribution function $\mathbf{F}$ connected with IF-state $\mathbf{m}$.

Definition 3.3. (Maximum domain of attraction for IF-state). *We say that the IF-distribution function $\mathbf{F}$ connected with an IF-state $\mathbf{m}$ of IF-observable x_i belongs to the maximum domain of attraction of the extreme value distributions H if there exist two sequences of real constants $a_n > 0$ and b_n, such that*

$$\lim_{n\to\infty} \mathbf{m}\bigg(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\bigg) = H(t)$$

*holds. We write $\mathbf{F} \in$ **MDA**(H).*

Definition 3.4. (Excess IF-distribution function for IF-state). *Let* $\mathbf{F}$ *be an IF-distribution function of IF-observable* x *connected with IF-state* $\mathbf{m}$ *with right endpoint* $t_{\mathbf{F}}$. *For fixed* $u < t_{\mathbf{F}}$, $u > 0$,

$$\mathbf{F}_u(t) = \frac{\mathbf{F}(t+u) - \mathbf{F}(u)}{1 - \mathbf{F}(u)}, \quad 0 \le t \le t_{\mathbf{F}} - u$$

is the excess IF-distribution function of the IF-observable x *connected with IF-state* $\mathbf{m}$ *(of the IF-distribution function* $\mathbf{F}$ *connected with IF-state* $\mathbf{m}$*) over the threshold* u.

Theorem 3.3. (Variation of Pickands–Balkema–de Haan theorem for IF-state). *For every* $\varepsilon \in R$,

$$\mathbf{F} \in \mathbf{MDA}(H_\varepsilon) \quad \Longleftrightarrow \quad \lim_{u \to t_{\mathbf{F}}} \sup_{0<t<t_{\mathbf{F}}-u} |\mathbf{F}_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0$$

for some positive function β.

Remark 3.1. *The previous theorem says that for some function* β *to be estimated from the data, the excess IF-distribution* $\mathbf{F}_u$ *connected with IF-state* $\mathbf{m}$ *converges to the generalised Pareto distribution* $G_{\varepsilon,\beta}$ *for large* u.

Recall that by *generalized Pareto distribution (GPD)* we understand the distribution function $G_{\varepsilon,\beta}$ defined by

$$G_{\varepsilon,\beta}(x) = \begin{cases} 1 - \left(1 + \varepsilon \cdot \frac{x}{\beta}\right)^{-\frac{1}{\varepsilon}}, & \text{if } \varepsilon \neq 0, \\ 1 - e^{-\frac{x}{\beta}}, & \text{if } \varepsilon = 0, \end{cases}$$

where

$$\begin{aligned} x \ge 0 \quad & \text{if } \varepsilon \ge 0, \\ 0 \le x \le -\frac{\beta}{\varepsilon} \quad & \text{if } \varepsilon < 0 \end{aligned}$$

and $\beta > 0$ is the scale parameter and ε is called the shape parameter, see [5].

3.2 Two theorems from extreme value theory in the sense of IF-probability

In this subsection we will study two theorems from extreme value theory, i.e., the variations of Fisher–Tippett–Gnedenko theorem and Pickands–Balkema–de Haan theorem in the sense of IF-probability. Since the IF-probability $\mathcal{P}$ can be decomposed into two IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$, we will try to use the results from the extreme value theory for IF-states presented in the previous subsection.

Let $(x_n)_1^\infty$ be a sequence of independent, equally distributed IF-observables on $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is an IF-probability. Denote $\mathbf{M}_n$ maximum of n IF-observables such that $\mathbf{M}_1 = x_1$, $\mathbf{M}_n = \max(x_1, \ldots, x_n)$ for $n \ge 2$.

Theorem 3.4. (Variation of Fisher–Tippett–Gnedenko theorem for IF-probability). *Let* $(\mathcal{F}, \mathcal{P})$ *be an IF-space with IF-probability* $\mathcal{P}$, $(x_n)_1^\infty$ *be a sequence of independent, equally distributed IF-observables such that* $\mathbf{D}^2(x_n) = [\mathbf{D}^2_\flat(x_n), \mathbf{D}^2_\sharp(x_n)] = [\sigma^2, \sigma^2] = \{\sigma^2\}$,

$\mathbf{E}(x_n) = [\mathbf{E}^\flat(x_n), \mathbf{E}^\sharp(x_n)] = [\mu, \mu] = \{\mu\}$, $n \in N$. *If there exist two sequences of real constants* $a_n > 0$ *and* b_n *and a non-degenerate distribution function* H*, such that*

$$\lim_{n\to\infty} \mathcal{P}\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\right) = [H(t), H(t)] = \{H(t)\},$$

then the distribution function H *is one of the following three types of distributions:*

1. *Gumbel*
$$H_{\mu,\sigma}(t) = \exp\left(-e^{-\left(\frac{t-\mu}{\sigma}\right)}\right), \quad t \in R,$$

2. *Fréchet*
$$H_{\mu,\sigma,\alpha}(t) = \begin{cases} 0, & \text{for } t \leq \mu, \\ \exp\left(-\left(\frac{t-\mu}{\sigma}\right)^{-\alpha}\right), & \text{for } t > \mu, \alpha > 0, \end{cases}$$

3. *Weibull*
$$H_{\mu,\sigma,\alpha}(t) = \begin{cases} \exp\left(-\left(-\frac{t-\mu}{\sigma}\right)^{\alpha}\right), & \text{for } t \leq \mu, \alpha > 0, \\ 1, & \text{for } t > \mu. \end{cases}$$

There a parameter $\mu \in R$ *is the location parameter and a parameter* $\sigma > 0$ *is the scale parameter.*

Proof. Let $\mathcal{P}$ be an IF-probability. It can be expressed using two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$ such that $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})]$ for each $\mathbf{A} \in \mathcal{F}$. Denote $y_n = \frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)$, $n \in N$. Then

$$\mathcal{P}\big(y_n((-\infty, t))\big) = \big[\mathcal{P}^\flat\big(y_n((-\infty, t))\big), \mathcal{P}^\sharp\big(y_n((-\infty, t))\big)\big] \tag{1}$$

for all $t \in R$.

Since a sequence $(x_n)_n$ is the sequence of independent IF-observables in IF-space $(\mathcal{F}, \mathcal{P})$, then by Theorem 2.1, the sequence $(x_n)_n$ is independent in IF-spaces $(\mathcal{F}, \mathcal{P}^\flat)$, $(\mathcal{F}, \mathcal{P}^\sharp)$. Moreover the sequence $(x_n)_n$ of independent IF-observables is equally distributed in IF-spaces $(\mathcal{F}, \mathcal{P}^\flat)$, $(\mathcal{F}, \mathcal{P}^\sharp)$ and

$$\mathbf{E}^\flat(x_n) = \mathbf{E}^\sharp(x_n) = a, \mathbf{D}^2_\flat(x_n) = \mathbf{D}^2_\sharp(x_n) = \sigma^2.$$

Hence using Theorem 3.2 (i.e., the variation of Fisher–Tippett–Gnedenko theorem for IF-state), we obtain that if there exist two sequences of real constants $a_n > 0$ and b_n and a non-degenerate distribution function H, such that

$$\lim_{n\to\infty} \mathcal{P}^\flat\big(y_n((-\infty, t))\big) = \lim_{n\to\infty} \mathcal{P}^\flat\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\right) = H(t), \tag{2}$$

$$\lim_{n\to\infty} \mathcal{P}^\sharp\big(y_n((-\infty, t))\big) = \lim_{n\to\infty} P^\sharp\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty, t)\big)\right) = H(t), \tag{3}$$

then the distribution function H is one of the following three types of distributions: Gumbel, Fréchet and Weibull.

Therefore, using (1), (2) and (3), we have that

$$\lim_{n\to\infty} \mathcal{P}\big(y_n((-\infty, t))\big) = \lim_{n\to\infty} \big[\mathcal{P}^\flat\big(y_n((-\infty, t))\big), \mathcal{P}^\sharp\big(y_n((-\infty, t))\big)\big] = [H(t), H(t)] = \{H(t)\}$$

for all $t \in R$. $\square$

Now we can define the notion of maximum domain of attraction for IF-probability.

Definition 3.5. (Maximum domain of attraction for IF-probability). *We say that the IF-distribution function* $\mathbf{F}$ *connected with IF-probability* $\mathcal{P}$ *of IF-observable* x_i *belongs to the maximum domain of attraction of the extreme value distributions* H *if there exist two sequences of real constants* $a_n > 0$ *and* b_n*, such that*

$$\lim_{n\to\infty} \mathcal{P}\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty,t)\big)\right) = [H(t),H(t)] = \{H(t)\}$$

holds. We write $\mathbf{F} \in \boldsymbol{MDA}(H)$ *in IF-space* $(\mathcal{F},\mathcal{P})$.

Let x be an IF-observable on $\mathcal{F}$ and $\mathbf{F}$ be an IF-distribution function of x connected with the IF-probability $\mathcal{P}$. We denote by

$$t_{\mathbf{F}} = \sup\{t \in R : \mathbf{F}^\flat(t) < 1 \wedge \mathbf{F}^\sharp(t) < 1\} \;\; (\text{i.e., } t_{\mathbf{F}} = \sup\{t \in R : \mathbf{F}(t) < [1,1]\})$$

the right endpoint of IF-distribution function $\mathbf{F}$ connected with IF-probability $\mathcal{P}$. There $\mathbf{F}^\flat$, $\mathbf{F}^\sharp$ are the corresponding IF-distributions connected with IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.

Remark 3.2. *If* $t_{\mathbf{F}}$ *is the right endpoint of the IF-distribution function* $\mathbf{F}$ *connected with IF-probability* $\mathcal{P}$ *and* $t_{\mathbf{F}^\flat}$, $t_{\mathbf{F}^\sharp}$ *be the right endpoints of IF-distribution functions* $\mathbf{F}^\flat$, $\mathbf{F}^\sharp$ *connected with IF-states* $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$*, then* $t_{\mathbf{F}} = t_{\mathbf{F}^\sharp}$ *and* $t_{\mathbf{F}^\sharp} \leq t_{\mathbf{F}^\flat}$. *This can be seen from the inequality* $\mathbf{F}^\flat(t) \leq \mathbf{F}^\sharp(t)$ *for all* $t \in R$.

Theorem 3.5. *Let* $\mathbf{F}$ *be an IF-distribution function of the IF-observable* x *connected with the IF-probability* $\mathcal{P}$ *and* $\mathbf{F}^\flat$, $\mathbf{F}^\sharp$ *are the IF-distribution functions of the IF-observable* x *connected with IF-states* $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$*, respectively. Then* $\mathbf{F} \in \boldsymbol{MDA}(H)$ *in IF-space* $(\mathcal{F},\mathcal{P})$ *if and only if* $\mathbf{F}^\flat, \mathbf{F}^\sharp \in \boldsymbol{MDA}(H)$ *in IF-spaces* $(\mathcal{F},\mathcal{P}^\flat)$, $(\mathcal{F},\mathcal{P}^\sharp)$.

Proof. Let $\mathbf{F}$ be an IF-distribution function of the IF-observable x connected with IF-probability $\mathcal{P} = [\mathcal{P}^\flat, \mathcal{P}^\sharp]$ given by $\mathbf{F}(t) = [\mathbf{F}^\flat(t), \mathbf{F}^\sharp(t)]$ for all $t \in R$, where $\mathbf{F}^\flat$, $\mathbf{F}^\sharp$ are the corresponding IF-distribution function of the IF-observable x connected with IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.

<u>“ $\Rightarrow$ ”:</u> If $\mathbf{F} \in \mathbf{MDA}(H)$ in IF-space $(\mathcal{F},\mathcal{P})$, then there there exists constants $a_n > 0$, $b_n \in R$ such that

$$\lim_{n\to\infty} \mathcal{P}\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty,t)\big)\right) = [H(t),H(t)] = \{H(t)\}.$$

But

$$\mathcal{P}\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty,t)\big)\right) = \left[\mathcal{P}^\flat\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty,t)\big)\right), \mathcal{P}^\sharp\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty,t)\big)\right)\right].$$

Therefore,

$$\lim_{n\to\infty} \mathcal{P}^\flat\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty,t)\big)\right) = H(t),$$
$$\lim_{n\to\infty} \mathcal{P}^\sharp\left(\frac{1}{a_n}\big(\mathbf{M}_n - b_n\big)\big((-\infty,t)\big)\right) = H(t),$$

i.e., $\mathbf{F}^\flat, \mathbf{F}^\sharp \in \mathbf{MDA}(H)$ in IF-spaces $(\mathcal{F},\mathcal{P}^\flat)$, $(\mathcal{F},\mathcal{P}^\sharp)$, respectively.

<u>“ $\Leftarrow$ ” :</u> The opposite direction can be proved in a similar way. □

Now we formulate excess IF-distribution function for IF-probability.

Definition 3.6. (Excess IF-distribution function for IF-probability). *Let* $\mathbf{F}$ *be an IF-distribution function of an IF-observable* x *connected with IF-probability* $\mathcal{P}$ *with right endpoint* $t_{\mathbf{F}}$. *For fixed* $u < t_{\mathbf{F}}$, $u > 0$,

$$\mathbf{F}_u(t) = \frac{\mathbf{F}(t+u) - \mathbf{F}(u)}{[1,1] - \mathbf{F}(u)}, \ \ 0 \le t \le t_{\mathbf{F}} - u$$

is the excess IF-distribution function of the IF-observable x *connected with IF-probability* $\mathcal{P}$ *over the threshold* u.

Remark 3.3. *The excess IF-distribution function* $\mathbf{F}_u$ *connected with IF-probability* $\mathcal{P}$ *can be expressed in the following form*

$$\mathbf{F}_u(t) = \frac{\mathbf{F}(t+u) - \mathbf{F}(u)}{[1,1] - \mathbf{F}(u)} = \left[\frac{\mathbf{F}^\flat(t+u) - \mathbf{F}^\flat(u)}{1 - \mathbf{F}^\flat(u)}, \frac{\mathbf{F}^\sharp(t+u) - \mathbf{F}^\sharp(u)}{1 - \mathbf{F}^\sharp(u)}\right] = \left[\mathbf{F}^\flat_u(t), \mathbf{F}^\sharp_u(t)\right]$$

for $0 \le t \le t_{\mathbf{F}} - u$. *There* $\mathbf{F}^\flat_u$, $\mathbf{F}^\sharp_u$ *are the excess IF-distribution functions connected with IF-states* $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$.

Next we formulate a variation of Pickands–Balkema–de Haan theorem for IF-probability.

Theorem 3.6. (Variation of Pickands–Balkema–de Haan theorem for IF-probability). *Let* $\mathbf{F}$ *be an IF-distribution function of an IF-observable* x *connected with IF-probability* $\mathcal{P}$ *with right endpoint* $t_{\mathbf{F}}$. *Let* $\mathbf{F}_u$ *be an excess IF-distribution function of the IF-observable* x *connected with the IF-probability* $\mathcal{P}$ *over the threshold* u *for fixed* $u < t_{\mathbf{F}}$, $u > 0$. *For every* $\varepsilon \in R$,

$$\mathbf{F} \in \mathbf{MDA}(H_\varepsilon) \iff \lim_{u \to t_{\mathbf{F}}} \sup_{0<t<t_{\mathbf{F}}-u} \left|\mathbf{F}_u(t) - \{G_{\varepsilon,\beta(u)}(t)\}\right| = [0,0]$$

for some positive function β. *There* $\{G_{\varepsilon,\beta(u)}(t)\} = [G_{\varepsilon,\beta(u)}(t), G_{\varepsilon,\beta(u)}(t)]$.

Proof. Let $(\mathcal{F}, \mathcal{P})$ be an IF-space with an IF-probability $\mathcal{P} = [\mathcal{P}^\flat, \mathcal{P}^\sharp]$. Then by Theorem 3.5, we have that

$$\mathbf{F} \in \mathbf{MDA}(H_\varepsilon) \iff \mathbf{F}^\flat, \mathbf{F}^\sharp \in \mathbf{MDA}(H_\varepsilon)$$

for every $\varepsilon \in R$. There $\mathbf{F}^\flat$, $\mathbf{F}^\sharp$ are the IF-distribution functions connected with IF-states $\mathcal{P}^\flat$, $\mathcal{P}^\sharp$, respectively.

Using a variation of Pickands–Balkema–de Haan theorem for IF-state (see Theorem 3.3), we obtain

$$\mathbf{F}^\flat \in \mathbf{MDA}(H_\varepsilon) \iff \lim_{u \to t_{\mathbf{F}^\flat}} \sup_{0<t<t_{\mathbf{F}^\flat}-u} |\mathbf{F}^\flat_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0,$$
$$\mathbf{F}^\sharp \in \mathbf{MDA}(H_\varepsilon) \iff \lim_{u \to t_{\mathbf{F}^\sharp}} \sup_{0<t<t_{\mathbf{F}^\sharp}-u} |\mathbf{F}^\sharp_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0$$

for some positive function β. But $t_{\mathbf{F}} = t_{\mathbf{F}^\sharp}$ and $t_{\mathbf{F}^\sharp} \le t_{\mathbf{F}^\flat}$(see Remark 3.2), hence

$$\lim_{u \to t_{\mathbf{F}}} \sup_{0<t<t_{\mathbf{F}}-u} |\mathbf{F}^\flat_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0,$$
$$\lim_{u \to t_{\mathbf{F}}} \sup_{0<t<t_{\mathbf{F}}-u} |\mathbf{F}^\sharp_u(t) - G_{\varepsilon,\beta(u)}(t)| = 0.$$

Finally, we have

$$\lim_{u\to t_{\mathbf{F}}} \sup_{0<t<t_{\mathbf{F}}-u} \left|\mathbf{F}_u(t) - \{G_{\varepsilon,\beta(u)}(t)\}\right| = \lim_{u\to t_{\mathbf{F}}} \sup_{0<t<t_{\mathbf{F}}-u} \left[\left|\mathbf{F}_u^{\flat}(t) - G_{\varepsilon,\beta(u)}(t)\right|, \left|\mathbf{F}_u^{\sharp}(t) - G_{\varepsilon,\beta(u)}(t)\right|\right]$$
$$= [0,0]$$

for some positive function β. □

Remark 3.4. *The previous theorem says that for some function β to be estimated from the data, the excess IF-distribution $\mathbf{F}_u$ connected with IF-probability $\mathcal{P}$ converges to the generalised Pareto distribution $G_{\varepsilon,\beta}$ for large u.*

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References

[1] Atanassov, K. T. (1983). Intuitionistic fuzzy sets. *VII ITKR Session*, Sofia, 20-23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted: *Int. J. Bioautomation*, 20(1), S1–S6.

[2] Atanassov, K. T. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*. Physica Verlag, New York.

[3] Atanassov, K. T. (2012). *On Intuitionistic Fuzzy Sets*. Springer, Berlin.

[4] Bartková, R., & Čunderlíková, K. (2018). About Fisher–Tippett–Gnedenko Theorem for Intuitionistic Fuzzy Events. *Advances in Fuzzy Logic and Technology 2017, J. Kacprzyk et al. eds. IWIFSGN 2017, EUSFLAT 2017. Advances in Intelligent Systems and Computing*, Vol. 641, Springer, Cham, 125–135.

[5] Coles, S. (2001). *An Introduction to Statistical Modeling of Extreme Values*. Springer, London.

[6] Čunderlíková, K. (2019). **m**-almost everywhere convergence of intuitionistic fuzzy observables induced by Borel measurable function. *Notes on Intuitionistic Fuzzy Sets*, 25(2), 29–40.

[7] Čunderlíková, K. (2022). Intuitionistic fuzzy probability and convergence of intuitionistic fuzzy observables. *Notes on Intuitionistic Fuzzy Sets*, 28(4), 381–396.

[8] Čunderlíková, K., & Bartková, R. (2018). The Pickands–Balkema–de Haan theorem for intuitionistic fuzzy events. *Notes on Intuitionistic Fuzzy Sets*, 24(2), 63–75.

[9] Čunderlíková, K., & Riečan, B. (2021). Convergence of intuitionistic fuzzy observables. In: *Advances in Intelligent Systems and Computing. Uncertainty and Imprecision in Decision Making and Decision Support: New Challenges, Solutions and Perspectives*, IWIFSGN 2018. Advances in Intelligent Systems and Computing, vol 1081. Springer, Cham, pp. 29–39.

[10] Lendelová, K. (2005). Convergence of IF-observables. *Issues in the Representation and Processing of Uncertain and Imprecise Information – Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets, and Related Topics*, EXIT, Warsaw, 232–240.

[11] Lendelová, K. (2006). Conditional IF-probability. *Advances in Soft Computing: Soft Methods for Integrated Uncertainty Modelling*, 37, Springer, Berlin, Heidelberg, 275–283.

[12] Lendelová, K. (2006). Strong law of large numbers for IF-events. *Proceedings of the Eleventh International Conference IPMU 2006*, 2–7 July 2006, Paris, France, 2363–2366.

[13] Riečan, B. (2003). A descriptive definition of the probability on intuitionistic fuzzy sets. *EUSFLAT '2003 (M. Wagenecht, R. Hampet eds.)*, Zittau-Goerlitz Univ. Appl. Sci., 210–213.

[14] Riečan, B. (2004). Representation of probabilities on IFS events. *Soft Methodology and Random Information Systems (López-Diáz et al. eds.)*, Springer, Berlin–Heidelberg–New York, 243–248.

[15] Riečan, B. (2005). On the probability on IF-sets and MV-algebras. *Notes on Intuitionistic Fuzzy Sets*, 11(6), 21–25.

[16] Riečan, B. (2006). On the probability and random variables on IF events. *Applied Artificial Intelligence, Proc. 7th FLINS Conference*, Genova (Ruan, D., et al. (Eds.). 138–145.

[17] Riečan, B. (2007). Probability theory on IF events. *Algebraic and Proof-theoretic Aspects of Non-classical Logics. Lecture Notes in Computer Science, vol 4460, Aguzzoli, S., Ciabattoni, A., Gerla, B., Manara, C., Marra, V. eds.*, Springer, Berlin–Heidelberg, 290–308.

[18] Riečan, B. (2012). Analysis of fuzzy logic models, *Intelligent systems (Koleshko, V. (Ed.).* INTECH, 219–244.

[19] Riečan, B., & Neubrunn, T. (1997). *Integral, Measure, and Ordering*. Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava.

[20] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338–358.

[21] Zadeh, L. A. (1968). Probability measures on fuzzy sets. *Journal of Mathematical Analysis and Applications*, 23, 421–427.