

Modal operator $F_{\alpha,\beta}$ on intuitionistic fuzzy BG-algebras

S. R. Barbhuiya

Department of Mathematics, Srikishan Sarda College

Hailakandi–788151, Assam, India

e-mail: saidurbarbhuiya@gmail.com

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Abstract: In this paper, we study the effect of modal operator(s) in intuitionistic fuzzy BG-algebras and the effect of modal operator(s) on intuitionistic fuzzy BG-algebras under homomorphism and obtained some interesting properties.

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1 Introduction

In 1966, Imai and Iseki [11] introduced two classes of abstract algebras, viz., *BCK*-algebras and *BCI*-algebras. It is known that the class of *BCK*-algebra is a proper subclass of the class of *BCI*-algebras. Neggers and Kim [15] introduced a new concept, called *B*-algebras, which are related to several classes of algebras such as *BCI*/*BCK*-algebras. Kim and Kim [13] introduced the notion of *BG*-algebra which is a generalization of *B*-algebra. The concept of intuitionistic fuzzy subset (IFS) was introduced by Atanassov [3], which is a generalization of the notion of fuzzy sets [17]. The intuitionistic fuzzy modal operators \square and \diamond were introduced by Atanassov [3] which are analogous to the modal logic operator of necessity and possibility and have no counterparts in ordinary fuzzy set theory. The extension on both the operators \square and \diamond is the new operator D_α which represents both of them. Further the extension of all the operators is the operator $F_{\alpha,\beta}$ called (α, β) -modal operator. The effect of all the modal operator on IFSs is again an IFSs. The modal operators play a very significant role in the study of IFSs. A lot of operators

were defined and studied in [2, 4–10, 14, 16]. The concept of fuzzy subalgebras of BG-algebras was introduced by Ahn and Lee in [1]. Here in this paper, we study the effect of modal operators in particular (α, β) -modal operator on intuitionistic fuzzy BG-algebra.

2 Preliminaries

Definition 2.1. A BG-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

- (i) $x * x = 0$,
- (ii) $x * 0 = x$,
- (iii) $(x * y) * (0 * y) = x, \forall x, y \in X$.

For brevity, we also call X a BG-algebra.

Example 2.2. Let $X = \{0, 1, 2, 3, 4\}$ with the following cayley table

Table 1: Example of BG-algebra.

*	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Then $(X, *, 0)$ is a BG-algebra.

Definition 2.3. A non-empty subset S of a BG-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 2.4. A fuzzy subset μ of a BG-algebra X is called a fuzzy subalgebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Definition 2.5. An intuitionistic fuzzy set (IFS) A of a non empty set X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$, where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the element x in set A . For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$. The function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$. is called the degree of uncertainty of $x \in A$. The class of IFSs on a universe X is denoted by $IFS(X)$.

Definition 2.6. If $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x) \rangle | x \in X\}$ are any two IFSs of a set X , then

$A \subseteq B$ if and only if for all $x \in X$, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$,

$A = B$ if and only if for all $x \in X$, $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$,

$A \cap B = \{\langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) \rangle | x \in X\}$,

where $(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$,

$A \cup B = \{\langle x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x) \rangle | x \in X\}$,

where $(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$.

Definition 2.7. If $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x) \rangle | x \in X\}$ are any two IFSs of a set X , then their cartesian product is defined by

$A \times B = \{\langle (x, y), (\mu_A \times \mu_B)(x, y), (\nu_A \times \nu_B)(x, y) \rangle | x, y \in X\}$,

where $(\mu_A \times \mu_B)(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ and $(\nu_A \times \nu_B)(x, y) = \max\{\nu_A(x), \nu_B(y)\}$.

Definition 2.8. For any IFS $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ of X and $\alpha \in [0, 1]$, the operators $\square : IFS(X) \rightarrow IFS(X)$, $\diamond : IFS(X) \rightarrow IFS(X)$, $D_\alpha : IFS(X) \rightarrow IFS(X)$ are defined as

(i) $\square(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle | x \in X\}$ is called necessity operator

(ii) $\diamond(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle | x \in X\}$ is called possibility operator

(iii) $D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1 - \alpha)\pi_A(x) \rangle | x \in X\}$ is called α -modal operator.

Clearly $\square(A) \subseteq A \subseteq \diamond(A)$ and the equality hold, when A is a fuzzy set also $D_0(A) = \square(A)$ and $D_1(A) = \diamond(A)$. Therefore the α -model operator $D_\alpha(A)$ is an extension of necessity operator $\square(A)$ and possibility operator $\diamond(A)$.

Definition 2.9. For any IFS $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ of X and for any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, the (α, β) -modal operator $F_{\alpha, \beta} : IFS(X) \rightarrow IFS(X)$ is defined as $F_{\alpha, \beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x) \rangle | x \in X\}$, where $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$. Therefore we can write

$F_{\alpha, \beta}(A)$ as $F_{\alpha, \beta}(A)(x) = (\mu_{F_{\alpha, \beta}(A)}(x), \nu_{F_{\alpha, \beta}(A)}(x))$

where $\mu_{F_{\alpha, \beta}(A)}(x) = \mu_A(x) + \alpha\pi_A(x)$ and $\nu_{F_{\alpha, \beta}(A)}(x) = \nu_A(x) + \beta\pi_A(x)$.

Clearly, $F_{0, 1}(A) = \square(A)$, $F_{1, 0}(A) = \diamond(A)$ and $F_{\alpha, 1-\alpha}(A) = D_\alpha(A)$.

Definition 2.10. Let X and Y be two non empty sets and $f : X \rightarrow Y$ be a mapping. Let A and B be IFS's of X and Y respectively. Then the image of A under the map f is denoted by $f(A)$ and is defined by $f(A)(y) = (\mu_{f(A)}(y), \nu_{f(A)}(y))$, where

$$\mu_{f(A)}(y) = \begin{cases} \vee\{\mu_A(x) : x \in f^{-1}(y)\} & \nu_{f(A)}(y) = \begin{cases} \wedge\{\nu_A(x) : x \in f^{-1}(y)\} \\ 1 & \text{otherwise} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

also pre image of B under f is denoted by $f^{-1}(B)$ and is defined as $f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)) = (\mu_B(f(x)), \nu_B(f(x)))$; $\forall x \in X$.

Remark 2.11. $\mu_A(x) \leq \mu_{f(A)}(f(x))$ and $\nu_A(x) \geq \nu_{f(A)}(f(x)) \quad \forall x \in X$ however equality hold when the map f is bijective.

Definition 2.12. An IFS A of a BG-algebra X is said to be an IF BG-subalgebra of X if

- (i) $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$,
- (ii) $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} \forall x, y \in X$.

Example 2.13. Consider a BG-algebra $X = \{0, 1, 2\}$ with the following cayley table:

Table 2: Example of IF BG-subalgebra.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

The intuitionistic fuzzy subset $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ given by $\mu_A(0) = \mu_A(1) = 0.6$, $\mu_A(2) = 0.2$ and $\nu_A(0) = \nu_A(1) = 0.3$, $\nu_A(2) = 0.5$ is an IF BG-subalgebra of X .

Definition 2.14. An IFS A of a BG-algebra X is said to be an IF normal subalgebra of X if

- (i) $\mu_A((x * a) * (y * b)) \geq \min\{\mu_A(x * y), \mu_A(a * b)\}$,
- (ii) $\nu_A((x * a) * (y * b)) \leq \max\{\nu_A(x * y), \nu_A(a * b)\}, \forall x, y \in X$.

3 Modal operator $F_{\alpha, \beta}$ on intuitionistic fuzzy subalgebras

In this section, we study the effect of modal operator on IF subalgebra of BG-algebra X .

Theorem 3.1. If A is an IF subalgebra of BG-algebra X , then $F_{\alpha, \beta}(A)$ is also an IF subalgebra of BG-algebra X .

Proof: Let $x, y \in X$, then $F_{\alpha, \beta}(x * y) = (\mu_{F_{\alpha, \beta}(A)}(x * y), \nu_{F_{\alpha, \beta}(A)}(x * y))$. Where $\mu_{F_{\alpha, \beta}(A)}(x * y) = \mu_A(x * y) + \alpha\pi_A(x * y)$ and $\nu_{F_{\alpha, \beta}(A)}(x * y) = \nu_A(x * y) + \beta\pi_A(x * y)$

Now

$$\begin{aligned}
\mu_{F_{\alpha, \beta}(A)}(x * y) &= \mu_A(x * y) + \alpha\pi_A(x * y) \\
&= \mu_A(x * y) + \alpha(1 - \mu_A(x * y) - \nu_A(x * y)) \\
&= \alpha + (1 - \alpha)\mu_A(x * y) - \alpha\nu_A(x * y) \\
&\geq \alpha + (1 - \alpha)\min(\mu_A(x), \mu_A(y)) - \alpha\max(\nu_A(x), \nu_A(y)) \\
&= \alpha\{1 - \max(\nu_A(x), \nu_A(y))\} + (1 - \alpha)\min(\mu_A(x), \mu_A(y)) \\
&= \alpha\min(1 - \nu_A(x), 1 - \nu_A(y))\} + (1 - \alpha)\min(\mu_A(x), \mu_A(y)) \\
&= \min\{\alpha(1 - \nu_A(x)) + (1 - \alpha)\mu_A(x), \alpha(1 - \nu_A(y)) + (1 - \alpha)\mu_A(y)\} \\
&= \min\{\mu_A(x) + \alpha(1 - \mu_A(x) - \nu_A(x)), \mu_A(y) + \alpha(1 - \mu_A(y) - \nu_A(y))\} \\
&= \min\{\mu_{F_{\alpha, \beta}(A)}(x), \mu_{F_{\alpha, \beta}(A)}(y)\}
\end{aligned}$$

$$\therefore \mu_{F_{\alpha,\beta}(A)}(x * y) \geq \min\{\mu_{F_{\alpha,\beta}(A)}(x), \mu_{F_{\alpha,\beta}(A)}(y)\}$$

Similarly we can prove

$$\nu_{F_{\alpha,\beta}(A)}(x * y) \leq \max\{\nu_{F_{\alpha,\beta}(A)}(x), \nu_{F_{\alpha,\beta}(A)}(y)\}$$

Hence $F_{\alpha,\beta}(A)$ is an IF subalgebra of BG-algebra X .

Remark 3.2. The converse of above Theorem need not be true as shown in Example below.

Example 3.3. Consider a BG-algebra $X = \{0, 1, 2\}$ with the following cayley table:

Table 3: Illustration of converse of Theorem 3.1.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

The IF subset $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ given by $\mu_A(0) = 0.48, \mu_A(1) = 0.5, \mu_A(2) = 0.3$ and $\nu_A(0) = 0.3, \nu_A(1) = 0.4, \nu_A(2) = 0.5$ is not an IF BG-subalgebra of X . Since $\mu_A(0) = 0.48 \not\geq \min\{\mu_A(1), \mu_A(2)\} = \mu_A(1) = 0.5$.

Now take $\alpha = 0.7, \beta = 0.3, \alpha + \beta \leq 1$, then $F_{\alpha,\beta}(A) = \{\langle x, \mu_{F_{\alpha,\beta}(A)}(x), \nu_{F_{\alpha,\beta}(A)}(x) \rangle | x \in X\}$ is $\mu_{F_{0.7,0.3}(A)}(0) = 0.63, \mu_{F_{0.7,0.3}(A)}(1) = 0.57, \mu_{F_{0.7,0.3}(A)}(2) = 0.44$ and $\nu_{F_{0.7,0.3}(A)}(0) = 0.36, \nu_{F_{0.7,0.3}(A)}(1) = 0.43, \nu_{F_{0.7,0.3}(A)}(2) = 0.56$. It can easily verified that $F_{0.7,0.3}(A)$ is an IF BG-subalgebra of X .

Corollary 3.4. If A is an IF subalgebra of BG-algebra X , then

- (i) $\square(A)$ is also an IF subalgebra of BG-algebra X ;
- (ii) $\diamond(A)$ is also an IF subalgebra of BG-algebra X ;
- (iii) $D_\alpha(A)$ is also an IF subalgebra of BG-algebra X .

Theorem 3.5. If A is an IF subalgebra of BG-algebra X , then

- (i) $\mu_{F_{\alpha,\beta}(A)}(0) \geq \mu_{F_{\alpha,\beta}(A)}(x)$
- (ii) $\nu_{F_{\alpha,\beta}(A)}(0) \leq \nu_{F_{\alpha,\beta}(A)}(x) \quad \forall x \in X$.

Proof: We have

$$\begin{aligned} \mu_{F_{\alpha,\beta}(A)}(0) &= \mu_{F_{\alpha,\beta}(A)}(x * x) \\ &\geq \min\{\mu_{F_{\alpha,\beta}(A)}(x), \mu_{F_{\alpha,\beta}(A)}(x)\} \\ &= \mu_{F_{\alpha,\beta}(A)}(x) \end{aligned}$$

and

$$\begin{aligned} \nu_{F_{\alpha,\beta}(A)}(0) &= \nu_{F_{\alpha,\beta}(A)}(x * x) \\ &\leq \max\{\nu_{F_{\alpha,\beta}(A)}(x), \nu_{F_{\alpha,\beta}(A)}(x)\} \\ &= \nu_{F_{\alpha,\beta}(A)}(x) \quad \forall x \in X. \end{aligned}$$

Theorem 3.6. *If A and B are two IF subalgebras of BG-algebra X , then*

(i) $A \cap B$ is also an IF subalgebra of BG-algebra X .

(ii) $A \times B$ is also an IF subalgebra of BG-algebra $X \times X$.

Proof: (i) We have $A \cap B = \{\langle x, \mu_{(A \cap B)}(x), \nu_{(A \cup B)}(x) \rangle | x \in X\}$,

where $\mu_{(A \cap B)}(x) = (\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ and

$\nu_{(A \cup B)}(x) = (\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$,

Let $x, y \in X$. Since both A, B are subalgebras of X , therefore

$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$

Also $\mu_B(x * y) \geq \min\{\mu_B(x), \mu_B(y)\}$ and $\nu_B(x * y) \leq \max\{\nu_B(x), \nu_B(y)\}$

Now

$$\begin{aligned} \mu_{(A \cap B)}(x * y) &= \min\{\mu_A(x * y), \mu_B(x * y)\} \\ &\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, \min\{\mu_B(x), \mu_B(y)\}\} \\ &= \min\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}\} \\ &= \min\{\mu_{(A \cap B)}(x), \mu_{(A \cap B)}(y)\} \\ \Rightarrow \mu_{(A \cap B)}(x * y) &\geq \min\{\mu_{(A \cap B)}(x), \mu_{(A \cap B)}(y)\} \end{aligned}$$

Similarly we can prove

$$\nu_{(A \cup B)}(x * y) \leq \max\{\nu_{(A \cup B)}(x), \nu_{(A \cup B)}(y)\}$$

Hence $A \cap B$ is also an IF subalgebra of BG-algebra X .

(ii) Similar to proof of (i)

Theorem 3.7. *If A and B are two IF subalgebras of BG-algebra X , then*

(i) $F_{\alpha, \beta}(A \cap B)$ is also an IF subalgebra of BG-algebra X .

(ii) $F_{\alpha, \beta}(A \times B)$ is also an IF subalgebra of BG-algebra $X \times X$.

Proof: (i) We have $F_{\alpha, \beta}(A \cap B)(x) = \{\langle x, \mu_{F_{\alpha, \beta}(A \cap B)}(x), \nu_{F_{\alpha, \beta}(A \cup B)}(x) \rangle | x \in X\}$, where

$\mu_{(A \cap B)}(x) = (\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ and

$\nu_{(A \cup B)}(x) = (\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$,

Let $x, y \in X$. Since both A, B are subalgebras of X , therefore

$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$

Also $\mu_B(x * y) \geq \min\{\mu_B(x), \mu_B(y)\}$ and $\nu_B(x * y) \leq \max\{\nu_B(x), \nu_B(y)\}$

Now

$$\begin{aligned} \mu_{F_{\alpha, \beta}(A \cap B)}(x * y) &= \mu_{(A \cap B)}(x * y) + \alpha \pi_{(A \cap B)}(x * y) \\ &= \mu_{(A \cap B)}(x * y) + \alpha \{1 - \mu_{(A \cap B)}(x * y) - \nu_{(A \cup B)}(x * y)\} \\ &= \alpha + (1 - \alpha) \mu_{(A \cap B)}(x * y) - \alpha \nu_{(A \cup B)}(x * y) \\ &= \alpha(1 - \max(\nu_A(x * y), \nu_A(y * y))) + (1 - \alpha) \min(\mu_A(x * y), \mu_A(x * y)) \\ &\geq (1 - \alpha) \min\{\min(\mu_A(x), \mu_A(y)), \min(\mu_B(x), \mu_B(y))\} + \alpha - \\ &\quad \alpha \max\{\max(\nu_A(x), \nu_A(y)), \max(\nu_B(x), \nu_B(y))\} \end{aligned}$$

$$\begin{aligned}
&= (1 - \alpha) \min\{\min(\mu_A(x), \mu_B(x)), \min(\mu_A(y), \mu_B(y))\} + \alpha - \\
&\quad \alpha \max\{\max(\nu_A(x), \nu_B(x)), \max(\nu_A(y), \nu_B(y))\} \\
&= (1 - \alpha) \min\{\mu_{(A \cap B)}(x), \mu_{(A \cap B)}(y)\} + \alpha - \alpha \max\{\nu_{(A \cup B)}(x), \nu_{(A \cup B)}(y)\} \\
&= (1 - \alpha) \min\{\mu_{(A \cap B)}(x), \mu_{(A \cap B)}(y)\} + \alpha \min\{1 - \nu_{(A \cup B)}(x), 1 - \nu_{(A \cup B)}(y)\} \\
&= \min\{(1 - \alpha)\mu_{(A \cap B)}(x) + \alpha(1 - \nu_{(A \cup B)}(x)), (1 - \alpha)\mu_{(A \cap B)}(y) + \alpha(1 - \nu_{(A \cup B)}(y))\} \\
&= \min\{\mu_{F_{\alpha, \beta}(A \cap B)}(x), \mu_{F_{\alpha, \beta}(A \cap B)}(y)\} \\
\text{Therefore } &\mu_{F_{\alpha, \beta}(A \cap B)}(x * y) \geq \min\{\mu_{F_{\alpha, \beta}(A \cap B)}(x), \mu_{F_{\alpha, \beta}(A \cap B)}(y)\}
\end{aligned}$$

Similarly we can prove

$$\nu_{F_{\alpha, \beta}(A \cup B)}(x * y) \leq \max\{\nu_{F_{\alpha, \beta}(A \cup B)}(x), \nu_{F_{\alpha, \beta}(A \cup B)}(y)\}$$

(ii) Similar to proof of (i)

Theorem 3.8. If $\{A_i : i = 1, 2, \dots, n\}$ be n IF subalgebras of X , then

(i) $F_{\alpha, \beta}(\cap_{i=1}^n A_i : i = 1, 2, \dots, n)$ is also an IF subalgebra of X .

(i) $F_{\alpha, \beta}(\times_{i=1}^n A_i : i = 1, 2, \dots, n)$ is also an IF subalgebra of $\times_{i=1}^n X_i$.

Theorem 3.9. If $F_{\alpha, \beta}(A) = \langle \mu_{F_{\alpha, \beta}A}, \nu_{F_{\alpha, \beta}A} \rangle$ is an IF subalgebra of X . Then the sets

$$X_{\mu_{F_{\alpha, \beta}}} = \{x \in X \mid \mu_{F_{\alpha, \beta}(A)}(x) = \mu_{F_{\alpha, \beta}(A)}(0)\}$$

$$X_{\nu_{F_{\alpha, \beta}}} = \{x \in X \mid \nu_{F_{\alpha, \beta}(A)}(x) = \nu_{F_{\alpha, \beta}(A)}(0)\}$$

are subalgebras of X .

Proof: Let $x, y \in \mu_{F_{\alpha, \beta}(A)}$, then $\mu_{F_{\alpha, \beta}(A)}(x) = \mu_{F_{\alpha, \beta}(A)}(y) = \mu_{F_{\alpha, \beta}(A)}(0)$

Now

$$\begin{aligned}
\mu_{F_{\alpha, \beta}(A)}(x * y) &\geq \min\{\mu_{F_{\alpha, \beta}(A)}(x), \mu_{F_{\alpha, \beta}(A)}(y)\} \\
&= \min\{\mu_{F_{\alpha, \beta}(A)}(0), \mu_{F_{\alpha, \beta}(A)}(0)\} \\
&= \mu_{F_{\alpha, \beta}(A)}(0)
\end{aligned}$$

$$\Rightarrow \mu_{F_{\alpha, \beta}(A)}(x * y) \geq \mu_{F_{\alpha, \beta}(A)}(0)$$

$$\text{Also } \mu_{F_{\alpha, \beta}(A)}(0) \geq \mu_{F_{\alpha, \beta}(A)}(x * y) \text{ by Theorem 3.5}$$

$$\text{Therefore } \mu_{F_{\alpha, \beta}(A)}(x * y) = \mu_{F_{\alpha, \beta}(A)}(0)$$

$$\Rightarrow x * y \in X_{\mu_{F_{\alpha, \beta}}}$$

$$\text{Again let } x, y \in \nu_{F_{\alpha, \beta}(A)} \text{ then } \nu_{F_{\alpha, \beta}(A)}(x) = \nu_{F_{\alpha, \beta}(A)}(y) = \nu_{F_{\alpha, \beta}(A)}(0)$$

$$\begin{aligned}
\nu_{F_{\alpha, \beta}(A)}(x * y) &\leq \max\{\nu_{F_{\alpha, \beta}(A)}(x), \nu_{F_{\alpha, \beta}(A)}(y)\} \\
&= \max\{\nu_{F_{\alpha, \beta}(A)}(0), \nu_{F_{\alpha, \beta}(A)}(0)\} \\
&= \nu_{F_{\alpha, \beta}(A)}(0)
\end{aligned}$$

$$\Rightarrow \nu_{F_{\alpha, \beta}(A)}(x * y) \leq \nu_{F_{\alpha, \beta}(A)}(0)$$

$$\text{also } \nu_{F_{\alpha, \beta}(A)}(0) \leq \nu_{F_{\alpha, \beta}(A)}(x * y) \text{ by Theorem 3.5}$$

$$\text{Therefore } \nu_{F_{\alpha, \beta}(A)}(x * y) = \nu_{F_{\alpha, \beta}(A)}(0)$$

$$\Rightarrow x * y \in X_{\nu_{F_{\alpha, \beta}}}$$

Hence $X_{\mu_{F_{\alpha,\beta}}}$ and $X_{\nu_{F_{\alpha,\beta}}}$ are subalgebras of X .

Proposition 3.10. *If A and B be two IFS sets of X and Y respectively and $f : X \rightarrow Y$ be a mapping, then*

- (i) $f^{-1}(F_{\alpha,\beta}(B)) = F_{\alpha,\beta}(f^{-1}(B))$
- (ii) $f(F_{\alpha,\beta}(A)) \subseteq F_{\alpha,\beta}(f(A))$

Theorem 3.11. *If A is an IF fuzzy normal subalgebra of BG-algebra X , then $F_{\alpha,\beta}(A)$ is also an IF normal subalgebra of X .*

Proof: Let $x, y, a, b \in X$, then $F_{\alpha,\beta}((x * a) * (y * b)) = (\mu_{F_{\alpha,\beta}(A)}((x * a) * (y * b)), \nu_{F_{\alpha,\beta}(A)}((x * a) * (y * b)))$ where $\mu_{F_{\alpha,\beta}(A)}((x * a) * (y * b)) = \mu_A((x * a) * (y * b)) + \alpha\pi_A((x * a) * (y * b))$ and $\nu_{F_{\alpha,\beta}(A)}((x * a) * (y * b)) = \nu_A((x * a) * (y * b)) + \beta\pi_A((x * a) * (y * b))$

Now

$$\begin{aligned}
& \mu_{F_{\alpha,\beta}(A)}((x * a) * (y * b)) \\
&= \mu_A((x * a) * (y * b)) + \alpha\pi_A((x * a) * (y * b)) \\
&= \mu_A((x * a) * (y * b)) + \alpha(1 - \mu_A((x * a) * (y * b)) - \nu_A((x * a) * (y * b))) \\
&= \alpha + (1 - \alpha)\mu_A((x * a) * (y * b)) - \alpha\nu_A((x * a) * (y * b)) \\
&\geq \alpha + (1 - \alpha)\min(\mu_A(x * y), \mu_A(a * b)) - \alpha\max(\nu_A(x * y), \nu_A(a * b)) \\
&= \alpha\{1 - \max(\nu_A(x * y), \nu_A(a * b))\} + (1 - \alpha)\min(\mu_A(x * y), \mu_A(a * b)) \\
&= \alpha\min(1 - \nu_A(x * y), 1 - \nu_A(a * b))\} + (1 - \alpha)\min(\mu_A(x * y), \mu_A(a * b)) \\
&= \min\{\alpha(1 - \nu_A(x * y)) + (1 - \alpha)\mu_A(x * y), \alpha(1 - \nu_A(a * b)) + (1 - \alpha)\mu_A(a * b)\} \\
&= \min\{\mu_A(x * y) + \alpha(1 - \mu_A(x * y) - \nu_A(x * y)), \mu_A(a * b) \\
&\quad + \alpha(1 - \mu_A(a * b) - \nu_A(a * b))\} \\
&= \min\{\mu_{F_{\alpha,\beta}(A)}(x * y), \mu_{F_{\alpha,\beta}(A)}(a * b)\}
\end{aligned}$$

$$\therefore \mu_{F_{\alpha,\beta}(A)}((x * a) * (y * b)) \geq \min\{\mu_{F_{\alpha,\beta}(A)}(x * y), \mu_{F_{\alpha,\beta}(A)}(a * b)\}$$

Similarly we can prove

$$\nu_{F_{\alpha,\beta}(A)}((x * a) * (y * b)) \leq \max\{\nu_{F_{\alpha,\beta}(A)}(x * y), \nu_{F_{\alpha,\beta}(A)}(a * b)\}$$

Hence $F_{\alpha,\beta}(A)$ is an IF normal subalgebra of BG-algebra X .

Theorem 3.12. *If A and B are two IF normal subalgebras of BG-algebra X , then*

- (i) $F_{\alpha,\beta}(A \cap B)$ is also an IF normal subalgebra of BG-algebra X .
- (ii) $F_{\alpha,\beta}(A \times B)$ is also an IF normal subalgebra of BG-algebra $X \times X$.

Theorem 3.13. *If $\{A_i : i = 1, 2, \dots, n\}$ be n IF normal subalgebras of X , then*

- (i) $F_{\alpha,\beta}(\bigcap_{i=1}^n A_i : i = 1, 2, \dots, n)$ is also an IF normal subalgebra of X .
- (ii) $F_{\alpha,\beta}(\times_{i=1}^n A_i : i = 1, 2, \dots, n)$ is also an IF normal subalgebra of $\times_{i=1}^n X_i$.

4 Effect of modal operators on intuitionistic fuzzy BG-algebra under homomorphism

Definition 4.1. Let X and Y be two BG-algebras, then a mapping $f : X \longrightarrow Y$ is said to be homomorphism if $f(x * y) = f(x) * f(y)$, $\forall x, y \in X$.

Theorem 4.2. Let $f : X \longrightarrow Y$ be a homomorphism of BG-algebras. If $F_{\alpha,\beta}(A)$ is an IF subalgebra of Y , then $f^{-1}(F_{\alpha,\beta}(A))$ is also an IF subalgebra of X .

Proof: Since $f^{-1}(F_{\alpha,\beta}(A)) = F_{\alpha,\beta}(f^{-1}(A))$

It is enough to show that $F_{\alpha,\beta}(f^{-1}(A))$ is an IF BG subalgebras of X .

Let $x, y \in X$, then

$$\begin{aligned} \mu_{F_{\alpha,\beta}(f^{-1}(A))}(x * y) &= \mu_{F_{\alpha,\beta}(A)}f(x * y) \\ &= \mu_{F_{\alpha,\beta}(A)}(f(x) * f(y)) \\ &\geq \min\{\mu_{F_{\alpha,\beta}(A)}(f(x)), \mu_{F_{\alpha,\beta}(A)}(f(y))\} \\ &= \min\{\mu_{F_{\alpha,\beta}(f^{-1}(A))}(x), \mu_{F_{\alpha,\beta}(f^{-1}(A))}(y)\} \\ \mu_{F_{\alpha,\beta}(f^{-1}(A))}(x * y) &\geq \min\{\mu_{F_{\alpha,\beta}(f^{-1}(A))}(x), \mu_{F_{\alpha,\beta}(f^{-1}(A))}(y)\} \end{aligned}$$

Similarly we can show

$$\nu_{F_{\alpha,\beta}(f^{-1}(A))}(x * y) \leq \max\{\nu_{F_{\alpha,\beta}(f^{-1}(A))}(x), \nu_{F_{\alpha,\beta}(f^{-1}(A))}(y)\}$$

Hence $f^{-1}(F_{\alpha,\beta}(A))$ is also an IF BG subalgebras of X .

Corollary 4.3. Let $f : X \longrightarrow Y$ be a homomorphism of BG-algebras.

(i) If $\Box(A)$ is an IF subalgebra of Y , then $f^{-1}(\Box(A))$ is also an IF subalgebra of X .

(ii) If $\Diamond(A)$ is an IF subalgebra of Y , then $f^{-1}(\Diamond(A))$ is also an IF subalgebra of X .

Theorem 4.4. Let $f : X \longrightarrow Y$ be a homomorphism of BG-algebras. If $F_{\alpha,\beta}(A)$ is an IF normal subalgebra of Y , then $f^{-1}(F_{\alpha,\beta}(A))$ is also an IF normal subalgebra of X .

Theorem 4.5. Let $f : X \longrightarrow Y$ be an onto homomorphism of BG-algebras. If $F_{\alpha,\beta}(A)$ is an IF fuzzy subalgebra of X , then $f(F_{\alpha,\beta}(A))$ is also an IF fuzzy subalgebra of Y .

Proof: Let $y_1, y_2 \in Y$. Since f is onto, therefore there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$, $f(x_2) = y_2$

$$f(F_{\alpha,\beta}(A))(y_1 * y_2) = (\mu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2), \nu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2))$$

Now $\mu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2) = \mu_{F_{\alpha,\beta}(A)}(x_1 * x_2)$ where $y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2)$

$$\begin{aligned} &\mu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2) \\ &= \mu_{F_{\alpha,\beta}(A)}(x_1 * x_2) \\ &= \mu_{F_{\alpha,\beta}(A)}(x_1 * x_2) \\ &= \mu_A(x_1 * x_2) + \alpha\pi_A(x_1 * x_2) \\ &= \mu_A(x_1 * x_2) + \alpha[1 - \mu_A(x_1 * x_2) - \nu_A(x_1 * x_2)] \end{aligned}$$

$$\begin{aligned}
&= \alpha + (1 - \alpha)\mu_A(x_1 * x_2) - \alpha\nu_A(x_1 * x_2) \\
&\geq \alpha + (1 - \alpha) \min\{\mu_A(x_1), \mu_A(x_2)\} - \alpha \max\{\nu_A(x_1), \nu_A(x_2)\} \\
&= \alpha\{1 - \max\{\nu_A(x_1), \nu_A(x_2)\}\} + (1 - \alpha) \min\{\mu_A(x_1), \mu_A(x_2)\} \\
&= \alpha \min\{1 - \nu_A(x_1), 1 - \nu_A(x_2)\} + (1 - \alpha) \min\{\mu_A(x_1), \mu_A(x_2)\} \\
&= \min\{(1 - \alpha)\mu_A(x_1) + \alpha(1 - \nu_A(x_1)), (1 - \alpha)\mu_A(x_2) + \alpha(1 - \nu_A(x_2))\} \\
&= \min\{\mu_A(x_1) + \alpha(1 - \nu_A(x_1) - \mu_A(x_1)), \mu_A(x_2) + \alpha(1 - \nu_A(x_2) - \mu_A(x_2))\} \\
&= \min\{\mu_A(x_1) + \alpha\pi_A(x_1), \mu_A(x_2) + \alpha\pi_A(x_2)\} \\
&= \min\{\mu_{F_{\alpha,\beta}(A)}(x_1), \mu_{F_{\alpha,\beta}(A)}(x_2)\} \\
&= \min\{\mu_{f(F_{\alpha,\beta}(A))}(f(x_1)), \mu_{f(F_{\alpha,\beta}(A))}(f(x_2))\} \\
&= \min\{\mu_{f(F_{\alpha,\beta}(A))}(y_1), \mu_{f(F_{\alpha,\beta}(A))}(y_2)\} \\
&\therefore \mu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2) \geq \min\{\mu_{f(F_{\alpha,\beta}(A))}(y_1), \mu_{f(F_{\alpha,\beta}(A))}(y_2)\}
\end{aligned}$$

Similarly we can show

$$\nu_{f(F_{\alpha,\beta}(A))}(y_1 * y_2) \leq \max\{\nu_{f(F_{\alpha,\beta}(A))}(y_1), \nu_{f(F_{\alpha,\beta}(A))}(y_2)\}$$

Hence $f(F_{\alpha,\beta}(A))$ is an IF fuzzy subalgebra of Y .

Corollary 4.6. *Let $f : X \rightarrow Y$ be a homomorphism of BG-algebras.*

(i) *If $\square(A)$ is an IF subalgebra of X , then $f(\square(A))$ is also an IF subalgebra of Y .*

(ii) *If $\diamond(A)$ is an IF subalgebra of X , then $f(\diamond(A))$ is also an IF subalgebra of Y .*

Theorem 4.7. *Let $f : X \rightarrow Y$ be an onto homomorphism of BG-algebras. If $F_{\alpha,\beta}(A)$ is an IF fuzzy normal subalgebra of X , then $f(F_{\alpha,\beta}(A))$ is also an IF fuzzy normal subalgebra of Y .*

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