Notes on Intuitionistic Fuzzy Sets
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Research on intuitionistic fuzzy implications. Part 4

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Abstract: Continuing the research from Parts 1, 2 [2, 3] where intuitionistic fuzzy implications, determined as implications with "good" properties, were investigated, here we correct the list of the implications that satisfy the *Modus Ponens* from Part 3, [4], and further select among them those implications that satisfy the *Modus Tollens*, as well. We discuss some applications of these implications and show the relationship between every two of them.

Keywords: Intuitionistic fuzzy implication, Intuitionistic fuzzy pair, *Modus Ponens*, *Modus Tollens*.
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1 Introduction

Logical conclusions, fundamental to both logic and mathematics, involve deriving a statement based on given premises. This process ensures that if the premises are true, the conclusion must also be true.

Two primary forms are widely used:

$$\textit{Modus Ponens (MP)}: \quad \frac{A, \quad A \to B}{B},$$
$$\textit{Modus Tollens (MT)}: \quad \frac{A \to B, \quad \neg B}{\neg A}.$$

These logical constructs provide a framework for validating arguments and ensuring consistency in reasoning [7].

Modus Ponens is fundamental in reasoning and has direct applications in both procedural and logical programming languages. Since the Procedural Programming Languages (like C, Python, Java, [8–10]) focus on a sequence of imperative commands that change a program's state, *Modus Ponens* is used for the conditional statements like "if-else".

Logical Programming Languages (like Prolog) are based on formal logic [6]. Programs consist of a set of logical statements, and the execution involves proving queries using these statements. In Prolog, rules are defined using clauses that directly implement *Modus Ponens*. Prolog and other logical languages use a process called resolution to prove goals, which mimics the *Modus Ponens* using Horn clause expressions.

For example:

```
parent(alice, bob).
parent(bob, charlie).
ancestor(X, Y) :- parent(X, Y).
ancestor(X, Y) :- parent(X, Z), ancestor(Z, Y).
```

To prove `ancestor(alice, charlie)`, Prolog resolves this goal by applying *Modus Ponens* to the premises `parent(alice, bob)` and `ancestor(bob, charlie).`

Something more, the logical languages use backtracking to explore different possibilities, applying *Modus Ponens* repeatedly to find all possible solutions.

Modus Tollens is another fundamental logical tool, although it might be less explicit than *Modus Ponens*. In procedural languages, *Modus Tollens* is used to infer the falsity of a condition based on the failure of an expected outcome. However, *Modus Tollens* is not directly applied in classical logic languages. To be applied indirectly, a negation can be introduced for the purpose of contradiction.

In intuitionistic fuzzy calculus (see, e.g., [5]), the Intuitionistic Fuzzy Pair (IFP) $\langle a, b\rangle$, i.e., pair for which $a, b, a + b \in [0, 1]$, is called a *tautological IFP (TIFP)* if and only if $a = 1, b = 0$ and *intuitionistic fuzzy tautological IFP (IFTIFP)* if and only if $a \geq b$.

Here, we will show which of the intuitionistic fuzzy implications that were proven in [3] to satisfy the *Modus Ponens*, satisfy *Modus Tollens* as well, in its standard form.

All necessary notations are given in Parts 1, 2 and 3, but below, we will repeat a part of them as appropriately.

The analysis of existing logical programming languages and the results of the current article will be used in the creation of a new logical language that can directly apply *Modus Tollens* as a part of its inference schemes.

2 Main results

As it was mentioned in [3], *in Part 1 of the present research [2], giving short remarks on the results related to the intuitionistic fuzzy implications, for a first time in fuzzy sets theory, we introduced the relations between separate implications.* In Part 2 (see, [3]), we selected those implications that satisfy a larger number of the basic properties of implications, discussed by different authors. In Part 3 (see, [4]), we gave the list of intuitionistic fuzzy implications that satisfy *Modus Ponens*. Preparing the present leg of the research, we found a couple of misprints in the list below which are here corrected. The *Modus Ponens*

$$\frac{A, \quad A \to_i B}{B}$$

is valid for $i = 1, 2, 3, 4, 5, 9, 11, 13, 14, 17, 18, 24, 28, 29, 61, 71, 76, 77, 79, 81, 100, 109, 110, 112, 125, 127, 166, 167, 176, 177$ when A and $A \to_i B$ are TIFPs. In [3], the list incorrectly included implication 186, while missing implications $167, 176$ and 177.

First, we study the intuitionistic fuzzy implications that satisfy the *Modus Tollens*.

Theorem 1. *For* $i = 1, 2, 3, 4, 5, 9, 11, 13, 14, 17, 18, 20, 22, 23, 24, 27, 28, 29, 61, 71, 74, 76, 77, 79, 81, 102, 105, 109, 110, 112, 125, 127, 166, 167, 170, 176, 177, 180, 186, 192, 198$ *the Modus Tollens*

$$\frac{A \to_i B \quad \neg_{\varphi(i)} B}{\neg_{\varphi(i)} A}$$

holds for every two TIFPs A and B, where $\neg_\varphi(i)$ is the intuitionistic fuzzy negation generated by the intuitionistic fuzzy implication $\to_i$.

Proof. First, we will show why for $i = 1$ the assertion is valid. From [5] we see that $\varphi(1) = 1$ and

$$\neg_1 \langle a, b \rangle = \langle b, a \rangle.$$

Because $\neg_1 B$ is an IFTP, it has the form $\langle 1, 0 \rangle$ and hence

$$\neg_1 B = \langle d, c \rangle = \langle 1, 0 \rangle,$$

i.e., $c = 0$ and $d = 1$. From

$$\langle a, b \rangle \to_1 \langle c, d \rangle = \langle \max(b, c), \min(a, d) \rangle = \langle 1, 0 \rangle$$

it follows that

$$\max(b, c) = 1,$$
$$\min(a, d) = 0,$$

i.e., $b = 1$ and $a = 0$ and hence $\neg_1 A = \langle 1, 0\rangle$ is a TIFP.

We will prove also the case, when $i = 2$. Now, $\varphi(2) = 2$ and

$$\neg_2 \langle a, b\rangle = \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle,$$

where

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \qquad \overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

Since $\neg_2 B$ is an IFTP, it has the form $\langle 1, 0\rangle$ and hence

$$\neg_2 B = \langle \overline{\text{sg}}(c), \text{sg}(c)\rangle = \langle 1, 0\rangle,$$

i.e., $c = 0$. We must mention that in this case we do not discuss the value of d.

Now, from formula (see [5]):

$$\begin{aligned} A \rightarrow_2 B &= \langle a, b\rangle \rightarrow_2 \langle c, d\rangle \\ &= \langle \overline{\text{sg}}(a - c), d.\text{sg}(a - c)\rangle \\ &= \langle \overline{\text{sg}}(a), d.\text{sg}(a)\rangle \\ &= \langle 1, 0\rangle \end{aligned}$$

it follows that

$$\begin{aligned} \overline{\text{sg}}(a) &= 1 \\ d.\text{sg}(a) &= 0 \end{aligned}$$

from where it follows that $a = 0$. Therefore,

$$\neg_2 A = \neg_2 \langle a, b\rangle = \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle = \langle 1, 0\rangle,$$

i.e., $\neg_2 A$ is a TIFP.

In the same way, we can check validity of the *Modus Tollens* for the remaining values of i from the Theorem.

Second, we will show why for the values of i that are not mentioned in the Theorem do not satisfy *Modus Tollens*. Let, for example, $i = 35$. Then $\varphi(35) = 8$ and MT has the form

$$\frac{A \rightarrow_{35} B \quad \neg_8 B}{\neg_8 A}.$$

Since

$$\neg_8 B = \neg_8 \langle c, d\rangle = \langle 1 - c, c\rangle$$

is an TIFP, then $c = 0$. Because

$$A \rightarrow_{35} B = \langle a, b\rangle \rightarrow_{35} \langle c, d\rangle = \langle 1 - ad, ad\rangle$$

is a TIFP, then

$$1 - ad = 1.$$

If $d = 0$, then we do not know anything about a, i.e., we cannot assert that $\neg_8 A$ is a TIFP.

Finally, we will discuss one very interesting case: when $i = 101$ and $\varphi(i) = 19$. Then

$$\neg_{19}B = \neg_{19}\langle c, d\rangle = \langle d.\text{sg}(c), 0\rangle.$$

Let $\neg_{19}B = \langle 1, 0\rangle$. Then $d.\text{sg}(c) = 1$. If we assume that $d = 1$, then $c = 0$ and $d.\text{sg}(c) = 0$. Therefore, $c > 0$, i.e., $d < 1$ (the opposite case, $d < 1$ does not imply that $c > 0$, but the case $c = 0$ is already discussed) and let $c > 0$. Then $d.\text{sg}(c) = d < 1$, which is in a contradiction with our assumption. □

Second, we will mention that intuitionistic fuzzy implications $\to_{192}$ and $\to_{198}$, that are not discussed in [4], satisfy *Modus Ponens*, too. Really, if A is a TIFP, i.e., $a = 1, b = 0$, then from

$$\langle a, b\rangle \to_{192} \langle c, d\rangle = \langle \max(c, \min(b, d)), \min(d, \max(a, c))\rangle$$

we obtain

$$\langle 1, 0\rangle \to_{192} \langle c, d\rangle = \langle \max(c, 0), \min(d, 1)\rangle = \langle 1, 0\rangle,$$

which means that $B = \langle 1, 0\rangle$, i.e., B is a TIFP.

Third, in the Table 1, we will give the list and the forms of the "good" intuitionistic fuzzy implications that simultaneously *Modus Ponens* and *Modus Tollens*.

Table 1. List of the intuitionistic fuzzy implications
that satisfy *Modus Ponens* and *Modus Tollens*

$\to_1$	$\langle \max(b, \min(a, c)), \min(a, d)\rangle$
$\to_2$	$\langle \overline{\text{sg}}(a - c), d\text{sg}(a - c)\rangle$
$\to_3$	$\langle 1 - (1 - c)\text{sg}(a - c)), d\text{sg}(a - c)\rangle$
$\to_4$	$\langle \max(b, c), \min(a, d)\rangle$
$\to_5$	$\langle \min(1, b + c), \max(0, a + d - 1)\rangle$
$\to_9$	$\langle b + a^2c, ab + a^2d\rangle$
$\to_{11}$	$\langle 1 - (1 - c)\text{sg}(a - c), d\text{sg}(a - c)\text{sg}(d - b)\rangle$
$\to_{13}$	$\langle b + c - bc, ad\rangle$
$\to_{14}$	$\langle 1 - (1 - c)\text{sg}(a - c) - d\overline{\text{sg}}(a - c)\text{sg}(d - b), d\text{sg}(d - b)\rangle$
$\to_{17}$	$\langle \max(b, c), \min(ab + a^2, d)\rangle$
$\to_{18}$	$\langle \max(b, c), \min(1 - b, d)\rangle$
$\to_{24}$	$\langle \overline{\text{sg}}(a - c)\overline{\text{sg}}(d - b), \text{sg}(a - c)\text{sg}(d - b)\rangle$
$\to_{28}$	$\langle \max(\overline{\text{sg}}(1 - b), c), \min(a, d)\rangle$
$\to_{29}$	$\langle \max(\overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - c)), \min(a, \overline{\text{sg}}(1 - d))\rangle$
$\to_{61}$	$\langle \max(c, \min(b, d)), \min(a, d)\rangle$
$\to_{71}$	$\langle \max(b, c), \min(cd + d^2, a)\rangle$
$\to_{76}$	$\langle \max(c, 1 - a), \min(1 - c, a)\rangle$
$\to_{77}$	$\langle (1 - \min(\overline{\text{sg}}(1 - a), \text{sg}(1 - c))), \min(\overline{\text{sg}}(1 - a), \text{sg}(1 - c))\rangle$

(Continued on the next page)

Table 1 *(Continued from previous page)*

$\to_{79}$	$\langle \max(\overline{\text{sg}}(1-c), \text{sg}(b)), \min(\text{sg}(d), \overline{\text{sg}}(1-a))\rangle$
$\to_{81}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min(d, \overline{\text{sg}}(1-a))\rangle$
$\to_{109}$	$\langle b+\min(\overline{\text{sg}}(1-a), c), ab+\min(\overline{\text{sg}}(1-a), d)\rangle$
$\to_{110}$	$\langle \max(b, c), \min(ab+\overline{\text{sg}}(1-a), d)\rangle$
$\to_{112}$	$\langle b+c-bc, ab+\overline{\text{sg}}(1-a)d\rangle$
$\to_{125}$	$\langle \max(b, c), \min(cd+\overline{\text{sg}}(1-d), a)\rangle$
$\to_{127}$	$\langle b+c-bc, (cd+\overline{\text{sg}}(1-d))a\rangle$
$\to_{166}$	$\langle \max(b, \min(a, c)), \min(a, \max(b, d))\rangle$
$\to_{167}$	$\langle \max(1-a, \min(a, c)), \min(a, 1-\max(a, d))\rangle$
$\to_{176}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\max(b, c), \text{sg}(a-c)\min(a, d)\rangle$
$\to_{177}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\max(1-a, c), \text{sg}(a-c)\min(a, 1-c)\rangle$
$\to_{192}$	$\langle \max(c, 0), \min(d, 1)\rangle = \langle 1, 0\rangle$
$\to_{198}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\frac{c}{a}, \text{sg}(a-c)\frac{a-c}{a}\rangle$

These implications generate the following negations, see Table 2 below.

Table 2. List of the intuitionistic fuzzy negations, generated by the intuitionistic fuzzy implications that satisfy both *Modus Ponens* and *Modus Tollens*

$\neg_1$	$\to_1, \to_4, \to_5, \to_{13}, \to_{61}, \to_{71}, \to_{125}, \to_{127}, \to_{166}, \to_{186}$	$\langle b, a\rangle$
$\neg_2$	$\to_2, \to_3, \to_{11}$	$\langle \overline{\text{sg}}(a), \text{sg}(a)\rangle$
$\neg_3$	$\to_9, \to_{17}$	$\langle b, ab+a^2\rangle$
$\neg_4$	$\to_{18}$	$\langle b, 1-b\rangle$
$\neg_5$	$\to_{14}$	$\langle \overline{\text{sg}}(1-b), \text{sg}(1-b)\rangle$
$\neg_6$	$\to_{24}$	$\langle \overline{\text{sg}}(1-b), \text{sg}(a)\rangle$
$\neg_7$	$\to_{28}, \to_{29}$	$\langle \overline{\text{sg}}(1-b), a\rangle$
$\neg_8$	$\to_{76}, \to_{167}, \to_{177}$	$\langle 1-a, a\rangle$
$\neg_{13}$	$\to_{77}$	$\langle \text{sg}(1-a), \overline{\text{sg}}(1-a)\rangle$
$\neg_{14}$	$\to_{79}$	$\langle \text{sg}(b), \overline{\text{sg}}(1-a)\rangle$
$\neg_{15}$	$\to_{81}$	$\langle \overline{\text{sg}}(1-b),\ os(1-a)\rangle$
$\neg_{18}$	$\to_{100}$	$\langle b.\text{sg}(a), a.\text{sg}(b)\rangle$
$\neg_{26}$	$\to_{109}, \to_{110}, \to_{112}$	$\langle b, ab+\overline{\text{sg}}(1-a)\rangle$
$\neg_{53}$	$\to_{176}$	$\langle \overline{\text{sg}}(a)+\text{sg}(a)b, a\rangle$

As a minor remark, in [5] the following mistake appears which is here corrected: implication $\to_{167}$ generates negation $\neg_{52}$, which however fully coincides with $\neg_8$.

The relationships between the intuitionistic fuzzy implications that satisfy as *Modus Ponens* as well as *Modus Tollens* are shown on Figure 1. All checks necessary for producing Tables 1, 2 and Figure 1 are done with the software IFSTOOL, [1].

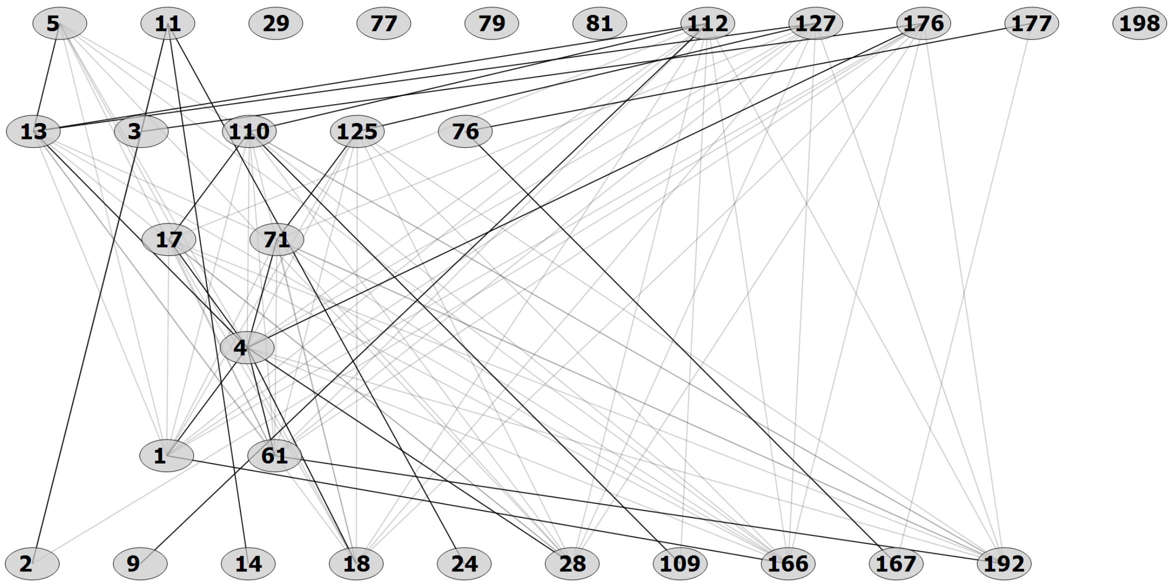

Figure 1. Relationships between the intuitionistic fuzzy implications that satisfy both *Modus Ponens* and *Modus Tollens*

3 Conclusion

Modus Ponens is a fundamental principle in logic that is widely applicable in various aspects of computer science. In programming, it manifests in conditional constructs; in expert systems, it is used to derive conclusions from rules and facts; logical programming, epitomized by languages like Prolog, relies on formal logic to solve problems. In these languages, *Modus Ponens* is a central mechanism for deriving new information from existing facts and rules. In future, some of the so collected implications will be used in a program realization of a new programming language. The key feature of the upcoming development will involve leveraging *Modus Tollens*, mirroring the role of *Modus Ponens* in traditional programming paradigms. This addition aims to enhance the logical inference capabilities of the new programming language, allowing it to derive conclusions by negating implications—a functionality which could be crucial for advancing problem-solving methodologies.

In a next research, we will give the list of the intuitionistic fuzzy conjunctions and disjunctions generated by the above implications and negations and will discuss some of their properties.

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