Notes on Intuitionistic Fuzzy Sets
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Group action on intuitionistic fuzzy ideals of Γ-ring

Poonam Kumar Sharma

Post-Graduate Department of Mathematics, D.A.V. College
Jalandhar, Punjab, India
e-mail: pksharma@davjalandhar.com

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Abstract: Group actions serve as a powerful tool for exploring the symmetry and automorphism properties of rings. In this paper, we examine group actions on intuitionistic fuzzy ideals (IFIs) within a Γ-ring $\mathcal{M}$. We introduce the concept of the intrinsic product of IFIs in $\mathcal{M}$ and explore various properties of intuitionistic fuzzy prime ideals under the influence of group actions. Further, we propose the notion of an intuitionistic fuzzy $\mathcal{G}$-prime ideal in $\mathcal{M}$. We demonstrate that for an IFI A of $\mathcal{M}$, the ideal $A^{\mathcal{G}} = \bigcap_{g\in\mathcal{G}} A^g$ represents the largest $\mathcal{G}$-invariant IFI contained within A. Additionally, we establish that the $\mathcal{G}$-primeness of $A^{\mathcal{G}}$ is uniquely characterized by the $\mathcal{G}$-primeness of A. Lastly, we examine the behavior of intuitionistic fuzzy $\mathcal{G}$-prime ideals of $\mathcal{M}$ under a $\mathcal{G}$-homomorphism.
Keywords: Γ-ring, Intuitionistic fuzzy prime ideal, $\mathcal{G}$-invariant intuitionistic fuzzy ideals, $\mathcal{G}$-prime intuitionistic fuzzy ideals, $\mathcal{G}$-homomorphism.
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1 Introduction

Algebraic structures form a cornerstone of modern mathematics, with wide-ranging applications in fields such as theoretical physics, computer science, control engineering, information theory, and coding theory. Among these, the theory of Γ-rings introduced by Nobusawa [10] as a natural generalization of classical ring theory has emerged as a significant area of study. The class of

Γ-rings includes all rings as well as Hestenes ternary rings, thereby offering a broader algebraic framework. Barnes [3] later refined Nobusawa's definition by slightly relaxing its conditions, leading to a more flexible theoretical foundation. Since then, extensive research has been carried out to develop and expand the theory of Γ-rings in the senses of both Nobusawa and Barnes. These efforts have led to meaningful generalizations of classical results in ring theory. Notable contributions include structural investigations by Barnes [3] and Kyuno [5, 6], Warsi's work on the decomposition of primary ideals [20], and Paul's study of various Γ-ideal types and their corresponding operator rings [15].

The study of group actions on rings led to the establishment of the Galois theory for rings. Lorenz and Passman [8]. Montgomery [9], and others researched the skew grouping approach in the context of the Galois theory, as well as the grouping and the fixed ring. The link between the $\mathcal{G}$-prime ideals of $\mathcal{R}$ and the prime ideals of skew grouping $\mathcal{RG}$ was identified by Lorenz and Passman [8]. Montgomery [9] investigated the relationship between the prime ideals of $\mathcal{R}$ and $\mathcal{R}^{\mathcal{G}}$, leading him to broaden the scope of the action of a group to Spec$\mathcal{R}$.

The idea of intuitionistic fuzzy sets was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set given by Zadeh [21]. Kim et al. in [4] considered the intuitionistic fuzzification of ideal of Γ-ring which were further studied by Palaniappan at al. in [13, 14, 11]. The notion of intuitionistic fuzzy prime ideal and semiprime were studied by Palaniappan and Ramachandran in [12]. Authors in [16] and [18] studied intuitionistic fuzzy characteristic ideals and intuitionistic fuzzy primary ideals in Γ-ring respectively.

Lee and Park [7] recently investigate the action of group on intuitionistic fuzzy ideal of a ring $\mathcal{R}$ and found a relationship between the intuitionistic fuzzy $\mathcal{G}$-prime ideals of $\mathcal{R}$ and the intuitionistic fuzzy prime ideal of $\mathcal{R}$. We define the action of group on an intuitionistic fuzzy ideal of Γ-ring $\mathcal{M}$ and investigate the action of group on intuitionistic fuzzy ideals and $\mathcal{G}$-invariant intuitionistic fuzzy ideals of $\mathcal{M}$. The homomorphic behaviour of group action on intuitionistic fuzzy ideals of Γ-ring $\mathcal{M}$ have also been analysed.

2 Preliminaries

Let us recall some definitions and results, which are necessary for the development of the paper. Throughout this paper unless stated otherwise all Γ-rings are commutative Γ-ring with unity.

Definition 2.1. ([10, 3]) If $(\mathcal{M}, +)$ and $(\Gamma, +)$ are additive Abelian groups, then $\mathcal{M}$ is called a Γ-ring (in the sense of Barnes [3]) if there exist mapping $\mathcal{M} \times \Gamma \times \mathcal{M} \to \mathcal{M}$ [image of (x, α, y) is denoted by $x\alpha y$, $x, y \in \mathcal{M}, \gamma \in \Gamma$] satisfying the following conditions:

(1) $x\alpha y \in \mathcal{M}$.

(2) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$.

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$. for all $x, y, z \in \mathcal{M}$, and $\gamma \in \Gamma$.

The Γ-ring $\mathcal{M}$ is called commutative if $x\gamma y = y\gamma x, \forall x, y \in \mathcal{M}, \gamma \in \Gamma$. An element $1 \in \mathcal{M}$ is said to be the unity of $\mathcal{M}$ if for each $x \in \mathcal{M}$ there exists $\gamma \in \Gamma$ such that $x\gamma 1 = 1\gamma x = x$.

A subset $\mathcal{N}$ of a Γ-ring $\mathcal{M}$ is a left (right) ideal of $\mathcal{M}$ if $\mathcal{N}$ is an additive subgroup of $\mathcal{M}$ and $\mathcal{M}\Gamma\mathcal{N} = \{x\alpha y | x \in \mathcal{M}, \alpha \in \Gamma, y \in \mathcal{N}\}$, ("$\mathcal{N}\Gamma\mathcal{M} = \{x\alpha y | x \in \mathcal{N}, \alpha \in \Gamma, y \in \mathcal{M}\}$") is contained in $\mathcal{N}$. If $\mathcal{N}$ is both a left and a right ideal then $\mathcal{N}$ is a two-sided ideal, or simply an ideal of $\mathcal{M}$. A Γ-ring $\mathcal{M}$ is said to be commutative if $a\gamma b = b\gamma a$ for all $a, b \in \mathcal{M}$ and $\gamma \in \Gamma$. A mapping $\sigma : \mathcal{M} \to \mathcal{M}'$ of Γ-rings is called a Γ-homomorphism [3] if $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(x\alpha y) = \sigma(x)\alpha\sigma(y)$ for all $x, y \in \mathcal{M}, \alpha \in \Gamma$.

Definition 2.2. ([20]) A proper ideal $\mathcal{I}$ of a Γ-ring $\mathcal{M}$ is called a prime if for any ideal $\mathcal{U}, \mathcal{V}$ of $\mathcal{M}$ $\mathcal{U}\Gamma\mathcal{V} \subseteq \mathcal{I}$ implies $\mathcal{U} \subseteq \mathcal{I}$ or $\mathcal{V} \subseteq \mathcal{I}$.

Definition 2.3. ([1, 2]) An intuitionistic fuzzy set A in $\mathcal{X}$ can be represented as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{X}\}$, where the functions $\mu_A, \nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in \mathcal{X}$ to A, respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in \mathcal{X}$.

Remark 2.4. ([1, 2]) When $\mu_A(x) + \nu_A(x) = 1, \forall x \in \mathcal{X}$, then A is called a fuzzy set.

If $A, B \in IFS(\mathcal{X})$, then $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in \mathcal{X}$. Also, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. For any subset $\mathcal{Y}$ of $\mathcal{X}$, the intuitionistic fuzzy characteristic function $\chi_{\mathcal{Y}}$ is an intuitionistic fuzzy set of $\mathcal{X}$, defined by: $\chi_{\mathcal{Y}}(x) = (1, 0), \forall x \in \mathcal{Y}$ and $\chi_{\mathcal{Y}}(x) = (0, 1), \forall x \in \mathcal{X}\backslash\mathcal{Y}$. Let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then the crisp set $A_{(\alpha,\beta)} = \{x \in \mathcal{X} : \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta\}$ is called the (α, β)-level cut subset of A. Also, the IFS $x_{(\alpha,\beta)}$ of $\mathcal{X}$ defined as $x_{(\alpha,\beta)}(y) = (\alpha, \beta)$, if $y = x$, otherwise $(0, 1)$ is called the intuitionistic fuzzy point (IFP) in $\mathcal{X}$ with support x. By $x_{(\alpha,\beta)} \in A$ we mean $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Furthermore, if $\sigma : \mathcal{X} \to \mathcal{Y}$ is a mapping, and A and B are IFS of $\mathcal{X}$ and $\mathcal{Y}$, respectively, then the image $\sigma(A)$, is an IFS of $\mathcal{Y}$, is defined as: $\mu_{\sigma(A)}(y) = \sup\{\mu_A(x) : \sigma(x) = y\}$, $\nu_{\sigma(A)}(y) = \inf\{\nu_A(x) : \sigma(x) = y\}$, for all $y \in \mathcal{Y}$. The inverse image $\sigma^{-1}(B)$, is an IFS of $\mathcal{X}$, is defined as: $\mu_{\sigma^{-1}(B)}(x) = \mu_B(\sigma(x))$, $\nu_{\sigma^{-1}(B)}(x) = \nu_B(\sigma(x))$, for all $x \in \mathcal{X}$, i.e., $\sigma^{-1}(B)(x) = B(\sigma(x))$, for all $x \in \mathcal{X}$. Furthermore, the IFS A of $\mathcal{X}$ is said to be σ-invariant if for any $x, y \in \mathcal{X}$, whenever $\sigma(x) = \sigma(y)$ implies $A(x) = A(y)$.

Definition 2.5. ([4, 14]) Let A and B be two IFSs of a Γ-ring $\mathcal{M}$, and $\gamma \in \Gamma$. Then the intrinsic product $A\Gamma B$ is defined by:

$$(\mu_{A\Gamma B}(x), \nu_{A\Gamma B}(x)) = \begin{cases} (\sup\limits_{x=y\gamma z} \min\{\mu_A(y), \mu_B(z)\}, \inf\limits_{x=y\gamma z} \max\{\nu_A(y), \nu_B(z)\}), & \text{if } x = y\gamma z \\ (0, 1), & \text{otherwise} \end{cases}$$

Definition 2.6. ([4, 14]) Let A be an IFS of a Γ-ring $\mathcal{M}$. Then A is called an intuitionistic fuzzy ideal of $\mathcal{M}$, if for all $x, y \in \mathcal{M}, \gamma \in \Gamma$, the following are satisfied:

(i) $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(ii) $\mu_A(x\gamma y) \geq \max\{\mu_A(x), \mu_A(y)\}$;

(iii) $\nu_A(x - y) \leq \max\{\nu_A(x), \nu_A(y)\}$;

(iv) $\nu_A(x\gamma y) \leq \min\{\nu_A(x), \nu_A(y)\}$.

The set of all intuitionistic fuzzy ideals of a Γ-ring $\mathcal{M}$ is denoted by $IFI(\mathcal{M})$. Note that if $A \in IFI(\mathcal{M})$, then $\mu_A(0_{\mathcal{M}}) \geq \mu_A(x) \geq \mu_A(1_{\mathcal{M}})$ and $\nu_A(0_{\mathcal{M}}) \leq \nu_A(x) \leq \nu_A(1_{\mathcal{M}}), \forall x \in \mathcal{M}$ (see [21]).

Definition 2.7. ([12, 18]) Let P be an IFI of a Γ-ring $\mathcal{M}$. Then P is said to be prime if it is non-constant, and for any IFIs A and B of $\mathcal{M}$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Remark 2.8. ([12, 18]) Let $x_{(p,q)}, y_{(t,s)} \in IFP(\mathcal{M})$. Then

$$x_{(p,q)}\Gamma y_{(t,s)} = (x\Gamma y)_{(p\wedge t, q\vee s)}$$

Theorem 2.9. ([12, 18]) *Let $\mathcal{M}$ be a commutative Γ-ring, and let A be an IFI of $\mathcal{M}$. Then following are equivalent:*

(i) *A is an intuitionistic fuzzy prime ideal of $\mathcal{M}$.*

(ii) *For any $x_{(p,q)}, y_{(t,s)} \in IFP(\mathcal{M})$ such that $x_{(p,q)}\Gamma y_{(t,s)} \subseteq A \Rightarrow x_{(p,q)} \subseteq A$ or $y_{(t,s)} \subseteq A$.*

Proposition 2.10. ([17]) *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings. If $\sigma : \mathcal{M} \to \mathcal{M}'$ is a surjective homomorphism, then for all $x \in \mathcal{M}$ and $p, q \in (0, 1]$ such that $p + q \leq 1$, we have*

$$\sigma(x_{(p,q)}) = (\sigma(x))_{(p,q)}$$

Theorem 2.11. ([12, 17]) *Let $f : \mathcal{M} \to \mathcal{M}'$ be a surjective Γ-homomorphism, and let A be an f-invariant IF prime ideal of $\mathcal{M}$, and B be an IF prime ideal of $\mathcal{M}'$. Then $f(A)$ and $f^{-1}(B)$ are IF prime ideal of $\mathcal{M}'$ and $\mathcal{M}$, respectively.*

3 Group action on intuitionistic fuzzy ideals of Γ-ring

In this section, we study the group action on the intuitionistic fuzzy ideals of the Γ-ring. Using this concept, we develop the notion of a $\mathcal{G}$-invariant intuitionistic fuzzy ideal of the Γ-ring, which is further utilized to define the concept of intuitionistic fuzzy $\mathcal{G}$-prime ideals. Various related properties of these notions will also be analysed.

Definition 3.1. ([1]) Let $\mathcal{G}$ be a group and $\mathcal{S}$ a non-empty set. Then the mapping $\phi : \mathcal{G} \times \mathcal{S} \to \mathcal{S}$, with $\phi(g, x)$ written as $g * x$, is an action of $\mathcal{G}$ on $\mathcal{S}$ if and only if, for all $g, h \in \mathcal{G}$ and $x \in \mathcal{S}$, the following conditions hold:

1. $g * (h * x) = (gh) * x$,
2. $e * x = x$, where e is the identity element of the group $\mathcal{G}$.

We assume that $\mathcal{M}$ is a Γ-ring and $\mathcal{G} = Aut(\mathcal{M})$ is the group of automorphism of $\mathcal{M}$. Then $\mathcal{G}$ acts on $\mathcal{M}$, defined by $\phi(g, x) = g(x), i.e., g * x = g(x)$. In this section, we define the group action of $\mathcal{G}$ on an intuitionistic fuzzy set (IFS) A of the Γ-ring $\mathcal{M}$.

Definition 3.2. The group action of $\mathcal{G}$ on an IFS A of a Γ-ring $\mathcal{M}$ is denoted by A^g and is defined as $A^g = \{\langle x, \mu_{A^g}(x), \nu_{A^g}(x)\rangle : x \in \mathcal{M}\}$, where $\mu_{A^g}(x) = \mu_A(x^g)$ and $\nu_{A^g}(x) = \nu_A(x^g)$ for every $x \in \mathcal{M}, g \in \mathcal{G}$.

Example 3.3. Consider $\mathcal{M} = \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$, $\Gamma = \{(0,0), (1,1)\}$ and $\mathcal{K} = \mathbf{Z}_2 \times \{0\} = \{(1,0), (0,0)\}$ and $\mathcal{H} = \{0\} \times \mathbf{Z}_2 = \{(0,0), (0,1)\}$, where $\mathbf{Z}_2$ is the ring of integers modulo 2. Clearly, $\mathcal{M}$ and Γ are additive abelian groups, and that $\mathcal{M}$ is a Γ-ring. Moreover, $\mathcal{K}$ and $\mathcal{H}$ are Γ-ideals of $\mathcal{M}$.

Consider the IFS A on $\mathcal{M}$ defined as follows:

$$\mu_A((x,y)) = \begin{cases} 1, & \text{if } (x,y) \in \mathcal{K} \\ 0.5, & \text{if } (x,y) \notin \mathcal{K} \end{cases}; \quad \nu_A((x,y)) = \begin{cases} 0, & \text{if } (x,y) \in \mathcal{K} \\ 0.3, & \text{if } (x,y) \notin \mathcal{K}. \end{cases}$$

Now, $\mathcal{G} = Aut(\mathcal{M}) = \{g_1 = i, g_2, g_3, g_4, g_5, g_6\}$ be the group of automorphisms of $\mathcal{M}$, where

$$\begin{aligned}
g_1 &= \{(0,0) \to (0,0); (0,1) \to (0,1); (1,0) \to (1,0); (1,1) \to (1,1)\} \\
g_2 &= \{(0,0) \to (0,0); (0,1) \to (1,1); (1,0) \to (1,0); (1,1) \to (0,1)\} \\
g_3 &= \{(0,0) \to (0,0); (0,1) \to (0,1); (1,0) \to (1,1); (1,1) \to (1,0)\} \\
g_4 &= \{(0,0) \to (0,0); (0,1) \to (1,0); (1,0) \to (0,1); (1,1) \to (1,1)\} \\
g_5 &= \{(0,0) \to (0,0); (0,1) \to (1,0); (1,0) \to (1,1); (1,1) \to (0,1)\} \\
g_6 &= \{(0,0) \to (0,0); (0,1) \to (1,1); (1,0) \to (0,1); (1,1) \to (1,0)\}.
\end{aligned}$$

Now, it is easy to see that, for $g = g_1$ or $g_2 \in \mathcal{G}$, $A^g = A$ and for $g = g_4$ or $g_5 \in \mathcal{G}$, A^g is given by

$$\mu_{A^g}((x,y)) = \begin{cases} 1, & \text{if } (x,y) \in \mathcal{H} \\ 0.5, & \text{if } (x,y) \notin \mathcal{H} \end{cases}; \quad \nu_{A^g}((x,y)) = \begin{cases} 0, & \text{if } (x,y) \in \mathcal{H} \\ 0.3, & \text{if } (x,y) \notin \mathcal{H}. \end{cases}$$

Also, for $g = g_3$ or $g_6 \in \mathcal{G}$, A^g is given by

$$\mu_{A^g}((x,y)) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,0),(1,1)\} \\ 0.5, & \text{if } (x,y) \notin \{(0,0),(1,1)\} \end{cases}; \quad \nu_{A^g}((x,y)) = \begin{cases} 0, & \text{if } (x,y) \in \{(0,0),(1,1)\} \\ 0.3, & \text{if } (x,y) \notin \{(0,0),(1,1)\}. \end{cases}$$

From the definition of group action on an IFS, the following results are easy to derive.

Lemma 3.4. *Let A and B be two IFSs of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on A and B. Then:*

1. $(A \cap B)^g = A^g \cap B^g, \forall g \in \mathcal{G}$;
2. $(A \cup B)^g = A^g \cup B^g, \forall g \in \mathcal{G}$;
3. $(A \times B)^g = A^g \times B^g, \forall g \in \mathcal{G}$;
4. *If* $A \subseteq B$, *then* $A^g \subseteq B^g, \forall g \in \mathcal{G}$;
5. $(A^g)^h = A^{gh}, \forall g, h \in \mathcal{G}$;
6. $(A^g)^{g^{-1}} = A^e, \forall g \in \mathcal{G}$.

Lemma 3.5. *Let $\mathcal{G}$ be a finite group which acts on a Γ-ring $\mathcal{M}$. Then for $x, y \in \mathcal{M}, \gamma \in \Gamma, g \in \mathcal{G}$, we have*

1. $(x - y)^g = x^g - y^g$;
2. $(x\gamma y)^g = x^g \gamma y^g$;
3. $(x, y)^g = (x^g, y^g)$;

Proposition 3.6. *Let A be an IFI of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on A. Then A^g is also an IFI of $\mathcal{M}$.*

Proof. Let $x, y \in \mathcal{M}, \gamma \in \Gamma, g \in \mathcal{G}$ be any elements. Then

$$\begin{aligned} \mu_{A^g}(x-y) &= \mu_A((x-y)^g) = \mu_A(x^g - y^g) \\ &\geq \min\{\mu_A(x^g), \mu_A(y^g)\} \\ &= \min\{\mu_{A^g}(x), \mu_{A^g}(y)\}. \end{aligned}$$

Thus $\mu_{A^g}(x-y) \geq \min\{\mu_{A^g}(x), \mu_{A^g}(y)\}$.
Likewise, it can be established that $\nu_{A^g}(x-y) \leq \max\{\nu_{A^g}(x), \nu_{A^g}(y)\}$. Also,

$$\begin{aligned} \mu_{A^g}(x\gamma y) &= \mu_A((x\gamma y)^g) = \mu_A(x^g\gamma y^g) \\ &\geq \max\{\mu_A(x^g), \mu_A(y^g)\} \\ &= \max\{\mu_{A^g}(x), \mu_{A^g}(y)\}. \end{aligned}$$

Thus $\mu_{A^g}(x\gamma y) \geq \max\{\mu_{A^g}(x), \mu_{A^g}(y)\}$.
Likewise, it can be proved that $\nu_{A^g}(x\gamma y) \leq \min\{\nu_{A^g}(x), \nu_{A^g}(y)\}$.
Thus A^g is an IFI of $\mathcal{M}$. □

Remark 3.7. It is easy to check that the IFS A as defined in Example 3.3 is an IFI of $\mathcal{M}$. Also, for $g = g_1$ or g_2 or g_4 or $g_5 \in \mathcal{G}$, A^g is also an IFI of $\mathcal{M}$. However, for $g = g_3$ or $g_6 \in \mathcal{G}$, A^g is not an IFI of $\mathcal{M}$, for

$$\mu_{A^{g_6}}((1,1)(1,1)(0,1)) = \mu_{A^{g_6}}((0,1)) = 0.5 \ngeq 1 = \max\{\mu_{A^{g_6}}((1,1)), \mu_{A^6}((0,1))\} = \{1, 0.5\};$$

$$\nu_{A^{g_6}}((1,1)(1,1)(0,1)) = \nu_{A^{g_6}}((0,1)) = 0.3 \nleq 0 = \min\{\nu_{A^{g_6}}((1,1)), \nu_{A^{g_6}}((0,1))\} = \{0, 0.3\}.$$

Definition 3.8. Let A be an IFI of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on A. Then A is called an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$ if A^g is an IFI of $\mathcal{M}$ for all $g \in \mathcal{G}$.

Proposition 3.9. *If A, B are two IFIs of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on A and B, then $(A \cap B)^g$ is also an IFI of $\mathcal{M}$.*

Proof. Follows from Lemma 3.4(1) and Proposition 3.6. □

Proposition 3.10. *If A, B are two IFIs of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on A and B, then $(A \times B)^g$ is also an IFI of $\mathcal{M}$.*

Proof. Follows from Lemma 3.4(3) and Proposition 3.6. □

Proposition 3.11. *If P is an intuitionistic fuzzy prime ideal of a Γ-ring $\mathcal{M}$, then P^g is also an intuitionistic fuzzy prime ideal of $\mathcal{M}$, where $g \in \mathcal{G}$ be any element.*

Proof. Let A, B be two IFIs of a near ring $\mathcal{M}$ with $A\Gamma B \subseteq P^g$, where $g \in \mathcal{G}$ be any element. Now, we claim that $A^{g^{-1}}\Gamma B^{g^{-1}} \subseteq P$. It is sufficient to show that $\mu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) \leq \mu_P(x)$ and $\nu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) \geq \nu_P(x), \forall \gamma \in \Gamma, x \in \mathcal{M}$.

$$\begin{aligned} \mu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) &= \sup_{x=a\gamma b}\{\min(\mu_{A^{g^{-1}}}(a), \mu_{B^{g^{-1}}}(b))\} \\ &= \sup_{x^{g^{-1}}=a^{g^{-1}}\gamma b^{g^{-1}}}\{\min(\mu_A(a^{g^{-1}}), \mu_B(b^{g^{-1}}))\} \\ &= \mu_{A\Gamma B}(x^{g^{-1}}) \\ &\leq \mu_{P^g}(x^{g^{-1}}) \\ &= \mu_P(x). \end{aligned}$$

Thus $\mu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) \leq \mu_P(x)$. Likewise, it can be established that $\nu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) \geq \nu_P(x)$. Hence $A^{g^{-1}}\Gamma B^{g^{-1}} \subseteq P$ which implies that either $A^{g^{-1}} \subseteq P$ or $B^{g^{-1}} \subseteq P$, i.e., either $A \subseteq P^g$ or $B \subseteq P^g$, for if, $A^{g^{-1}} \subseteq P$, then $\mu_A(x) = \mu_A((x^g)^{g^{-1}}) = \mu_{A^{g^{-1}}}(x^g) \leq \mu_P(x^g) = \mu_{P^g}(x)$. Similarly, we have $\nu_A(x) \geq \nu_{P^g}(x)$. Thus $A \subseteq P^g$. Hence P^g is an IF prime ideal of $\mathcal{M}$. □

By using the definition of $\mathcal{G}$-invariant ideal of a ring $\mathcal{M}$, we define $\mathcal{G}$-invariant intuitionistic fuzzy ideal and $\mathcal{G}$-invariant intuitionistic fuzzy prime ideal of Γ-ring $\mathcal{M}$.

Definition 3.12. Let A be an IFS of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a group which acts on $\mathcal{M}$. Then A is said to be $\mathcal{G}$-invariant IFS of $\mathcal{M}$ if and only if

$$\mu_{A^g}(x) = \mu_A(x^g) \geq \mu_A(x), \nu_{A^g}(x) = \nu_A(x^g) \leq \nu_A(x), \forall x \in \mathcal{M} \text{ and } \forall g \in \mathcal{G}.$$

Proposition 3.13. *Let A be an IFS of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$. Then A is $\mathcal{G}$-invariant IFS of $\mathcal{M}$ if and only if $A^g = A$, for all $g \in \mathcal{G}$.*

Proof. For $x \in \mathcal{M}, g \in \mathcal{G}$, we have $\mu_A(x) = \mu_A((x^g)^{g^{-1}}) \geq \mu_A(x^g) \geq \mu_A(x)$ implies that $\mu_A(x) = \mu_A(x^g) = \mu_{A^g}(x)$. Similarly, we have $\nu_A(x) = \nu_{A^g}(x)$. Hence $A^g = A$. □

Theorem 3.14. *Let A be an IFS of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$. Let $A^{\mathcal{G}} = \bigcap_{g\in\mathcal{G}} A^g$. Then $A^{\mathcal{G}} = (\mu_{A^{\mathcal{G}}}, \nu_{A^{\mathcal{G}}})$, where $\mu_{A^{\mathcal{G}}}(x) = \min\{\mu_A(x^g) : g \in \mathcal{G}\}$ and $\nu_{A^{\mathcal{G}}}(x) = \max\{\nu_A(x^g) : g \in \mathcal{G}\}, \forall x \in \mathcal{M}$. Moreover, $A^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IFS of $\mathcal{M}$ contained in A.*

Proof. Since A be an IFS of $\mathcal{M}$ and so A^g is IFS of $\mathcal{M}$ for all $g \in \mathcal{G}$. Also, intersection of IFSs of $\mathcal{M}$ is an IFS of $\mathcal{M}$ and so $A^{\mathcal{G}}$ is an IFS of $\mathcal{M}$. Next, we show that $A^{\mathcal{G}}$ is $\mathcal{G}$-invariant IFS of $\mathcal{M}$. Now,

$$\begin{aligned} \mu_{A^{\mathcal{G}}}(x^g) &= \min\{\mu_{A^h}(x^g) : h \in \mathcal{G}\} \\ &= \min\{\mu_A\{(x^g)^h\} : h \in \mathcal{G}\} \\ &= \min\{\mu_A(x^{gh}) : h \in \mathcal{G}\} \\ &= \min\{\mu_A(x^{g'}) : g' \in \mathcal{G}\} \\ &= \mu_{A^{\mathcal{G}}}(x). \end{aligned}$$

Likewise, it can be proved that $\nu_{A^{\mathcal{G}}}(x^g) = \nu_{A^{\mathcal{G}}}(x), \forall x \in \mathcal{M}$. Thus $A^{\mathcal{G}}$ is $\mathcal{G}$-invariant IFS of $\mathcal{M}$.

Further, let B be any $\mathcal{G}$-invariant IFS of $\mathcal{M}$ such that $B \subseteq A$. Then for any $x \in \mathcal{M}, g \in \mathcal{G}$, we get $\mu_B(x^g) = \mu_B(x) \leq \mu_A(x)$ and $\nu_B(x^g) = \nu_B(x) \geq \nu_A(x)$. Now,

$$\mu_B(x^g) = \mu_B(x) = \mu_B\{(x^g)^{g^{-1}}\} \leq \mu_A(x^g) \Rightarrow \mu_B(x) \leq \min\{\mu_A(x^g) : g \in \mathcal{G}\} = \mu_{A^{\mathcal{G}}}(x).$$

Similarly,

$$\nu_B(x^g) = \nu_B(x) = \nu_B\{(x^g)^{g^{-1}}\} \geq \nu_A(x^g) \Rightarrow \nu_B(x) \geq \max\{\nu_A(x^g) : g \in G\} = \nu_{A^{\mathcal{G}}}(x).$$

Thus $B \subseteq A^{\mathcal{G}}$. Hence $A^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IFS of $\mathcal{M}$ contained in A. □

Proposition 3.15. *Let A be an IFI of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$. Then $A^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IFI of $\mathcal{M}$ contained in A.*

Proof. Let $x, y \in \mathcal{M}, \gamma \in \Gamma, g \in \mathcal{G}$ be any element, then

$$\begin{aligned}
\mu_{A^{\mathcal{G}}}(x-y) &= \min\{\mu_A((x-y)^g) : g \in \mathcal{G}\} \\
&= \min\{\mu_A(x^g - y^g) : g \in \mathcal{G}\} \\
&\geq \min\{\min\{\mu_A(x^g), \mu_A(y^g)\} : g \in \mathcal{G}\} \\
&= \min\{\min\{\mu_A(x^g) : g \in \mathcal{G}\}, \min\{\mu_A(y^g) : g \in \mathcal{G}\}\} \\
&= \min\{\mu_{A^{\mathcal{G}}}(x), \mu_{A^{\mathcal{G}}}(y)\}.
\end{aligned}$$

Likewise, it can be established that $\nu_{A^{\mathcal{G}}}(x-y) \leq \max\{\nu_{A^{\mathcal{G}}}(x), \nu_{A^{\mathcal{G}}}(y)\}$. Also,

$$\begin{aligned}
\mu_{A^{\mathcal{G}}}(x\gamma y) &= \min\{\mu_A((x\gamma y)^g) : g \in \mathcal{G}\} \\
&= \min\{\mu_A(x^g \gamma y^g) : g \in \mathcal{G}\} \\
&\geq \min\{\max\{\mu_A(x^g), \mu_A(y^g)\} : g \in \mathcal{G}\} \\
&= \max\{\min\{\mu_A(x^g), \mu_A(y^g)\} : g \in \mathcal{G}\} \\
&= \max\{\min\{\mu_A(x^g) : g \in \mathcal{G}\}, \min\{\mu_A(y^g) : g \in \mathcal{G}\}\} \\
&= \max\{\mu_{A^{\mathcal{G}}}(x), \mu_{A^{\mathcal{G}}}(y)\}.
\end{aligned}$$

Hence $A^{\mathcal{G}}$ is an IFI of $\mathcal{M}$. Further, $A^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IFI of $\mathcal{M}$ contained in A can be proved similar to Theorem 3.14. □

Proposition 3.16. *An IFI A of a Γ-ring $\mathcal{M}$ is $\mathcal{G}$-invariant IFI of $\mathcal{M}$ if and only if $A^{\mathcal{G}} = A$.*

Proof. From Proposition 3.15 we get $A^{\mathcal{G}} \subseteq A$. Also, because A is $\mathcal{G}$-invariant IFS of $\mathcal{M}$ and $A \subseteq A$ implies that $A \subseteq A^{\mathcal{G}}$. Hence $A^{\mathcal{G}} = A$. □

Example 3.17. Consider $\mathcal{M}, \Gamma, \mathcal{G}$ and the IFS A as in Example (3.3).
Then it is easy to obtain $A^{\mathcal{G}}$ as:

$$\mu_{A^{\mathcal{G}}}((x,y)) = \begin{cases} 1, & \text{if } (x,y) = (0,0) \\ 0.5, & \text{if } (x,y) \neq (0,0) \end{cases}; \quad \nu_{A^{\mathcal{G}}}((x,y)) = \begin{cases} 0, & \text{if } (x,y) = (0,0) \\ 0.3, & \text{if } (x,y) \neq (0,0). \end{cases}$$

Since $A^{\mathcal{G}} \neq A$. So A is not a $\mathcal{G}$-invariant IFI of $\mathcal{M}$.

Example 3.18. Consider $\mathcal{M} = \mathbf{Z_6} = \{0,1,2,3,4,5\}$, $\Gamma = \mathbf{Z_2} = \{0,1\}$. Then $\mathcal{M}$ is a Γ-ring. Take $\mathcal{K} = \{0,2,4\}$ be a Γ-ideal of $\mathcal{M}$. Define an IFS A on $\mathcal{M}$ as

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in \mathcal{K} \\ 0.5, & \text{if } x \notin \mathcal{K} \end{cases}; \quad \nu_A((x,y)) = \begin{cases} 0, & \text{if } x \in \mathcal{K} \\ 0.3, & \text{if } x \notin \mathcal{K}. \end{cases}$$

It is easy to check that A is an IFI of $\mathcal{M}$. Let $\mathcal{G} = Aut(\mathcal{M}) = \{i, g\}$ be the automorphism group of $\mathcal{M}$, where $g = \{1 \to 5, 2 \to 4, 3 \to 3, 4 \to 2, 5 \to 1, 0 \to 0\}$. Now it is easy to check that $A^g = A, \forall g \in \mathcal{G}$. Moreover, we also get that $A^{\mathcal{G}} = A$. Hence A is a $\mathcal{G}$-invariant IFI of $\mathcal{M}$.

Theorem 3.19. *If A and B are $\mathcal{G}$-invariant IFIs of a Γ-ring $\mathcal{M}$, then $A+B$ is also a $\mathcal{G}$-invariant IFI of $\mathcal{M}$.*

Proof. Suppose $x \in \mathcal{M}, g \in \mathcal{G}$ be any elements, then

$$\begin{aligned}
\mu_{(A+B)^g}(x) &= \mu_{A+B}(x^g) \\
&= \sup_{x^g=a+b} \{\mu_A(a), \mu_B(b)\} \\
&= \sup_{x^g=a+b} \{\mu_A(a^{g^{-1}}), \mu_B(b^{g^{-1}})\} \\
&= \sup_{x=a^{g^{-1}}+b^{g^{-1}}} \{\mu_A(a^{g^{-1}}), \mu_B(b^{g^{-1}})\} \\
&= \mu_{A+B}(x).
\end{aligned}$$

Likewise, it can be established that $\nu_{(A+B)^g}(x) = \nu_{A+B}(x)$. Thus $(A+B)^g = A+B, \forall g \in \mathcal{G}$. Hence $A+B$ is also a $\mathcal{G}$-invariant IFI of $\mathcal{M}$. □

Theorem 3.20. *If A and B are $\mathcal{G}$-invariant IFIs of a Γ-ring $\mathcal{M}$, then $A\Gamma B$ is also a $\mathcal{G}$-invariant IFI of $\mathcal{M}$.*

Proof. Let $x \in \mathcal{M}, \gamma \in \Gamma, g \in \mathcal{G}$ be any elements, then

$$\begin{aligned}
\mu_{(A\Gamma B)^g}(x) &= sup_{x^g=a\gamma b} \min\{\mu_A(a), \mu_B(b)\} \\
&= \sup_{x^g=a\gamma b} \min\{\mu_A(a^{g^{-1}}), \mu_B(b^{g^{-1}})\} \\
&= \sup_{x=a^{g^{-1}}\gamma b^{g^{-1}}} \min\{\mu_A(a^{g^{-1}}), \mu_B(b^{g^{-1}})\} \\
&= \mu_{A\Gamma B}(x).
\end{aligned}$$

Likewise, it can be established that $\nu_{(A\Gamma B)^g}(x) = \nu_{A\Gamma B}(x)$. Thus $(A\Gamma B)^g = A\Gamma B, \forall g \in \mathcal{G}$. Hence $A\Gamma B$ is also a $\mathcal{G}$-invariant IFI of $\mathcal{M}$. □

Definition 3.21. Let P be a non-constant IFI of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on P. Then P is termed as an IF $\mathcal{G}$-prime ideal of $\mathcal{M}$ if P is $\mathcal{G}$-invariant IF prime ideal of $\mathcal{M}$.

Proposition 3.22. *Let P be an IF $\mathcal{G}$-prime ideal of $\mathcal{M}$. Then $P_{(s,t)}$ is a $\mathcal{G}$-prime ideal of $\mathcal{M}$, where $s \in [\mu_P(1), \mu_P(0)]$ and $t \in [\nu_P(0), \nu_P(1)]$ such that $s+t \leq 1$.*

Proof. It is easy to show that $P_{(s,t)}$ is an ideal of $\mathcal{M}$. We show that $P_{(s,t)}$ is a $\mathcal{G}$-invariant.
Let $x \in P_{(s,t)}, g \in \mathcal{G}$ be any element. Since P is $\mathcal{G}$-invariant intuitionistic fuzzy prime ideal of $\mathcal{M}$, so $\mu_P(x^g) = \mu_P(x) \geq s$ and $\nu_P(x^g) = \nu_P(x) \leq t, \forall g \in \mathcal{G}$ implies that $x^g \in P_{(s,t)}, \forall g \in \mathcal{G}$. Hence $P_{(s,t)}$ is $\mathcal{G}$-invariant.

Next we show that $P_{(s,t)}$ is a prime ideal of $\mathcal{M}$. Let I and J be two $\mathcal{G}$-invariant ideals of $\mathcal{M}$ such that $I\Gamma J \subseteq P_{(s,t)}$. Define two IFSs $A = \chi_I$ and $B = \chi_J$. It is easy to check that A and B are $\mathcal{G}$-invariant IFIs of $\mathcal{M}$ (as I and J are $\mathcal{G}$-invariant ideals). We claim that $A\Gamma B \subseteq P$. Let

$x \in \mathcal{M}$, $\gamma \in \Gamma$ be any element. If $A\Gamma B(x) = (0,1)$, there is nothing to prove. If $A\Gamma B(x) \neq (0,1)$, then $\mu_{A\Gamma B}(x) = \sup_{x=y\gamma z} \min\{\mu_A(y), \mu_B(z)\} = \sup_{x=y\gamma z} \min\{\mu_{\chi_I}(y), \mu_{\chi_I}(z)\} \neq 0$ and $\nu_{A\Gamma B}(x) = \inf_{x=y\gamma z} \max\{\nu_A(y), \nu_B(z)\} = \inf_{x=y\gamma z} \max\{\nu_{\chi_I}(y), \nu_{\chi_I}(z)\} \neq 1$. This implies that there exist $y \in I, z \in J$ such that $x = y\gamma z$. Moreover, $A\Gamma B(x) = (s,t)$. Thus $x = y\gamma z \in I\Gamma J \subseteq P_{(s,t)}$. So $\mu_P(x) \geq s, \nu_P(x) \leq t$. Hence $A\Gamma B \subseteq P$. Since P is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}$, either $A \subseteq P$ or $B \subseteq P$. Suppose that, $A \subseteq P$, then $I \subseteq P_{(s,t)}$. For, if $I \supset P_{(s,t)}$, then there is an element $a \in \mathcal{M}$ such that $a \in I$, but $a \notin P_{(s,t)}$. This implies that $\mu_A(a) = \mu_{\chi_I}(a) = s$ and $\nu_A(a) = \nu_{\chi_I}(a) = t$, but $\mu_P(a) < s$ and $\nu_P(a) > t$. Thus $\mu_A(a) = s > \mu_P(a)$ and $\nu_A(a) = t < \nu_P(a)$. Hence $A \supset P$, a contradiction. Similarly, we have $B \subseteq P$. Hence $P_{(s,t)}$ is $\mathcal{G}$-prime ideal of $\mathcal{M}$. □

From the above discussion on the results on intuitionistic fuzzy prime ideals that are $\mathcal{G}$-invariant also. We can also define intuitionistic fuzzy $\mathcal{G}$-prime ideals in the following ways too

Definition 3.23. A non-constant $\mathcal{G}$-invariant IFI P of a Γ-ring $\mathcal{M}$ is said to be $\mathcal{G}$-prime IFI if for any two $\mathcal{G}$-invariant IFIs A and B of $\mathcal{M}$ such that $A\Gamma B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.

Proposition 3.24. *Let P be an intuitionistic fuzzy $\mathcal{G}$-invariant ideal of Γ-ring $\mathcal{M}$. Then the following are equivalent:*

1. *P is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}$;*
2. *For any $x_{(p,q)}, y_{(s,t)} \in IFP(\mathcal{M})$, where $x, y \in \mathcal{M}$ are $\mathcal{G}$-invariant points such that $x_{(p,q)}\Gamma y_{(s,t)} \subseteq P$ implies that either $x_{(p,q)} \subseteq P$ or $y_{(s,t)} \subseteq P$.*

Proof. (1) ⇒ (2): Assume that P is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}$.
Let $x_{(p,q)}, y_{(s,t)} \in IFP(\mathcal{M})$, where $x, y \in \mathcal{M}$ are $\mathcal{G}$-invariant points such that $x_{(p,q)}\Gamma y_{(s,t)} \subseteq P$.
Then $x_{(p,q)}\Gamma y_{(s,t)} = (x\Gamma y)_{(p\wedge s, q\vee t)}$, where $\mu_P(x\Gamma y) \geq p \wedge s$ and $\nu_P(x\Gamma y) \leq q \vee t$.

Define IFSs $A = \chi_{\langle x\rangle}$ and $B = \chi_{\langle y\rangle}$ of $\mathcal{M}$. Clearly, A and B are $\mathcal{G}$-invariant IFIs of $\mathcal{M}$. Now $\mu_{A\Gamma B}(z) = \sup\limits_{z=u\gamma v} \min\{\mu_A(u), \mu_B(v)\} = p \wedge s$; $\nu_{A\Gamma B}(z) = \inf\limits_{z=u\gamma v} \max\{\nu_A(u), \nu_B(v)\} = q \vee t$, where $u \in \langle x\rangle$ and $v \in \langle y\rangle$. Thus $\mu_{A\Gamma B}(z) = p \wedge s \leq \mu_P(z)$ and $\nu_{A\Gamma B}(z) = q \vee t \geq \nu_P(z)$, when $z = u\gamma v$, where $u \in \langle x\rangle, v \in \langle y\rangle$ and $\gamma \in \Gamma$. Otherwise $A\Gamma B(z) = (0,1)$, i.e., $A\Gamma B \subseteq P$. As P is an intuitionistic fuzzy $\mathcal{G}$-prime ideal so either $A \subseteq P$ or $B \subseteq P$. Then $x_{(p,q)} \subseteq A \subseteq P$ or $y_{(s,t)} \subseteq B$.

(2) ⇒ (1): Let A and B be two $\mathcal{G}$-invariant IFIs of $\mathcal{M}$ such that $A\Gamma B \subseteq P$. Suppose $A \supset P$. Then there exists a $\mathcal{G}$-invariant element $x \in \mathcal{M}$ such that $\mu_A(x) > \mu_P(x)$ and $\nu_A(x) < \nu_P(x)$. Let $\mu_A(x) = p, \nu_A(x) = q$. Let $y \in \mathcal{M}$ be $\mathcal{G}$-invariant element of $\mathcal{M}$ with $\mu_A(x) = r, \nu_A(x) = s$. If $z = x\gamma y$, then $x_{(p,q)}\Gamma y_{(s,t)} = (x\Gamma y)_{(p\wedge s, q\vee t)}$.
Hence $\mu_P(z) = \mu_P(x\gamma y) \geq \mu_{A\Gamma B}(x\gamma y) \geq \min\{\mu_A(x), \mu_B(y)\} = p \wedge s = \mu_{(x\gamma y)_{(p\wedge s, q\vee t)}}(z)$.
Similarly, we have $\nu_P(z) \leq \nu_{(x\gamma y)_{(p\wedge s, q\vee t)}}(z)$. Hence $x_{(p,q)}\Gamma y_{(s,t)} \subseteq P$, then by (1), we get either $x_{(p,q)} \subseteq P$ or $y_{(s,t)} \subseteq P$, i.e., either $\mu_P(x) \geq p, \nu_P(x) \leq q$ or $\mu_P(x) \geq s, \nu_P(x) \leq t$.
Since $\mu_P(x) < p, \nu_P(x) > q$ and $\mu_B(y) = s \leq \mu_P(y), \nu_B(y) = t \geq \nu_P(y)$. So, $B \subseteq P$.
Hence P is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}$. □

Similarly, we can prove the following proposition.

Proposition 3.25. *If P is an IF prime ideal of a Γ-ring $\mathcal{M}$, then $P^{\mathcal{G}}$ is $\mathcal{G}$-prime IFI ideal of $\mathcal{M}$. Conversely, if Q is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}$, then there exists an IF prime ideal P of $\mathcal{M}$ such that $P^{\mathcal{G}} = Q$.*

Proof. Let P be an intuitionistic fuzzy prime ideal of the Γ-ring $\mathcal{M}$ and let A and B be two $\mathcal{G}$-invariant IFIs of $\mathcal{M}$ such that $A\Gamma B \subseteq P^{\mathcal{G}}$. Then $A\Gamma B \subseteq P$ (since $P^{\mathcal{G}} \subseteq P$ always). So, either $A \subseteq P$ or $B \subseteq P$. But $P^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IFI of $\mathcal{M}$ contained in P. So, either $A \subseteq P^{\mathcal{G}}$ or $B \subseteq P^{\mathcal{G}}$. Hence $P^{\mathcal{G}}$ is $\mathcal{G}$-prime IFI ideal of $\mathcal{M}$.

For the converse part, suppose that Q is an IF $\mathcal{G}$-prime ideal of $\mathcal{M}$. Therefore, $Q^{\mathcal{G}} = Q$. Let $\mathcal{S} = \{P | P \text{ is an IFI of } \mathcal{M} \text{ with } P^{\mathcal{G}} \subseteq Q\}$. By Zorn′s lemma, there exists an intuitionistic fuzzy maximal ideal P such that $P^{\mathcal{G}} \subseteq Q$. Let A and B be two IFIs of $\mathcal{M}$ such that $A\Gamma B \subseteq P$. Then $(A\Gamma B)^{\mathcal{G}} \subseteq P^{\mathcal{G}} \subseteq Q$. Since $A^{\mathcal{G}}$ and $B^{\mathcal{G}}$ are largest IFIs of $\mathcal{M}$ contained in A and B, respectively. We claim that $A^{\mathcal{G}}\Gamma B^{\mathcal{G}} \subseteq A\Gamma B$ is $\mathcal{G}$-invariant.

$$\begin{aligned}\mu_{A^{\mathcal{G}}\Gamma B^{\mathcal{G}}}(x^g) &= \sup_{x^g = u\gamma v} \min\{\mu_{A^{\mathcal{G}}}(u), \mu_{B^{\mathcal{G}}}(v)\} \\ &= \sup_{x = u^{g^{-1}}\gamma v^{g^{-1}}} \min\{\mu_{A^{\mathcal{G}}}(u^{g^{-1}}), \mu_{B^{\mathcal{G}}}(v^{g^{-1}})\} \\ &= \mu_{A^{\mathcal{G}}\Gamma B^{\mathcal{G}}}(x).\end{aligned}$$

Similarly, we can show that $\nu_{A^{\mathcal{G}}\Gamma B^{\mathcal{G}}}(x^g) = \nu_{A^{\mathcal{G}}\Gamma B^{\mathcal{G}}}(x)$. Hence $A^{\mathcal{G}}\Gamma B^{\mathcal{G}} \subseteq (A\Gamma B)^{\mathcal{G}} \subseteq Q$. Since Q is an IF $\mathcal{G}$ prime ideal of $\mathcal{M}$, then we have either $A^{\mathcal{G}} \subseteq Q$ or $B^{\mathcal{G}} \subseteq Q$. By maximality of P either $A \subseteq P$ or $B \subseteq P$. This implies that P is an IF prime ideal of $\mathcal{M}$. As $Q^{\mathcal{G}} = Q$, we have $Q \in \mathcal{S}$. But maximality of P gives that $Q \subseteq P$. Since P and $Q^{\mathcal{G}}$ are $\mathcal{G}$ invariant and $P^{\mathcal{G}}$ is largest in P, we get $Q \subseteq P^{\mathcal{G}}$. Hence $P^{\mathcal{G}} = Q$. □

4 $\mathcal{G}$-homomorphism of intuitionistic fuzzy $\mathcal{G}$–ideals

In this part of paper, we explore the image and preimage of intuitionistic fuzzy $\mathcal{G}$-ideals under the Γ-ring homomorphism.

Definition 4.1. A Γ-ring homomorphism $\phi : \mathcal{M} \to \mathcal{M}'$ from a Γ-ring $\mathcal{M}$ to a Γ-ring $\mathcal{M}'$ with unity is called $\mathcal{G}$-homomorphism, if for all $g \in \mathcal{G}, x \in \mathcal{M}, \phi(g * x) = g * \phi(x)$, where group $\mathcal{G}$ acts on both the Γ-rings.

Lemma 4.2. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ be a function defined by $f(x^g) = (f(x))^g, \forall x \in \mathcal{M}, g \in \mathcal{G}$. Then f is a Γ-ring homomorphism. Moreover, f is also a $\mathcal{G}$-homomorphism.*

Proof. Let $x, y \in \mathcal{M}, \gamma \in \Gamma, g \in \mathcal{G}$ be any elements, then we have

$$f(x^g + y^g) = f((x+y)^g) = (f(x) + f(y))^g = (f(x))^g + (f(y))^g = f(x^g) + f(y^g) \text{ and}$$
$$f(x^g\gamma y^g) = f((x\gamma y)^g) = (f(x)\gamma f(y))^g = (f(x))^g\gamma(f(y))^g = f(x^g)\gamma f(y^g).$$

$$\begin{aligned} f(g*(x_1+x_2)) &= f((x_1+x_2)^g) = (f(x_1^g+x_2^g)) \\ &= f(x_1^g)+f(x_2^g) = (f(x_1))^g+(f(x_2))^g \\ &= g*f(x_1)+g*f(x_2) \\ &= g*(f(x_1)+f(x_2)) \\ &= g*f(x_1+x_2). \end{aligned}$$

Also,

$$\begin{aligned} f(g*(x_1\gamma x_2)) &= f((x_1\gamma x_2)^g) = (f(x_1^g\gamma x_2^g)) \\ &= f(x_1^g)\gamma f(x_2^g) = (f(x_1))^g\gamma(f(x_2))^g \\ &= (g*f(x_1))\gamma(g*f(x_2)) \\ &= g*(f(x_1)\gamma f(x_2)) \\ &= g*f(x_1\gamma x_2). \end{aligned}$$

Therefore f is a $\mathcal{G}$-homomorphism. □

Lemma 4.3. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f:\mathcal{M}\to\mathcal{M}'$ be a $\mathcal{G}$-homomorphism and A,B are IFSs of $\mathcal{M}$ and $\mathcal{M}'$, respectively. Then*
(1) $f^{-1}(B^g)=(f^{-1}(B))^g, \forall g\in\mathcal{G}$;
(2) $f(A^g)=(f(A))^g, \forall g\in\mathcal{G}$.

Proof. (1) Let $x\in\mathcal{M}$ and $g\in\mathcal{G}$ be any element. Then

$$f^{-1}(B^g)(x)=B^g(f(x))=B((f(x))^g)=B(f(x^g)))=f^{-1}(B)(x^g)=(f^{-1}(B))^g(x)$$

Hence $f^{-1}(B^g)=(f^{-1}(B))^g, \forall g\in\mathcal{G}$.

(2) Let $y\in\mathcal{M}'$ and $g\in\mathcal{G}$ be any element. Then $f(A^g)(y)=(\mu_{f(A^g)}(y),\nu_{f(A^g)}(y))$. Now

$$\begin{aligned} \mu_{f(A^g)}(y) &= \sup\{\mu_{A^g}(x):f(x)=y\}=\sup\{\mu_A(x^g):f(x)=y\} \\ &= \sup\{\mu_A(x^g):f(x^g)=y^g\} \\ &= \mu_{f(A)}(f(x^g))=\mu_{f(A)}((f(x))^g)=\mu_{(f(A))^g}(f(x))=\mu_{(f(A))^g}(y). \end{aligned}$$

Similarly, we can show that $\nu_{f(A^g)}(y)=\mu_{(f(A))^g}(y)$. Hence $f(A^g)=(f(A))^g, \forall g\in\mathcal{G}$. □

Theorem 4.4. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f:\mathcal{M}\to\mathcal{M}'$ be a $\mathcal{G}$-homomorphism. If B is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}'$, then $f^{-1}(B)$ is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$.*

Proof. Let B be an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}'$. To show that $f^{-1}(B)$ is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$. For this we show that $(f^{-1}(B))^g$ is an IFI of $\mathcal{M}$ for all $g\in\mathcal{G}$. In view of Lemma 4.2(1), we show that $f^{-1}(B^g)$ is an IFI of $\mathcal{M}$ for all $g\in\mathcal{G}$.
For $x,y\in\mathcal{M},\gamma\in\Gamma$ and $g\in\mathcal{G}$, $f^{-1}(B^g)(x+y)=(\mu_{f^{-1}(B^g)}(x+y),\nu_{f^{-1}(B^g)}(x+y))$, where

$$\begin{aligned} \mu_{f^{-1}(B^g)}(x+y) &= \mu_{B^g}\{f(x+y)\}=\mu_{B^g}\{f(x)+f(y)\} \\ &\geq \min\{\mu_{B^g}(f(x)),\mu_{B^g}(f(y))\} \\ &= \min\{\mu_{f^{-1}(B^g)}(x),\mu_{f^{-1}(B^g)}(y)\}. \end{aligned}$$

Thus $\mu_{f^{-1}(B^g)}(x+y)\geq\min\{\mu_{f^{-1}(B^g)}(x),\mu_{f^{-1}(B^g)}(y)\}$.

Likewise, it can be established that $\nu_{f^{-1}(B^g)}(x+y) \leq \max\{\nu_{f^{-1}(B^g)}(x), \nu_{f^{-1}(B^g)}(y)\}$.
Also, $f^{-1}(B^g)(x\gamma y) = (\mu_{f^{-1}(B^g)}(x\gamma y), \nu_{f^{-1}(B^g)}(x\gamma y))$, where

$$\begin{aligned} \mu_{f^{-1}(B^g)}(x\gamma y) &= \mu_{B^g}\{f(x\gamma y)\} = \mu_{B^g}\{f(x)\gamma f(y)\} \\ &\geq \max\{\mu_{B^g}(f(x)), \mu_{B^g}(f(y))\} \\ &= \max\{\mu_{f^{-1}(B^g)}(x), \mu_{f^{-1}(B^g)}(y)\}. \end{aligned}$$

Thus $\mu_{f^{-1}(B^g)}(x\gamma y) \geq \max\{\mu_{f^{-1}(B^g)}(x), \mu_{f^{-1}(B^g)}(y)\}$.
Likewise, it can be proved that $\nu_{f^{-1}(B^g)}(x\gamma y) \leq \min\{\nu_{f^{-1}(B^g)}(x), \nu_{f^{-1}(B^g)}(y)\}$. Therefore, $f^{-1}(B^g) = (f^{-1}(B))^g$ is an IFI of $\mathcal{M}$.
Hence $f^{-1}(B)$ is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$. □

Theorem 4.5. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\mathcal{G}$-epimorphism. If A is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$ which is constant on* $\operatorname{Ker} f$ *of $\mathcal{M}$, then $f(A)$ is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}'$.*

Proof. Let A be an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$. To show that $f(A)$ is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}'$. For this we show that $(f(A))^g$ is IFI of $\mathcal{M}'$ for all $g \in \mathcal{G}$. In view of Lemma 4.2(2), we show that $f(A^g)$ is an IFI of $\mathcal{M}'$ for all $g \in \mathcal{G}$.

Let $x', y' \in \mathcal{M}', \gamma \in \Gamma, g \in \mathcal{G}$. As f is epimorphism, therefore there exist $x, y \in \mathcal{M}$ such that $f(x) = x'$ and $f(y) = y'$. Now, $f(A^g)(x'+y') = (\mu_{f(A^g)}(x'+y'), \nu_{f(A^g)}(x'+y'))$, where

$$\begin{aligned} \mu_{f(A^g)}(x'+y') &= \mu_{f(A^g)}(f(x)+f(y)) = \mu_{f(A^g)}(f(x+y)) \\ &= \mu_{A^g}(x+y) \\ &\geq \min\{\mu_{A^g}(x), \mu_{A^g}(y)\} \\ &= \min\{\mu_{f(A^g)}(f(x)), \mu_{f(A^g)}(f(y))\} \\ &= \min\{\mu_{f(A^g)}(x'), \mu_{f(A^g)}(y')\}. \end{aligned}$$

Thus, $\mu_{f(A^g)}(x'+y') \geq \min\{\mu_{f(A^g)}(x'), \mu_{f(A^g)}(y')\}$.
Similarly, we can show that $\nu_{f(A^g)}(x'+y') \leq \max\{\nu_{f(A^g)}(x'), \nu_{f(A^g)}(y')\}$.
Also, $f(A^g)(x'\gamma y') = (\mu_{f(A^g)}(x'\gamma y'), \nu_{f(A^g)}(x'\gamma y'))$, where

$$\begin{aligned} \mu_{f(A^g)}(x'\gamma y') &= \mu_{f(A^g)}(f(x)\gamma f(y)) = \mu_{f(A^g)}(f(x\gamma y)) \\ &= \mu_{A^g}(x\gamma y) \\ &\geq \max\{\mu_{A^g}(x), \mu_{A^g}(y)\} \\ &= \max\{\mu_{f(A^g)}(f(x)), \mu_{f(A^g)}(f(y))\} \\ &= \max\{\mu_{f(A^g)}(x'), \mu_{f(A^g)}(y')\}. \end{aligned}$$

Thus, $\mu_{f(A^g)}(x'\gamma y') \geq \max\{\mu_{f(A^g)}(x'), \mu_{f(A^g)}(y')\}$.
Similarly, we can show that $\nu_{f(A^g)}(x'\gamma y') \leq \min\{\nu_{f(A^g)}(x'), \nu_{f(A^g)}(y')\}$. Therefore, $f(A^g)$ and so $(f(A))^g$ is an IFI of $\mathcal{M}'$. Hence $f(A)$ is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}'$. □

Theorem 4.6. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\mathcal{G}$-homomorphism. If P be an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}'$, then $f^{-1}(P)$ is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}$.*

Proof. Since P be an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}'$ so by Theorem 2.11 $f^{-1}(P)$ is also an intuitionistic fuzzy prime ideal of $\mathcal{M}$. So, it remains to show that $f^{-1}(P)$ is $\mathcal{G}$-invariant. For this consider $x \in \mathcal{M}, g \in \mathcal{G}$ be any elements. Then we have $\mu_{f^{-1}(P)}(x^g) = \mu_P(f(x^g)) = \mu_P((f(x))^g) = \mu_P((f(x))) = \mu_{f^{-1}(P)}(x)$. Likewise, it can be proved that $\nu_{f^{-1}(P)}(x^g) = \nu_{f^{-1}(P)}(x)$. Thus $f^{-1}(P)$ is $\mathcal{G}$-invariant. Hence $f^{-1}(P)$ is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}$. □

Theorem 4.7. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\mathcal{G}$-epimorphism. If P is an intuitionistic fuzzy $\mathcal{G}$-prime ideal which is constant on $Kerf$ of $\mathcal{M}$, then $f(P)$ is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}'$.*

Proof. Since P be an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}$ which is constant on $Kerf$ of $\mathcal{M}$ so by Theorem 2.11 $f(P)$ is also an intuitionistic fuzzy prime ideal of $\mathcal{M}'$. So, it remains to show that $f(P)$ is $\mathcal{G}$-invariant. For this consider $y \in \mathcal{M}', g \in \mathcal{G}$ be any element. As f is an epimorphism so, there exists $x \in \mathcal{M}$ such that $f(x) = y$. Then we have
$\mu_{f(P)}(y^g) = \mu_{(f(P))^g}(y) = \mu_{f(P^g)}(y) = \mu_{P^g}(f^{-1}(y)) = \mu_{P^g}(x) = \mu_P(x^g) = \mu_P(x) = \mu_P(f^{-1}(y)) = \mu_{f(P)}(y)$. Similarly, we can show that $\nu_{f(P)}(y^g) = \nu_{f(P)}(y)$. Thus $f(P)$ is $\mathcal{G}$-invariant. Hence $f(P)$ is an intuitionistic fuzzy $\mathcal{G}$-prime ideal of $\mathcal{M}'$. □

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