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A variant of Craig's interpolation theorem for intuitionistic fuzzy formulas. Part 2

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Abstract: A variant of Craig's interpolation theorem related to so-called sg-implication in intuitionistic fuzzy logic, is given. An open problem is formulated.
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1 Preliminaries

Here we formulate and prove an analogue of W. Craig's interpolation theorem [1, 2] in the terms of Intuitionistic Fuzzy Propositional Calculus (IFPC; see [3]). The present research is a second part of [4], where the case of $max - min$-implication is discussed. Now, we shall study the case of sg-implications.

To each proposition p in IFPC (see [3]) we can assign a "truth degree" $\mu(p) \in [0, 1]$ and a "falsity degree" $\nu(p) \in [0, 1]$, such that $\mu(p) + \nu(p) \leq 1$.

Let this assignment be provided by an evaluation function V defined over a set of propositions S in such a way that $V(p) = \langle \mu(p),\ \nu(p) \rangle$. When the values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be extended also for the operation "$\rightarrow$" by the definition :

$$V(p \supset q) = \langle 1 - (1 - \mu(q)).sg(\mu(p) - \mu(q)), \nu(q).sg(\mu(p) - \mu(q)).sg(\nu(q) - \nu(p)) \rangle,$$

where:

$$sg(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

and let $V(p) \supset V(q) = V(p \supset q)$.

Proposition A is a tautology, if and only if $V(A) = \langle 1, 0 \rangle$.

All the above notions for propositions are analogically extended for the case of formulas.

2 Main result

Let $\mathcal{F}$ be a set of formulas, with the property that for all $\langle a, b\rangle \in [0,1] \times [0,1]$ such that $a+b \leq 1$, there exists a formula $f \in \mathcal{F}$ such that $V(F) = \langle a, b\rangle$.

Theorem: Let F and G be different formulas in $\mathcal{F}$ and let $F \to G$ be a tautology. Then, there exists a formula $H \in \mathcal{F}$ different from F and G, such that $F \to H$ and $H \to G$ are tautologies.
Proof: Let $V(F) = \langle \mu_F, \nu_F\rangle, V(G) = \langle \mu_G, \nu_G\rangle$. Then,

$$V(F \to G) = \langle 1 - (1-\mu_G).sg(\mu_F - \mu_G), \nu_G.sg(\mu_F - \mu_G).sg(\nu_G - \nu_F)\rangle$$

and by condition,

$$1 - (1-\mu_G).sg(\mu_F - \mu_G) = 1$$

$$\nu_G.sg(\mu_F - \mu_G).sg(\nu_G - \nu_F) = 0$$

Hence,

$$(1-\mu_G).sg(\mu_F - \mu_G) = 0. \tag{1}$$

Let $V(H) = \langle \mu_H, \nu_H\rangle$, where, e.g.,

$$\mu_H = \frac{\mu_F + \mu_G}{2}, \nu_H = \frac{\nu_F + \nu_G}{2}.$$

Having in mind that F and G are different formulas in $\mathcal{F}$, we see that the above defined formula H will be different than formulas in F and G, too. Now, we obtain

$$(1-\mu_H).sg(\mu_F - \mu_H) = \frac{1}{2}(2 - \mu_F - \mu_G).sg(\mu_F - \mu_G) = 0, \tag{2}$$

because, if $\mu_F \leq \mu_G$, then $sg(\mu_F - \mu_G) = 0$ and therefore (2) is valid. On the other hand, if we assume that $1 \geq \mu_F > \mu_G$, then $sg(\mu_F - \mu_G) = 1$ and from (1) it follows that $1 - \mu_G = 0$, i.e., $\mu_G = 1$, which is impossible. If $\mu_F = \mu_G = 1$, then $2 - \mu_F - \mu_G = 0$ and $sg(\mu_F - \mu_G) = 0$. Therefore, in all cases (2) is vaild, i.e.,

$$V(F \to H) = \langle 1 - (1-\mu_H).sg(\mu_F - \mu_H), \nu_H.sg(\mu_F - \mu_H).sg(\nu_H - \nu_F)\rangle = \langle 1, 0\rangle.$$

Analogically, it is checked that $H \to G$ is a tautology, that proves the Theorem.

Open problem: Check whether a similar result is valid for all other types of IF implications.

References

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