Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy group subalgebras

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Abstract: This paper presents a systematic investigation of intuitionistic fuzzy algebraic structures associated with group algebras. We introduce the concept of an intuitionistic fuzzy group subalgebra (IFGSA) of an intuitionistic fuzzy group algebra (IFGA) constructed from the group algebra $K[G]$, where G is a finite group and K is a field. Structural properties of IFGSAs are examined, with particular emphasis on their behavior under intuitionistic fuzzy group algebra homomorphisms (IFGA-homomorphisms). The image and inverse image of IFGSAs are studied, and it is proved that the intersection of an arbitrary family of IFGSAs is again an IFGSA. Furthermore, the notion of an intuitionistic fuzzy augmentation ideal (IFAI) in an IFGA is introduced and analyzed. It is shown that the intersection of an arbitrary family of IFAIs remains an IFAI. The image and inverse image of IFAIs under IFGA-homomorphisms are also investigated. Finally, isomorphism theorems for IFGAs are established, extending classical group algebra results to the intuitionistic fuzzy framework.
Keywords: Intuitionistic fuzzy algebra, Group algebra, Intuitionistic fuzzy group subalgebra, Intuitionistic fuzzy augmentation ideal, IFGA-homomorphism.
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1 Introduction

The theory of fuzzy sets, introduced by Zadeh [23], has provided a powerful framework for modeling uncertainty and vagueness in mathematical structures. As a natural generalization, intuitionistic fuzzy sets, proposed by Atanassov [1–3], incorporate both degrees of membership and non-membership, thereby offering greater flexibility in the representation of imprecise information. Over the past few decades, intuitionistic fuzzy concepts have been extensively studied and successfully applied to various branches of algebra, including groups, rings, vector spaces, ideals, and modules (see [4–6, 8–12, 14–18, 20, 21]).

Group algebras form an important class of algebraic structures that establish a deep connection between group theory and ring theory. Given a finite group G and a field K, the group algebra $K[G]$ plays a central role in representation theory and related areas (see [7, 13]). Recently, the author in [19] has extended classical algebraic notions associated with group algebras to the intuitionistic fuzzy settings, leading to the development of an intuitionistic fuzzy group algebras. These generalizations have opened new directions for studying algebraic systems under uncertainty.

The interaction between intuitionistic fuzzy theory and group algebras motivates the investigation of intuitionistic fuzzy substructures and their homomorphic properties. While intuitionistic fuzzy subalgebras and ideals have been examined in broader algebraic contexts, a systematic study of intuitionistic fuzzy group subalgebras and augmentation ideals within intuitionistic fuzzy group algebras has not yet been developed. In particular, properties related to intersections, homomorphic images, inverse images, and isomorphism theorems remain largely unexplored in this specialized setting.

The primary objective of this paper is to address these gaps by introducing and studying intuitionistic fuzzy algebraic structures associated with group algebras. We define the notion of an intuitionistic fuzzy group subalgebra of an intuitionistic fuzzy group algebra constructed from the group algebra $K[G]$. Fundamental properties of these subalgebras are investigated, with special emphasis on their behavior under intuitionistic fuzzy group algebra homomorphisms. We prove that the intersection of an arbitrary family of intuitionistic fuzzy group subalgebras is again an intuitionistic fuzzy group subalgebra.

Furthermore, we introduce the concept of an intuitionistic fuzzy augmentation ideal in an intuitionistic fuzzy group algebra and examine its structural properties. We show that the intersection of an arbitrary family of intuitionistic fuzzy augmentation ideals remains an intuitionistic fuzzy augmentation ideal. The image and the inverse image of intuitionistic fuzzy augmentation ideals under intuitionistic fuzzy group algebra homomorphisms are also analyzed. In addition, we establish isomorphism theorems for intuitionistic fuzzy group algebras, thereby extending classical results from group algebra theory to the intuitionistic fuzzy framework.

The paper is organized as follows. Section 2 recalls basic definitions and preliminary results on intuitionistic fuzzy sets, group algebras, and intuitionistic fuzzy group algebras. In Section 3, we introduce intuitionistic fuzzy group subalgebras and investigate their fundamental properties. Section 4 is devoted to the study of intuitionistic fuzzy augmentation ideals and their behavior under IFGA-homomorphisms. In Section 5, we establish isomorphism theorems for intuitionistic fuzzy group algebras. Finally, concluding remarks and possible directions for future research are presented.

2 Preliminaries

In this section, we recall basic definitions and results that are required throughout the paper. Unless otherwise stated, all groups considered are finite and K denotes a field.

Definition 2.1. ([2]) Let X be a non-empty set. An intuitionistic fuzzy set (IFS) A of X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and of non-membership of the element $x \in X$, respectively, and for any $x \in X$, we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Remark 2.2.

(i) If $\mu_A(x) + \nu_A(x) = 1$ for all $x \in X$, then A is called a fuzzy set.

(ii) For convenience, we write the IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ by $A = (\mu_A, \nu_A)$.

(iii) For $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, the (α, β)-cut set of an IFS $A = (\mu_A, \nu_A)$ is defined as the crisp set $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$. We denote the set $A_{(0,1)}$ by A^*.

Definition 2.3. ([2]) Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two IFSs of X, then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)\ \forall\, x \in X$;

(ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)\ \forall\, x \in X$;

(iii) $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$, where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_{A\cap B}(x) = \mu_A(x) \vee \mu_B(x)$;

(iv) $A \cup B = (\mu_{A\cup B}, \nu_{A\cup B})$, where $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$ and $\nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x)$.

Definition 2.4. ([5]) Let $(G, \cdot)$ be a group, and let A be an IFS in G. Then A is called an intuitionistic fuzzy group (IFG) if, for all $g_1, g_2, g \in G$, the following hold:

(i) $\mu_A(g_1g_2) \geq \min\{\mu_A(g_1), \mu_A(g_2)\}$; (ii) $\mu_A(g^{-1}) = \mu_A(g)$;

(iii) $\nu_A(g_1g_2) \leq \max\{\nu_A(g_1), \nu_A(g_2)\}$; (iv) $\nu_A(g^{-1}) = \nu_A(g)$.

Further, A is called an intuitionistic fuzzy normal subgroup (IFNSG) of G, if in addition to above conditions it satisfies $\mu_A(g_1g_2) = \mu_A(g_2g_1)$ and $\nu_A(g_1g_2) = \nu_A(g_2g_1)$, for all $g_1, g_2 \in G$.

Definition 2.5. ([21]) Let $f : G_1 \to G_2$ be a group homomorphism. If A and B are IFGs on G_1 and G_2, respectively, then f is called a weak intuitionistic fuzzy homomorphism of A into B if $f(A) \subseteq B$. It is called an intuitionistic fuzzy homomorphism of A onto B if $f(A) = B$. Further, if $f : G_1 \to G_2$ is an isomorphism, then f is called a weak intuitionistic fuzzy isomorphism if $f(A) \subseteq B$, and an intuitionistic fuzzy isomorphism if $f(A) = B$.

Definition 2.6. ([7]) Let G be a group and M be a vector space over a field K. Then M is called a G-module if for every $g \in G$ and $m \in M$, there exists a product (called the action of G on M), $g \cdot m \in M$ that satisfies the following axioms

(i) $1_G.m = m$, for all $m \in M$ (1_G being the identity of G)

(ii) $(gh) \cdot m = g \cdot (h \cdot m)$, for all $m \in M, g, h \in G$

(iii) $g.(k_1m_1 + k_2m_2) = k_1(g.m_1) + k_2(g.m_2)$, for all $k_1, k_2 \in K; m_1, m_2 \in M$ and $g \in G$.

Definition 2.7. ([7]) Let M be a G-module. A vector subspace N of M is a G-submodule if N is also a G-module under the same action of G.

Definition 2.8. ([7, 13]) Let M and M^* be G-modules. A mapping f : M $\to M^*$ is a G-module homomorphism if
(i) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$
(ii)$f(g \cdot m) = g \cdot f(m)$, for all $k_1, k_2 \in K; m, m_1, m_2 \in M$ and $g \in G$.

Definition 2.9. ([20]) Let G be a group and M be a G-module over K, which is a subfield of C. Then an intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of M such that for all $a, b \in K, g \in G; m, x, y \in M$, the following conditions are satisfied

(i) $\mu_A(ax + by) \geq \min\{\mu_A(x), \mu_A(y)\}$; (ii) $\nu_A(ax + by) \leq \max\{\nu_A(x), \nu_A(y)\}$;

(iii) $\mu_A(g \cdot m) \geq \mu_A(m)$; (iv) $\mu_A(g \cdot m) \leq \mu_A(m)$.

Recall that for $x = \sum_{g\in G} a_g g \in K[G]$, where $a_g \in K$, the set $\{g \in G : a_g \neq 0\}$ is called the support of x and is denoted by $supp(x)$. Moreover, $supp(hx) = h(supp(x))$ for all $h \in G$ (see Exercise 1, page 107 of [13]).

Definition 2.10. ([19]) Let G be a finite group, and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy group (IFG) on G. Then the intuitionistic fuzzy group algebra (IFGA) induced by A is an intuitionistic fuzzy set $K[A] = (\mu_{K[A]}, \nu_{K[A]})$ on the group algebra $K[G]$ defined as

$$(\mu_{K[A]}(x), \nu_{K[A]}(x)) = \begin{cases} (1, 0), & \text{if } supp(x) = \varnothing \\ (\min\{\mu_A(g) : g \in supp(x)\}, \max\{\nu_A(g) : g \in supp(x)\}), & \text{if } supp(x) \neq \varnothing. \end{cases}$$

where $supp(x) = \{g \in G : a_g \neq 0\}$ denotes the support of the element $x = \sum_{g\in G} a_g g \in K[G]$.

Remark 2.11. Let $K[A]$ be an intuitionistic fuzzy group algebra on the group algebra $K[G]$. For an element $x = \sum_{g\in G} a_g g \in K[G]$ with $supp(x) = S = \{g \in G : a_g \neq 0\}$, we write $x \in K[A]$ to mean that $\mu_{K[A]}(x) = \min\{\mu_A(g) : g \in S\}$ and $\nu_{K[A]}(x) = \max\{\nu_A(g) : g \in S\}$.

Proposition 2.12. ([19]) *If A, B are two intuitionistic fuzzy groups defined on a group G, then*

(i) $K[A \cap B] = K[A] \cap K[B]$; *(ii) If $A \subseteq B$, then $K[A] \subseteq K[B]$.*

Proposition 2.13. ([19]) *If $K[A]$ is an IFGA on $K[G]$ corresponding to the IFG A on G, then for $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, we have $(K[A])_{(\alpha,\beta)} = K[A_{(\alpha,\beta)}]$.*

Definition 2.14. ([7]) Let $\psi : G_1 \to G_2$ be a group homomorphism. The induced K-algebra homomorphism $f : K[G_1] \to K[G_2]$ is defined by $f(\sum_{g\in G_1} a_g g) = \sum_{g\in G_1} a_g \psi(g), a_g \in K$.

Theorem 2.15. *([19]) Let $f : K[G_1] \to K[G_2]$ be a group algebra homomorphism induced by a group homomorphism $\psi : G_1 \to G_2$. Then*

(i) If B is an IFG on G_2 and $K[B]$ is the corresponding IFGA on $K[G_2]$, then

$$f^{-1}(K[B]) = K[\psi^{-1}(B)] \text{ is an IFGA on } K[G_1].$$

(ii) If A is an IFG on G_1 and $K[A]$ is the corresponding IFGA on $K[G_1]$, then

$$f(K[A]) = K[\psi(A)] \text{ is an IFGA on } K[G_2].$$

Definition 2.16. ([19]) Let G_1 and G_2 be groups, and let K be a field. Suppose A is an IFG of G_1 and B is an IFG of G_2. Let $K[A]$ and $K[B]$ denote the corresponding IFGAs over $K[G_1]$ and $K[G_2]$, respectively. Then $f : K[A] \to K[B]$ is called an *intuitionistic fuzzy group algebra homomorphism* (IFGA-homomorphism) if the following hold:

(i) f is an induced group algebra homomorphism

(ii) $\mu_{K[B]}(f(\alpha)) \geq \mu_{K[A]}(\alpha)$ and $\nu_{K[B]}(f(\alpha)) \leq \nu_{K[A]}(\alpha)$, for all $\alpha = \sum_{g \in G_1} a_g g \in K[G_1]$.

To avoid confusion, we denote an IFGA-homomorphism by $\overline{f} : K[A] \to K[B]$, while its underlying K-algebra homomorphism is written as $f : K[G_1] \to K[G_2]$.

Theorem 2.17. ([19]) *Let $K[A], K[B], K[C]$ be IFGAs corresponding to IFGs A on G_1, B on G_2, and C on G_3, respectively. If $\overline{f} : K[A] \to K[B]$ and $\overline{h} : K[B] \to K[C]$ are IFGA-homomorphisms, then the composition $\overline{h} \circ \overline{f} : K[A] \to K[C]$ is also an IFGA-homomorphism.*

Remark 2.18. ([22]) An IFGA-homomorphism $\bar{f} : K[A] \to K[B]$ is called an

(1) IFGA-monomorphism if $f : K[G_1] \to K[G_2]$ is an injection;

(2) IFGA-epimorphism if $f : K[G_1] \to K[G_2]$ is an surjection;

(3) IFGA-isomorphism if $f : K[G_1] \to K[G_2]$ is an bijection.

Remark 2.19. ([22]) For all $\alpha \in K[G_1]$, we have:

(1) $\mu_{K[B]}(f(\alpha)) = \sup\{\mu_{K[A]}(f^{-1}(f(\alpha)))\}$ and $\nu_{K[B]}(f(\alpha)) = \inf\{\nu_{K[A]}(f^{-1}(f(\alpha)))\}$, if $f : K[G_1] \to K[G_2]$ is a K-algebra homomorphism;

(2) $\mu_{K[B]}(f(\alpha)) = \mu_{K[A]}(\alpha)$ and $\nu_{K[B]}(f(\alpha)) = \nu_{K[A]}(\alpha)$, if $f : K[G_1] \to K[G_2]$ is a K-algebra isomorphism.

3 Intuitionistic fuzzy group subalgebra (IFGSA)

Definition 3.1. ([19]) For any intuitionistic fuzzy group A on a finite group G and field K, the intuitionistic fuzzy group algebra $K[A]$ on $K[G]$ is an intuitionistic fuzzy algebra. Futhermore, if $K[G]$ is considered as a G-module, then the intuitionistic fuzzy group algebra $K[A]$ is an intuitionistic fuzzy G-module on $K[G]$.
For $x, y \in K[G]$ and $a, b \in K, g \in G$, we have

(i) $\mu_{K[A]}(ax + by) \geq \min\{\mu_{K[A]}(x), \mu_{K[A]}(y)\}$ and $\nu_{K[A]}(ax + by) \leq \max\{\nu_{K[A]}(x), \nu_{K[A]}(y)\}$

(ii) $\mu_{K[A]}(xy) \geq \min\{\mu_{K[A]}(x), \mu_{K[A]}(y)\}$ and $\nu_{K[A]}(g \cdot m) \leq \nu_{K[A]}(m)$.

(iii) $\mu_{K[A]}(g \cdot m) \geq \mu_{K[A]}(m)$ and $\nu_{K[A]}(g \cdot m) \leq \nu_{K[A]}(m)$.

Conditions (i) and (ii) show that $K[A]$ is an intuitionistic fuzzy group algebra on $K[G]$ and conditions (i) and (iii) show that $K[A]$ is an intuitionistic fuzzy G-module on $K[G]$.

Definition 3.2. Let G be a finite group, K be a field, and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy group on G such that $K[A]$ is an intuitionistic fuzzy group algebra of $K[G]$. Let S be an intuitionistic fuzzy group of G with $S \subseteq A$. Then the IFS $K[S]$ on $K[G]$ is called an intuitionistic fuzzy group subalgebra of $K[A]$ if, for all $x, y \in K[G], a, b \in K$, the following conditions hold:

i) $\mu_{K[S]}(ax+by) \geq \min\{\mu_{K[S]}(x), \mu_{K[S]}(y)\}$ and $\nu_{K[S]}(ax+by) \leq \max\{\nu_{K[S]}(x), \nu_{K[S]}(y)\}$

ii) $\mu_{K[S]}(xy) \geq \min\{\mu_{K[S]}(x), \mu_{K[S]}(y)\}$ and $\nu_{K[S]}(xy) \leq \max\{\nu_{K[S]}(x), \nu_{K[S]}(y)\}$.

Remark 3.3. The intuitionistic fuzzy group $A = (\mu_A, \nu_A)$ on G plays a central role in the construction of intuitionistic fuzzy group algebras. The set A provides the initial intuitionistic fuzzy structure on the group G, and its extension to the group algebra $K[G]$ gives rise to the intuitionistic fuzzy group algebra $K[A]$. More precisely, the membership and non-membership values of an element $x = \sum_{g\in G} \alpha_g g \in K[G]$ are determined from the values of μ_A and ν_A on the support of x. Thus, the behavior of $K[A]$ is entirely governed by the underlying intuitionistic fuzzy group A. Furthermore, any intuitionistic fuzzy group subalgebra or intuitionistic fuzzy ideal of $K[A]$ is required to be contained within A, ensuring consistency of the fuzzy structure. In this way, A acts as the ambient universe of discourse, and all subsequent substructures are defined relative to it.

Theorem 3.4. *Let G be a finite group and H be a subgroup of G. Let K be a field and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy group on G. Suppose $K[A] = (\mu_{K[A]}, \nu_{K[A]})$ be an intuitionistic fuzzy group algebra induced by A. If $A_H = (\mu_{A_H}, \nu_{A_H})$ is the restriction of A to H, then the intuitionistic fuzzy set $K[A_H] = (\mu_{K[A_H]}, \nu_{K[A_H]})$ on $K[H]$ is an intuitionistic fuzzy group subalgebra of $K[A]$.*

Proof. Let $A_H = A|_H$ be the restriction of IFG A to H. It is defined as

$$\mu_{A_H}(h) = \begin{cases} \mu_A(h), & \text{if } h \in H \\ 0, & \text{if } h \notin H \end{cases}; \quad \nu_{A_H}(h) = \begin{cases} \nu_A(h), & \text{if } h \in H \\ 1, & \text{if } h \notin H \end{cases}$$

It is evident that A_H is an IFG on G such that $A_H \subseteq A$.

Since $K[H]$ is a subalgebra of $K[G]$, so sums, products and scalar multiples of the elements of $K[H]$ remain in $K[H]$. Let $x, y \in K[H], a, b \in K$. Therefore, $x, y \in K[G]$ also. As $K[A]$ is an IFGA on $K[G]$, so following assertions hold:

i) $\mu_{K[A]}(ax+by) \geq \min\{\mu_{K[A]}(x), \mu_{K[A]}(y)\}$ and $\nu_{K[A]}(ax+by) \leq \max\{\nu_{K[A]}(x), \nu_{K[A]}(y)\}$

ii) $\mu_{K[A]}(xy) \geq \min\{\mu_{K[A]}(x), \mu_{K[A]}(y)\}$ and $\nu_{K[A]}(xy) \leq \max\{\nu_{K[A]}(x), \nu_{K[A]}(y)\}$.

Restricting to $K[H]$ and using the fact that

$$\mu_{K[A_H]}(x) = \begin{cases} \mu_{K[A]}(x), & \text{if } x \in K[H] \\ 0, & \text{if } x \notin K[H] \end{cases}; \quad \nu_{K[A_H]}(x) = \begin{cases} \nu_{K[A]}(x), & \text{if } x \in K[H] \\ 1, & \text{if } x \notin K[H] \end{cases}$$

i) $\mu_{K[A_H]}(ax+by) \geq \min\{\mu_{K[A_H]}(x), \mu_{K[A_H]}(y)\}$ and $\nu_{K[A_H]}(ax+by) \leq \max\{\nu_{K[A_H]}(x), \nu_{K[A_H]}(y)\}$

ii) $\mu_{K[A_H]}(xy) \geq \min\{\mu_{K[A_H]}(x), \mu_{K[A_H]}(y)\}$ and $\nu_{K[A_H]}(xy) \leq \max\{\nu_{K[A_H]}(x), \nu_{K[A_H]}(y)\}$.

Hence $K[A_H]$ is an IFGSA of $K[A]$. □

Example 3.5. Let $G = C_3 = \{e, a, a^2\}$ with $a^3 = e$, and $K = \mathbb{R}$.

First, define an IFG $A = (\mu_A, \nu_A)$ on G by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = e \\ 0.8, & \text{if } x = a, a^2 \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = e \\ 0.1, & \text{if } x = a, a^2. \end{cases}$$

Then $K[A]$ is the corresponding IFGA of $K[G]$.

Now let $S = (\mu_S, \nu_S)$ be the IFSG of G given by

$$\mu_S(x) = \begin{cases} 1, & \text{if } x = e \\ 0.6, & \text{if } x = a, a^2 \end{cases}; \quad \nu_S(x) = \begin{cases} 0, & \text{if } x = e \\ 0.3, & \text{if } x = a, a^2. \end{cases}$$

Clearly, $S \subseteq A$.

For $x = \sum_{g \in G} a_g g \in K(G)$ with $\mathrm{supp}(x) \neq \varnothing$ we have

$$\mu_{K[S]}(x) = \min_{g \in \mathrm{supp}(x)} \mu_S(g), \qquad \nu_{K[S]}(x) = \max_{g \in \mathrm{supp}(x)} \nu_S(g).$$

For example:

$$\begin{aligned} x &= 2e + 3a & \mu_{K(S)}(x) &= 0.6, & \nu_{K(S)}(x) &= 0.3, \\ y &= a - a^2 & \mu_{K(S)}(y) &= 0.6, & \nu_{K(S)}(y) &= 0.3, \\ x + y &= 2e + 4a - a^2 & \mu_{K(S)}(x+y) &= 0.6, & \nu_{K(S)}(x+y) &= 0.3. \end{aligned}$$

Multiplying

$$xy = (2e + 3a)(a - a^2) = -3e + 2a + a^2,$$

with support $\{e, a, a^2\}$, gives $\mu_{K[S]}(xy) = 0.6$, $\nu_{K[S]}(xy) = 0.3$. Hence $K[S]$ is an IFGSA of $K[A]$.

Example 3.6. Let $G = D_4 = \langle r, s \mid r^4 = e,\ s^2 = e,\ srs = r^{-1} \rangle$ and $K = \mathbb{Q}$.

Define an IFG as $A = \chi_G$ on G. Then $K[A]$ is the maximal IFGA of $K(G)$.

Now let $H = \langle r \rangle = \{e, r, r^2, r^3\}$ and define an IFSG $S = (\mu_S, \nu_S)$ by

$$\mu_S(g) = \begin{cases} 1, & g \in H, \\ 0.5, & g \notin H, \end{cases} \qquad \nu_S(g) = \begin{cases} 0, & g \in H, \\ 0.4, & g \notin H. \end{cases}$$

Clearly $S \subseteq A$.

For instance:

$$\begin{aligned} x &= e + r + s & \mu_{K[S]}(x) &= 0.5, & \nu_{K[S]}(x) &= 0.4, \\ y &= r^2 + rs & \mu_{K[S]}(y) &= 0.5, & \nu_{K[S]}(y) &= 0.4. \end{aligned}$$

Both addition and multiplication of such elements respect Definition 3.2, hence $K[S]$ is an IFGSA of $K[A]$.

Proposition 3.7. *Let $\{K[S_j] : j \in J\}$ be a family of IFGSAs of an IFGA $K[A]$ on $K[G]$. Then*

$$K[T] := \bigcap_{j \in J} K[S_j]$$

is also an IFGSA of $K[A]$, where $T = \bigcap_{j \in J} S_j$ is an IFG on G.

Proof. For each $j \in J$, write $S_j = (\mu_{S_j}, \nu_{S_j})$. Define an IFS $T = (\mu_T, \nu_T)$ on G by

$$\mu_T(g) = \inf_{j\in J} \mu_{S_j}(g), \qquad \nu_T(g) = \sup_{j\in J} \nu_{S_j}(g), \qquad (\forall g \in G).$$

Then T is an IFG on G (see [16]). Since $S_j \subseteq A$ for every $j \in J$, it follows that

$$\mu_T(g) \leq \mu_A(g), \qquad \nu_T(g) \geq \nu_A(g),$$

hence $T \subseteq A$.

Now, let $x = \sum_{g\in G} a_g g \in K[G]$. Recall that $supp(x) = \{g \in G : a_g \neq 0\}$.
If $supp(x) = \varnothing$, then $(\mu_{K[T]}(x), \nu_{K[T]}(x)) = (1, 0)$, so the claim holds.
If $supp(x) \neq \varnothing$, then

$$\begin{aligned} \mu_{K[T]}(x) &= \min_{g\in supp(x)} \mu_T(g) = \min_{g\in supp(x)} \Big(\inf_{j\in J} \mu_{S_j}(g)\Big) \\ &= \inf_{j\in J} \Big(\min_{g\in supp(x)} \mu_{S_j}(g)\Big) \\ &= \inf_{j\in J} \mu_{K[S_j]}(x). \end{aligned}$$

and similarly

$$\nu_{K[T]}(x) = \max_{g\in supp(x)} \nu_T(g) = \sup_{j\in J} \nu_{K[S_j]}(x).$$

Now let $x, y \in K[G], a, b \in K$.
(i) Since each $K[S_j]$ is an IFGSA,

$$\mu_{K[S_j]}(ax + by) \geq \min\{\mu_{K[S_j]}(x), \mu_{K[S_j]}(y)\}.$$

Taking infimum over j, we obtain

$$\mu_{K[T]}(ax + by) = \inf_j \mu_{K[S_j]}(ax + by) \geq \min\{\mu_{K[T]}(x), \mu_{K[T]}(y)\}.$$

Similarly,

$$\nu_{K[T]}(ax + by) = \sup_j \nu_{K[S_j]}(ax + by) \leq \max\{\nu_{K[T]}(x), \nu_{K[T]}(y)\}.$$

(ii) By the same reasoning,

$$\mu_{K[T]}(xy) \geq \min\{\mu_{K[T]}(x), \mu_{K[T]}(y)\}, \qquad \nu_{K[T]}(xy) \leq \max\{\nu_{K[T]}(x), \nu_{K[T]}(y)\}.$$

Thus $K[T]$ satisfies the two defining conditions of an IFGSA. Hence $K[T] = \bigcap_{j\in J} K[S_j]$ is an IFGSA of $K[A]$. □

Proposition 3.8. *Let $\bar{f} : K[A] \to K[B]$ be an IFGA-homomorphism from an IFGA $K[A]$ on $K[G_1]$ to an IFGA $K[B]$ on $K[G_2]$ with the corresponding group algebra homomorphism $f : K[G_1] \to K[G_2]$. Then*

1. *if $K[S_2]$ is an IFSGA of $K[B]$, then $(\bar{f})^{-1}(K[S_2])$ is an IFSGA of $K[A]$.*

2. *if $K[S_1]$ is an IFSGA of $K[A]$, then $\bar{f}(K[S_1])$ is an IFSGA of $K[B]$.*

Proof. The result follows from Theorem 2.15. □

4 Intuitionistic fuzzy ideals in intuitionistic fuzzy group algebras

We know that a normal subgroup N of a group G gives rise to a special two-sided ideal in $K[G]$, called the augmentation ideal associated with N. This ideal is denoted by I_N, and it consists of all elements in $K[G]$ whose coefficients sum is zero on each coset of N. Moreover, the group algebra of the quotient group G/N is isomorphic to the quotient algebra of $K[G]$ by this augmentation ideal, i.e., $K[G/N] \cong K[G]/I_N$. In this section, we try to investigate the analogous of this result in the intuitionistic fuzzy group algebra.

Definition 4.1. Let G be a finite group, K a field, and A an intuitionistic fuzzy group (IFG) on G. Let B be an intuitionistic fuzzy normal subgroup (IFNSG) of G with $supp(B) = B^*$ such that $B \subseteq A$ and $B^* \trianglelefteq G$. The natural quotient map $\psi : G \to G/B^*$ given by $\psi(g) = gB^*$ induces a K-algebra homomorphism

$$f : K[G] \longrightarrow K[G/B^*], \qquad f\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g\,(gB^*).$$

The kernel of this map,

$$ker(f) = \left\{ \sum_{g\in G} a_g g \in K[G] : \sum_{g\in G} a_g = 0, \text{ for all cosets } gB^* \in G/B^* \right\}$$

is the two-sided ideal of $K[G]$ generated by $\{g^* - 1 : g^* \in B^*\}$.
If C is an IFG on G/B^* satisfying

$$\mu_C(gB^*) \geq \max\{\mu_A(x) : x \in gB^*\} \text{ and } \nu_C(gB^*) \leq \min\{\nu_A(x) : x \in gB^*\}$$

and $K[C]$ is the corresponding IFGA on $K[G/B^*]$, then f induces an IFGA-homomorphism $\overline{f} : K[A] \to K[C]$ such that

i) $f : K[G] \to K[G/B^*]$ is an induced K-algebra homomorphism

ii) $\mu_{K[C]}(f(\alpha)) \geq \mu_{K[A]}(\alpha)$ and $\nu_{K[C]}(f(\alpha)) \leq \nu_{K[A]}(\alpha)$ (for all $\alpha \in K[G]$).

The kernel of this map, $I_B := \ker(\overline{f}) = K[A]|_{ker(f)} = (\mu_{K[A]|_{ker(f)}}, \nu_{K[A]|_{ker(f)}})$, is called an *intuitionistic fuzzy augmentation ideal (IFAI)* of $K[A]$ associated with B.

Remark 4.2.

1. When $B = \chi_G$, then the ideal I_B coincides with the intuitionistic fuzzy augmentation ideal of $K[A]$ associated with the whole group G.
2. When $B = \chi_{\{e\}}$, where e is the identity element of G, then I_B is the intuitionistic fuzzy zero ideal of $K[A]$.

Example 4.3. Let $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and $K = \mathbb{R}$. Define the IFG A on G as $A = \chi_G$ that is, A is the universal IFG on G. Further, let $N = A_3 = \{e, (123), (132)\} \trianglelefteq S_3$ and consider the IFNSG on G by $B = (\mu_B, \nu_B)$ given by

$$\mu_B(g) = \begin{cases} 1, & g \in N, \\ 0, & g \notin N, \end{cases} \qquad \nu_B(g) = \begin{cases} 0, & g \in N, \\ 1, & g \notin N. \end{cases}$$

Clearly, $B \subseteq A$ and $supp(B) = B^* = N$ is a normal subgroup of G. So we have $G/B^* = G/N = \{N, tN\}$, where t is any transposition in S_3. The natural quotient map $\psi : G \to G/N$ given by

$$\psi(g) = \begin{cases} N, & g \in N, \\ tN, & g \notin N, \end{cases}$$

induces a K-algebra homomorphism

$$f : K[G] \longrightarrow K[G/N], \qquad f\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g\,(\psi(g)).$$

with $\ker(f)$ generated by $\{g^* - 1 : g^* \in N\}$ is a two sided ideal of $K[G]$, i.e., the kernel is

$$\ker(f) = \Big\{ \sum_{g\in S_3} a_g g \in \mathbb{R}[S_3] \;:\; a_e + a_{(123)} + a_{(132)} = 0,\;\; a_{(12)} + a_{(13)} + a_{(23)} = 0 \Big\}.$$

Equivalently,

$$\ker(f) = \langle (123) - e,\ (132) - e,\ (12) - (13),\ (12) - (23) \rangle.$$

Now, it is easy to check that $K[A] = \chi_{\mathbb{R}[S_3]}$ and on taking the IFS $C = \chi_{G/N}$ on G/N, we find that the map f induces an IFGA-homomorphism $\bar{f} : K[A] \to K[C]$.

By Definition 4.1, the IFAI of $K[A]$ associated with B is given by

$$I_B = \ker(\bar{f}) = K[A]|_{\ker(f)} = \chi_{\mathbb{R}[S_3]}|_{\ker(f)} = \chi_{\ker(f)}.$$

Proposition 4.4. *Let G be a finite group, K be a field, and A be an IFG on G. Let $\{B_i : i \in J\}$ be a family of IFNSGs of G such that $B_i \subseteq A$ and $supp(B_i) = B_i^* \trianglelefteq G$ for all $i \in J$. Let I_{B_i} be the IFAI of $K[A]$ associated with B_i. Set $B = \cap_{i\in J} B_i$, $B^* = \cap_{i\in J} B_i^*$. Then B is an IFNSG of G, and $I_B = \cap_{i\in J} I_{B_i}$.*

Proof. Define $B = (\mu_B, \nu_B)$ by $\mu_B(g) = \inf\{\mu_{B_i}(g) : i \in J\}$, $\nu_B(g) = \sup\{\nu_{B_i}(g) : i \in J\}$, $g \in G$. Then $B \subseteq A$, $supp(B) = \cap_{i\in J} supp(B_i) = \cap_{i\in J} B_i^* = B^*$. Since each B_i is an IFNSG of G and intersections of IFNSGs are again an IFNSG of G. Moreover, as each $B_i^* \trianglelefteq G$, their intersection B^* is also a normal subgroup of G.

For each $i \in J$, let $\psi_i : G \to G/B_i^*$ be the natural quotient homomorphism, and let $f_i : K[G] \to K[G/B_i^*]$ be the induced K- algebra homomorphism.

Let C_i denote the IFG induced by A on G/B_i^*, so that $K[C_i]$ is the induced IFGA on $K[G/B_i^*]$.

Then f_i induces an IFGA-homomorphism $\bar{f}_i : K[A] \to K[C_i]$, and $I_{B_i} := \ker(\bar{f}_i)$.

Now, let $\psi : G \to G/B^*$ be the natural quotient homomorphism, and let $f : K[G] \to K[G/B^*]$ be the induced K-algebra homomorphism.

Let C denote the IFG induced by A on G/B^*, so that $K[C]$ is the induced IFGA on $K[G/B^*]$.

Then f induces an IFGA-homomorphism $\bar{f} : K[A] \to K[C]$, and $I_B := \ker(\bar{f})$.

Since $B^* = \cap_{i\in J} B_i^*$, an element of $x \in K[A]$ lies in $\ker(\bar{f})$ if and only if $x \in \ker(\bar{f}_i)$ for all $i \in J$, hence $I_B = \ker(\bar{f}) = \cap_{i\in J} \ker(\bar{f}_i) = \cap_{i\in J} I_{B_i}$.
This completes the proof. □

Theorem 4.5. *Let $\bar{f} : K[A_1] \to K[A_2]$ be an IFGA-homomorphism from an IFGA $K[A_1]$ on $K[G_1]$ to an IFGA $K[A_2]$ on $K[G_2]$, with a corresponding group algebra homomorphism $f : K[G_1] \to K[G_2]$. If $J_B = (\mu_{J_B}, \nu_{J_B})$ is an IFAI of $\bar{f}(K[A_1])$, then the preimage $\bar{f}^{-1}(J_B)$ is an IFAI of $K[A_1]$.*

Proof. Since A_1 is an IFG on G_1, we have $\bar{f}(K[A_1]) = K[\psi(A_1)]$ (Theorem 2.15(ii)), where $\psi : G_1 \to G_2$ is the group homomorphism inducing f.

Let $J_B = \ker(\bar{f}_B)$, where $\bar{f}_B : K[\psi(A_1)] \to K[C]$ is the IFGA-homomorphism induced by the quotient $G_2 \to G_2/B^*$ corresponding to the IFNSG $B \subseteq \psi(A_1)$ (Definition 4.1).

Then the preimage is

$$\bar{f}^{-1}(J_B) = \ker(\bar{f}_B \circ \bar{f}),$$

which is an IFGA in $K[A_1]$. By Theorem 2.15(i),

$$\ker(\bar{f}_B \circ \bar{f}) = K[\psi^{-1}(B)],$$

where $\psi^{-1}(B) \subseteq A_1$ is an IFNSG of G_1. Hence $\bar{f}^{-1}(J_B)$ is an IFAI of $K[A_1]$. □

This relationship can be visualized in the following commutative diagram:

$$\begin{array}{ccc} K[A_1] & \xrightarrow{\bar{f}} & K[\psi(A_1)] \\ & \searrow^{\bar{f}_B \circ \bar{f}} & \downarrow \bar{f}_B \\ & & K[C] \end{array}$$

Here, $\bar{f}_B \circ \bar{f}$ is the map whose kernel gives the IFAI $\bar{f}^{-1}(J_B)$ in $K[A_1]$.

Theorem 4.6. *Let $\bar{f} : K[A_1] \to K[A_2]$ be an IFGA-homomorphism from an IFGA $K[A_1]$ on $K[G_1]$ to an IFGA $K[A_2]$ on $K[G_2]$, with a corresponding group algebra homomorphism $f : K[G_1] \to K[G_2]$. Let $I_B = (\mu_{I_B}, \nu_{I_B})$ be an IFAI of $K[A_1]$ associated with an IFNSG $B \subseteq A_1$, with $B^* \trianglelefteq G_1$.*

If $\psi(B^*) \trianglelefteq G_2$, where $\psi : G_1 \to G_2$ is the group homomorphism inducing f, then $\bar{f}(I_B)$ is an IFAI of $K[A_2]$.

Proof. By definition, $I_B = K[A_1]|_{\ker(f_B)} = K[B]$, where $f_B : K[G_1] \to K[G_1/B^*]$ is the quotient-induced algebra homomorphism.

Since $\bar{f}$ is an IFGA-homomorphism induced by $\psi : G_1 \to G_2$, Theorem 2.15(ii) gives

$$\bar{f}(K[B]) = K[\psi(B)] \subseteq K[\psi(A_1)] \subseteq K[A_2].$$

By the assumption $\psi(B^*) \trianglelefteq G_2$, the set $\psi(B)$ is an IFNSG of G_2. Hence $K[\psi(B)]$ defines an IFAI of $K[A_2]$ associated with $\psi(B)$. Consequently, $\bar{f}(I_B) = \bar{f}(K[B]) = K[\psi(B)]$ is an IFAI of $K[A_2]$. □

Remark 4.7. The normality condition $\psi(B^*) \trianglelefteq G_2$ is essential; without it, the image may fail to form an IFNSG, and thus $\bar{f}(I_B)$ would not be a valid IFAI.

5 Isomorphism theorems for IFGAs

Theorem 5.1 (First Isomorphism Theorem for IFGAs)**.** *Let G be a finite group and K be a field. Let $A = (\mu_A, \nu_A)$ be an IFG on G. Let $B = (\mu_B, \nu_B)$ be an IFNSG of G such that $B \subseteq A$ and $B^* = \operatorname{supp}(B) \trianglelefteq G$. Let $f : K[G] \longrightarrow K[G/B^*]$ be the K-algebra homomorphism induced by the natural quotient map $\psi : G \to G/B^*$, $\psi(g) = gB^*$. Let $I_B = \big(\mu_{K[A]}|_{\ker(f)}, \nu_{K[A]}|_{\ker(f)}\big)$ be the IFAI of $K[A]$ associated with B. Then*

$$K[A]/I_B \cong K[A/B]$$

as intuitionistic fuzzy group algebras.

Proof. Define the map

$$f : K[G] \longrightarrow K[G/B^*], \qquad f\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g(gB^*).$$

Since $B^* \trianglelefteq G$, the quotient group G/B^* is well defined and f is a surjective K-algebra homomorphism induced by the quotient map $\psi : G \to G/B^*$. By the classical First Isomorphism Theorem for algebras,

$$K[G]/\ker(f) \cong K[G/B^*].$$

Next, define the intuitionistic fuzzy group algebra $K[A]$ on $K[G]$.
For $\alpha = \sum_{g\in G} a_g g \in K[G]$ with support $S = \{g \in G : a_g \neq 0\}$, define

$$\mu_{K[A]}(\alpha) = \min\{\mu_A(g) : g \in S\}, \qquad \nu_{K[A]}(\alpha) = \max\{\nu_A(g) : g \in S\}.$$

Define the quotient intuitionistic fuzzy group A/B on G/B^* by

$$\mu_{A/B}(gB^*) = \max\{\mu_A(x) : x \in gB^*\}, \qquad \nu_{A/B}(gB^*) = \min\{\nu_A(x) : x \in gB^*\},$$

and extend these functions to the group algebra $K[G/B^*]$ in the same manner to obtain the intuitionistic fuzzy group algebra $K[A/B]$.

This gives rise to a surjective IFGA-homomorphism

$$\bar{f} : K[A] \longrightarrow K[A/B].$$

For this, let $\alpha \in K[A]$ be an element having support S, and let $S^* = \{gB^* \in G/B^* : g \in S\}$ be the support of $f(\alpha)$. Then

$$\begin{aligned}
\mu_{K[A/B]}(f(\alpha)) &= \min\{\mu_{A/B}(gB^*) : gB^* \in S^*\} \\
&= \min\left\{\max\{\mu_A(x) : x \in gB^*\} : gB^* \in S^*\right\} \\
&= \min\{\mu_A(g) : g \in S\} = \mu_{K[A]}(\alpha),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\nu_{K[A/B]}(f(\alpha)) &= \max\{\nu_{A/B}(gB^*) : gB^* \in S^*\} \\
&= \max\left\{\min\{\nu_A(x) : x \in gB^*\} : gB^* \in S^*\right\} \\
&= \max\{\nu_A(g) : g \in S\} = \nu_{K[A]}(\alpha).
\end{aligned}$$

Thus, f preserves both the membership and non-membership functions and is an intuitionistic fuzzy group algebra homomorphism.

The kernel of f is given by

$$\ker(f) = \left\{ \sum_{g \in G} a_g g \in K[G] : \sum_{x \in gB^*} a_x = 0 \text{ for all } gB^* \in G/B^* \right\},$$

which is a two-sided ideal of $K[G]$.

Define $I_B = \left(\mu_{K[A]}|_{\ker(f)}, \nu_{K[A]}|_{\ker(f)}\right)$, the IFAI of $K[A]$ associated with B. Then the map

$$\phi : K[A]/I_B \to K[A/B],$$

defined by $\phi(\alpha + I_B) = f(\alpha)$, is well defined.

Now

$$\begin{aligned} \mu_{K[A/B]}(\phi(\alpha + I_B)) &= \mu_{K[A/B]}(f(\alpha)) \\ &= \mu_{K[A]}(\alpha) \\ &= \mu_{K[A]/I_B}(\alpha + I_B). \end{aligned}$$

Similarly, we can show that $\nu_{K[A/B]}(\phi(\alpha + I_B)) = \nu_{K[A]/I_B}(\alpha + I_B)$. Moreover, ϕ is a bijective. Hence, by Remark 2.18, ϕ is an IFGA-isomorphism and so, $K[A]/I_B \cong K[A/B]$. □

Theorem 5.2 (Second Isomorphism Theorem for IFGAs)**.** *Let G be a finite group and K be a field. Let $A = (\mu_A, \nu_A)$ be an IFG on G, and let $B = (\mu_B, \nu_B)$ be an IFNSG of G such that $B \subseteq A$ and $B^* = supp(B) \trianglelefteq G$. Let $C = (\mu_C, \nu_C)$ be an IFG on G satisfying $B \subseteq C \subseteq A$. Let $I_B = \left(\mu_{K[C]}|_{\ker(f)}, \nu_{K[C]}|_{\ker(f)}\right)$ be the IFAI of $K[C]$ associated with B. Then*

$$K[C]/I_B \cong K[C/B]$$

as intuitionistic fuzzy group algebras.

Proof. Let $\psi : G \to G/B^*$ be the natural quotient map defined by $\psi(g) = gB^*$, and let

$$f : K[G] \longrightarrow K[G/B^*]$$

be the induced K-algebra homomorphism. Since $B^* \trianglelefteq G$, the map f is surjective. Proceeding as in the First Isomorphism Theorem for IFGAs, this give rise to a surjective IFGA-homomorphism

$$\bar{f} : K[A] \longrightarrow K[A/B].$$

Restricting $\bar{f}$ to the IFSA $K[C]$ of $K[G]$ such that $K[C] \subseteq K[A]$, we obtain an induced IFGA-homomorphism

$$\bar{g} = \bar{f}|_{K[C]} : K[C] \longrightarrow K[C/B],$$

where C/B denotes the quotient intuitionistic fuzzy subgroup of G/B^* contained in A/B. For $\alpha = \sum_{g \in C} a_g g \in K[C]$ with support $S = \{g \in G : a_g \neq 0\}$, the membership and non-membership functions satisfy

$$\mu_{K[C/B]}(f(\alpha)) = \mu_{K[C]}(\alpha), \quad \nu_{K[C/B]}(f(\alpha)) = \nu_{K[C]}(\alpha),$$

as in Theorem 5.1. Thus, $\bar{g}$ is an IFGA-homomorphism.

The kernel of this map is exactly $\ker(\bar{g}) = I_B \subseteq K[C]$, the IFAI of $K[C]$ associated with B.

By the First Isomorphism Theorem for IFGAs, the induced map

$$\bar{\phi} : K[C]/I_B \to K[C/B], \quad \bar{\phi}(\alpha + I_B) = f(\alpha),$$

is an IFGA-isomorphism.

Hence, we conclude $K[C]/I_B \cong K[C/B]$ as intuitionistic fuzzy group algebras. □

Theorem 5.3 (Third Isomorphism Theorem for IFGAs)**.** *Let G be a finite group and K be a field. Let $A = (\mu_A, \nu_A)$ be an IFG on G. Let $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be IFNSGs of G such that $B \subseteq C \subseteq A$, with $B^* = \operatorname{supp}(B) \lhd G$ and $C^* = \operatorname{supp}(C) \lhd G$. Then C/B is an IFNSG of A/B and*

$$\frac{K[A/B]}{I_{C/B}} \cong K[A/C]$$

as IFGAs, where $I_{C/B}$ is the IFAI of $K[A/B]$ associated with C/B.

Proof. Since $B^* \subseteq C^*$ and both are normal subgroups of G, it follows that $C^*/B^* \lhd G/B^*$. Hence the quotient group $(G/B^*)/(C^*/B^*) \cong G/C^*$ is well defined. Define the natural quotient homomorphisms

$$\psi_1 : G \to G/B^*, \quad \psi_1(g) = gB^*,$$

and

$$\psi_2 : G/B^* \to (G/B^*)/(C^*/B^*), \quad \psi_2(gB^*) = (gB^*)(C^*/B^*)$$

These induce surjective K-algebra homomorphisms

$$K[G] \xrightarrow{f_1} K[G/B^*] \xrightarrow{f_2} K[G/C^*].$$

By the definition of quotient intuitionistic fuzzy groups, A/B is an IFG on G/B^* and C/B is an IFNSG of A/B with support C^*/B^*.

Hence f_2 induces a surjective IFGA-homomorphism $\bar{f}_2 : K[A/B] \to K[A/C]$.

Now, we identify the kernel of $\bar{f}_2$: An element of $K[A/B]$ lies in $\ker(\bar{f}_2)$ if and only if the image of $K[A/B]$ is zero, which happens precisely when it vanishes outside the cosets corresponding to C^*/B^*. Therefore,

$$\ker(\bar{f}_2) = I_{C/B}.$$

where $I_{C/B}$ denotes the IFAI of $K[A/B]$ associated with the IFNSG C/B of G/B^*.

By the First Isomorphism Theorem for intuitionistic fuzzy group algebras, we obtain

$$\frac{K[A/B]}{I_{C/B}} \cong K[A/C]$$

as intuitionistic fuzzy group algebras. □

6 Conclusion

In this paper, we have developed a systematic framework for studying intuitionistic fuzzy algebraic structures arising from group algebras. By introducing the notion of an intuitionistic fuzzy group subalgebra of an intuitionistic fuzzy group algebra associated with $K[G]$, we extended classical concepts from group algebra theory into the intuitionistic fuzzy setting. Fundamental structural properties of IFGSAs were established, particularly their stability under IFGA-homomorphisms. The analysis of images and inverse images under such homomorphisms, together with the result that arbitrary intersections of IFGSAs remain IFGSAs, demonstrates that this class of substructures is robust and well behaved.

Furthermore, the concept of an intuitionistic fuzzy augmentation ideal was defined and examined in detail. We showed that IFAIs inherit key closure properties analogous to those of classical augmentation ideals, including preservation under arbitrary intersections and under homomorphic images and preimages. The isomorphism theorems proved for IFGAs provide a natural intuitionistic fuzzy extension of the corresponding results in ordinary group algebra theory, thereby reinforcing the consistency and algebraic soundness of the proposed framework. Overall, these results contribute to a deeper understanding of intuitionistic fuzzy group algebras and lay a foundation for further research, such as the study of intuitionistic fuzzy representations, modules over IFGAs, and potential applications in areas where uncertainty and algebraic structures interact.

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