Notes on Intuitionistic Fuzzy Sets
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A general approach to modal topological structures illustrated by intuitionistic fuzzy objects

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In memory of my colleague and friend Prof. Marin Marinov

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Abstract: A new, general approach to introducing of the concept of a Modal Topological Structure (MTS) is given. The basic properties of the MTS are studied. The MTSs are illustrated by intuitionistic fuzzy objects - intuitionistic fuzzy sets, operations, relations and operators. So, 38 different intuitionistic fuzzy MTSs are described.
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1 Introduction

During the last two years, in a series of papers, the author introduced the concept of a Modal Topological Structure (MTS), studied its basic properties, introduced some of the MTS-modifications, and illustrated them by intuitionistic fuzzy objects - intuitionistic fuzzy sets (IFSs) and operations, relations and operators defined over them.

In the present paper, all notations and symbols, related to the IFSs are used from [2]. Only these, that are published after [2] are given here.

2 Intuitionistic fuzzy modal topological structures

2.1 Modal topological structures

Here, following and extending [3], we will define the concept of a MTS.

Let us have a fixed set X and let everywhere below

$$\mathcal{P}(X) = \{Y | Y \subseteq X\}.$$

Let O be the minimal element of $\mathcal{P}(X)$ and let for $Y \in \mathcal{P}(X)$,

$$\neg Y = X - Y,$$

where “ $-$ ” is the set-theoretical operation “subtraction”.

Let us have two operations $\Delta, \nabla : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined in such a way that for every two sets $A, B \in \mathcal{P}(X)$:

$$A \nabla B = \neg(\neg A \Delta \neg B), \tag{1}$$

$$A \Delta B = \neg(\neg A \nabla \neg B). \tag{2}$$

Let the operation Δ generate the topological operator $\mathcal{O}$ and the operation ∇ generate the topological operator $\mathcal{Q}$.

Now, following [26], we will say that the operator $\mathcal{O}$ is a topological operator of closure (cl)-type, if for any $A, B \in \mathcal{P}$:

C1 $\mathcal{O}(A \Delta B) = \mathcal{O}(A) \Delta \mathcal{O}(B)$,

C2 $A \subseteq \mathcal{O}(A)$,

C3 $\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$,

C4 $\mathcal{O}(O) = O$,

and that the operator $\mathcal{Q}$ is a topological operator of interior (in)-type, if for any $A, B \in \mathcal{P}$:

I1 $\mathcal{Q}(A \nabla B) = \mathcal{Q}(A) \nabla \mathcal{Q}(B)$,

I2 $\mathcal{Q}(A) \subseteq A$,

I3 $\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$,

I4 $\mathcal{Q}(X) = X$.

We will assume that for every set $A \in \mathcal{P}(X)$:

$$\mathcal{O}(A) = \neg(\mathcal{Q}(\neg A)), \tag{3}$$

$$\mathcal{Q}(A) = \neg(\mathcal{O}(\neg A)). \tag{4}$$

For example, if the operation Δ is operation ”union” ($\cup$), then operator $\mathcal{O}$ will be the cl-topological operator $\mathcal{C}$, and if operation ∇ is the operation ”intersection” ($\cap$), then operator $\mathcal{Q}$ will be in-topological operator $\mathcal{I}$ (cf. [16, 26]).

We must mention that in the general case, for every two sets $A, B \in \mathcal{P}(X)$ the equalities

$$\begin{aligned} \mathcal{O}(A\Delta B) &= \mathcal{O}(A)\Delta\mathcal{O}(B) \\ \mathcal{Q}(A\nabla B) &= \mathcal{Q}(A)\nabla\mathcal{Q}(B) \end{aligned} \tag{5}$$

hold, but this is not always the case with equalities

$$\begin{aligned} \mathcal{O}(A\nabla B) &= \mathcal{O}(A)\nabla\mathcal{O}(B), \\ \mathcal{Q}(A\Delta B) &= \mathcal{Q}(A)\Delta\mathcal{Q}(B). \end{aligned} \tag{6}$$

For example, if we have two sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ and if, e.g., $a_1 = 1, a_2 = 3, a_3 = 5$, $b_1 = 4, b_2 = 3, b_3 = 2$, then

$$\begin{aligned} \max_{1\le i\le 3}(\max(a_i, b_i)) &= \max(\max(1,4), \max(3,3), \max(5,2)) \\ &= \max(4,3,5) \\ &= \max(5,4) \\ &= \max(\max(1,3,5), \max(4,3,2)) \\ &= \max(\max_{1\le i\le 3} a_i, \max_{1\le i\le 3} b_i), \end{aligned}$$

but

$$\begin{aligned} \max_{1\le i\le 3}(\min(a_i, b_i)) &= \max(\min(1,4), \min(3,3), \min(5,2)) \\ &= \max(1,3,2) \\ &< \min(5,4) \\ &= \min(\max(1,3,5), \max(4,3,2)) \\ &= \min(\max_{1\le i\le 3} a_i, \max_{1\le i\le 3} b_i). \end{aligned}$$

Having in mind that the modal operators $\Diamond$ and $\Box$ (see, e.g., [15,22,23,35]) satisfy conditions C1–C4 and I1–I4, respectively, we can see that operator $\Diamond$ is from a cl-type and operator $\Box$ is from an in-type. More general, if the modal operator $\circ$ is from cl-type, i.e., if it satisfies conditions C1–C4 with symbols $\circ$ instead of $\mathcal{O}$, and if the modal operator $\bullet$ is from in-type, i.e., if it satisfies conditions I1–I4 with a symbol $\bullet$ instead of $\mathcal{Q}$, then we will assume that for each $A \in \mathcal{P}(X)$

$$\circ A = \neg(\bullet\neg A), \tag{7}$$

$$\bullet A = \neg(\circ\neg A). \tag{8}$$

We will call such a pair of modal operators *dual (modal) operators.* If they satisfy the equalities

$$\circ\bullet A = \bullet A, \tag{9}$$

$$\bullet\circ A = \circ A, \tag{10}$$

we will call them *idempotently interior* and when they satisfy the equalities

$$\circ \bullet A = \circ A, \tag{11}$$

$$\bullet \circ A = \bullet A, \tag{12}$$

we will call them *idempotently exterior.*

We must mention that in some cases (as we will see in some of the next sections, when we will discuss the intuitionistic fuzzy cases), the pairs of modal operators $(\circ, \bullet)$ satisfies both the equalities

$$\circ(A\Delta B) = \circ(A)\Delta \circ (B)$$

$$\bullet(A\nabla B) = \bullet(A)\nabla \bullet (B)$$

and the equalities

$$\begin{array}{c}\circ(A\nabla B) = \circ(A)\nabla \circ (B)\\ \bullet(A\Delta B) = \bullet(A)\Delta \bullet (B)\end{array}.$$

Having in mind that the topological structures can be from cl- or in-type, we can mention that a given structure is from φ-type, where $\varphi \in \{cl, in\}$.

Now, following the definition of the topological structure from [16] and the definition from [3], we define that the object $\langle \mathcal{P}(X), \mathcal{E}, \zeta, *, \eta \rangle$ is a χ-Modal φ-Topological Structure (χ-Mφ-TS) over the set X, where $\mathcal{E} \in \{\mathcal{O}, \mathcal{Q}\}$ is a topological operator from φ-type generated by operation $\zeta \in \{\Delta, \nabla\}$ and $* \in \{\circ, \bullet\}$ is a modal operator from χ-type related to the operation $\eta \in \{\Delta, \nabla\}$, where $\varphi, \chi \in \{cl, in\}$. Therefore, each of the both operators (the topological and the modal) must satisfy the respective C- or the respective I-conditions.

Extending the idea for the conditions of the topological operators that must satisfy only equalities (5), in the case of modal operators, we change conditions C1 and I1 with

$$*(A\ \eta\ B) = *A\ \eta\ *B,$$

where $\eta \in \{\Delta, \nabla\}$.

Therefore, we can try to construct topological structures satisfying each one of the equalities

$$\circ(A\Delta B) = \circ A\Delta \circ B,$$

$$\circ(A\nabla B) = \circ A\nabla \circ B,$$

$$\bullet(A\Delta B) = \bullet A\Delta \bullet B,$$

$$\bullet(A\nabla B) = \bullet A\nabla \bullet B.$$

Of course, this will not always be possible.

Hence, when the pair of modal operators $(\circ, \bullet)$ satisfies the equalities (7) and (8), then we can have both the structure $\langle \mathcal{P}(X), \mathcal{E}, \eta, *, \nabla \rangle$ and the structure $\langle \mathcal{P}(X), \mathcal{E}, \eta, *, \Delta \rangle$. In this case, it will be suitable to denote this structure as (χ, ∇)-Mφ-TS or (χ, Δ)-Mφ-TS. The modal operator χ must satisfy the condition C1 if the operation is Δ and the condition I1 if the operation is ∇.

In addition, these operators must satisfy the following additional condition $(*)$ for each $A \in \mathcal{P}(X)$:

$$*\mathcal{E}(A) = \mathcal{E}(*A) \tag{$*$}$$

that it will be cited in the next sections below.

2.2 IFMTSs with standard intuitionistic fuzzy topological operators $\mathcal{C}$ and $\mathcal{I}$

In this section, following [3], we combine the ideas and definitions from the areas of (general) topology (see, e.g., [16, 26, 37]), of (standard) modal logic (see, e.g., [15, 22, 23, 35]) and of intuitionistic fuzziness, and introduce the concept of an Intuitionistic Fuzzy Modal Topological Structure (IFMTS) as a particular case of a MTS.

We must mention immediately, that during the last years, the Intuitionistic Fuzzy Topology (IFT) has developed very actively. In [14, 17–21, 24, 25, 27–34, 36, 38–47], the first steps in this process were published. It will be interesting, in the future, to conduct a systematic research on IFT development. On the other hand, all research in the area of topology are related to set-theoretical operations "union" and/or "intersection", i.e., on the level of first order logic, but not to higher logical objects, e.g., modal logic operators. With the present research, we would like to introduce and illustrate this direction of future development of the topology and, in the present case, intuitionistic fuzzy topology.

Let everywhere below:

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$

$$U^* = \{\langle x, 0, 0\rangle | x \in E\},$$

$$E^* = \{\langle x, 1, 0\rangle | x \in E\}.$$

Then,

$$\begin{aligned} \mathcal{P}(O^*) &= \{O^*\}, \\ \mathcal{P}(E^*) &= \{A | A \subseteq E^*\}, \end{aligned}$$

where

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}.$$

When the topological structure is related to IFSs, it will be denoted as IFβ-Mα-TS for concrete types of α and β for the topological and the modal operators.

Now, we formulate four theorems related to IFMTSs. Two of them are given in [3] and their proofs will be omitted.

Theorem 1. [3] *$\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \Box, \cap\rangle$ is an IF$(in, \cap)$-Mcl-TS.*

Originally, this Theorem is formulated and proved in [3], but the following one is not discussed there, because the definition of a MTS did not have the extended form given above. The same is the situation with Theorem 3 (that is given in [3]) and Theorem 4 (that does not exist in [3]).

Theorem 2. *$\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \Diamond, \cap\rangle$ is an IF$(cl, \cap)$-Mcl-TS.*

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. Then, we check sequentially:

- the conditions C1–C4 for the topological operator:

C1:

$$\begin{aligned}\mathcal{C}(A\cup B) &= \mathcal{C}(\{\langle x,\mu_A(x),\nu_A(x)\rangle|x\in E\}\cup\{\langle x,\mu_B(x),\nu_B(x)\rangle|x\in E\})\\ &= \mathcal{C}(\{\langle x,\max(\mu_A(x),\mu_B(x)),\min(\nu_A(x),\nu_B(x))\rangle|x\in E\})\\ &= \{\langle x,\sup_{y\in E}\max(\mu_A(y),\mu_B(y)),\inf_{y\in E}\min(\nu_A(y),\nu_B(y))\rangle|x\in E\}\\ &= \{\langle x,\max(\sup_{y\in E}\mu_A(y),\sup_{y\in E}\mu_B(y)),\min(\inf_{y\in E}\nu_A(y),\inf_{y\in E}\nu_B(y))\rangle|x\in E\}\\ &= \mathcal{C}(A)\cup\mathcal{C}(B);\end{aligned}$$

C2:

$$\begin{aligned}A &= \{\langle x,\mu_A(x),\nu_A(x)\rangle|x\in E\}\\ &\subseteq \{\langle x,\sup_{y\in E}\mu_A(y),\inf_{y\in E}\nu_A(y)\rangle|x\in E\}\\ &= \mathcal{C}(A);\end{aligned}$$

C3:

$$\begin{aligned}\mathcal{C}(\mathcal{C}(A)) &= \mathcal{C}(\{\langle x,\sup_{y\in E}\mu_A(y),\inf_{y\in E}\nu_A(y)\rangle|x\in E\})\\ &= \{\langle x,\sup_{y\in E}\mu_A(y),\inf_{y\in E}\nu_A(y)\rangle|x\in E\}\\ &= \mathcal{C}(A);\end{aligned}$$

C4:

$$\begin{aligned}\mathcal{C}(O^*) &= \mathcal{C}(\{\langle x,0,1\rangle|x\in E\})\\ &= \{\langle x,\sup_{y\in E}0,\inf_{y\in E}1\rangle|x\in E\}\\ &= \{\langle x,0,1\rangle|x\in E\}\\ &= O^*;\end{aligned}$$

- the conditions C1–C4 for the modal (cl-)operator

C1:

$$\begin{aligned}\Diamond(A\cap B) &= \Diamond(\{\langle x,\mu_A(x),\nu_A(x)\rangle|x\in E\}\cap\{\langle x,\mu_B(x),\nu_B(x)\rangle|x\in E\})\\ &= \Diamond(\{\langle x,\min(\mu_A(x),\mu_B(x)),\max(\nu_A(x),\nu_B(x))\rangle|x\in E\})\\ &= \{\langle x,1-\max(\nu_A(x),\nu_B(x)),\max(\nu_A(x),\nu_B(x))\rangle|x\in E\}\\ &= \{\langle x,\min(1-\nu_A(x),1-\nu_B(x)),\max(\nu_A(x),\nu_B(x))\rangle|x\in E\}\\ &= \{\langle x,1-\nu_A(x),\nu_A(x)\rangle|x\in E\}\cap\{\langle x,1-\nu_B(x),\nu_B(x)\rangle|x\in E\}\\ &= \Diamond A\cap\Diamond B;\end{aligned}$$

C2:

$$\begin{aligned}A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}\\ &\subseteq \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\}\\ &= \Diamond A;\end{aligned}$$

C3:

$$\begin{aligned}\Diamond\Diamond A &= \Diamond\{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\}\\ &= \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\}\\ &= \Diamond A;\end{aligned}$$

C4:

$$\begin{aligned}\Diamond O^* &= \Diamond\{\langle x, 0, 1\rangle | x \in E\}\\ &= O^*;\end{aligned}$$

- and the condition $(*)$:

$$\begin{aligned}\Diamond\mathcal{C}(A) &= \Diamond\{\langle x, \sup_{y\in E}\mu_A(y), \inf_{y\in E}\nu_A(y)\rangle | x \in E\}\\ &= \{\langle x, 1-\inf_{y\in E}\nu_A(y), \inf_{y\in E}\nu_A(y)\rangle | x \in E\}\\ &= \{\langle x, \sup_{y\in E}(1-\nu_A(y)), \inf_{y\in E}\nu_A(y))\rangle | x \in E\}\\ &= \mathcal{C}(\Diamond A).\end{aligned}$$

This completes the proof. □

Theorem 3. $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \Box, \cup\rangle$ is an IF$(in, \cup)$-Min-TS.

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. Then, we check sequentially:

- the conditions I1–I4 for the topological operator

I1:

$$\begin{aligned}\mathcal{I}(A \cap B) &= \mathcal{I}(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cap \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\})\\ &= \mathcal{I}(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\})\\ &= \{\langle x, \inf_{y\in E}\min(\mu_A(y), \mu_B(y)), \sup_{y\in E}\max(\nu_A(y), \nu_B(y))\rangle | x \in E\}\\ &= \{\langle x, \min(\inf_{y\in E}\mu_A(y), \inf_{y\in E}\mu_B(y)), \max(\sup_{y\in E}\nu_A(y), \sup_{y\in E}\nu_B(y))\rangle | x \in E\}\\ &= \mathcal{I}(A) \cap \mathcal{I}(B);\end{aligned}$$

I2:

$$\begin{aligned}\mathcal{I}(A) &= \{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}\\ &\subseteq \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}\\ &= A;\end{aligned}$$

I3:

$$\begin{aligned}\mathcal{I}(\mathcal{I}(A)) &= \mathcal{I}(\{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\})\\ &= \{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}\\ &= A;\end{aligned}$$

I4:

$$\begin{aligned}\mathcal{I}(E^*) &= \mathcal{I}(\{\langle x, 1, 0\rangle | x \in E\})\\ &= \{\langle x, \sup_{y\in E} 1, \inf_{y\in E} 0\rangle | x \in E\}\\ &= \{\langle x, 1, 0\rangle | x \in E\}\\ &= E^*;\end{aligned}$$

- the conditions I1 – I4 for the modal (in-)operator

I1:

$$\begin{aligned}\Box\,(A \cup B) &= \Box\,(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cup \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\})\\ &= \Box\,(\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\})\\ &= \{\langle x, \max(\mu_A(x), \mu_B(x)), 1 - \max(\mu_A(x), \mu_B(x))\rangle | x \in E\}\\ &= \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(1 - \mu_A(x), 1 - \mu_B(x))\rangle | x \in E\}\\ &= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}\\ &\cup \{\langle x, \mu_B(x), 1 - \mu_B(x)\rangle | x \in E\}\\ &= A \cup B;\end{aligned}$$

I2:

$$\begin{aligned}\Box\,A &= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}\\ &\subseteq \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}\\ &= A;\end{aligned}$$

C3:

$$\begin{aligned}\Box\,\Box\,A &= \Box\,\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}\\ &= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}\\ &= \Box\,A;\end{aligned}$$

C4:

$$\Box E^* = \Box\{\langle x, 1, 0\rangle | x \in E\}$$
$$= E^*;$$

and condition $(*)$:

$$\Box \mathcal{I}(A) = \Box\{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}$$
$$= \{\langle x, \inf_{y\in E} \mu_A(y), 1 - \inf_{y\in E} \mu_A(y)\rangle | x \in E\}$$
$$= \{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} 1 - \mu_A(y))\rangle | x \in E\}$$
$$= \mathcal{I}(\Box A).$$

This completes the proof. □

Theorem 4. [3] $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \Diamond, \cup\rangle$ is an IF($cl, \cup$)-Min-TS.

Proof. Conditions I1 – I4 for the topological operator $\mathcal{I}$ are checked in Theorem 3. Conditions C2 – C4 for the modal operator $\Diamond$ are checked in Theorem 2. Therefore, we must check only condition C1 for the modal operator and condition (*). These checks are as follows:

C1:

$$\Diamond(A \cup B) = \Diamond(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cup \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\})$$
$$= \Diamond(\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\})$$
$$= \{\langle x, 1 - \min(\nu_A(x), \nu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}$$
$$= \{\langle x, \max(1 - \nu_A(x), 1 - \nu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}$$
$$= \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\} \cup \{\langle x, 1 - \nu_B(x), \nu_B(x)\rangle | x \in E\}$$
$$= \Diamond A \cup \Diamond B;$$

(*):

$$\Diamond \mathcal{I}(A) = \Diamond\{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}$$
$$= \{\langle x, 1 - \sup_{y\in E} \nu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}$$
$$= \{\langle x, \inf_{y\in E}(1 - \nu_A(y)), \inf_{y\in E} \nu_A(y))\rangle | x \in E\}$$
$$= \mathcal{I}\Diamond A.$$

□

Now, we formulate four new assertions, modifying the four above theorems. They are proved in the same manner.

Theorem 5. $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \Box, \cup \rangle$ is an IF$(in, \cup)$-Mcl-TS.

Theorem 6. $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \Diamond, \cup \rangle$ is an IF$(cl, \cup)$-Mcl-TS.

Theorem 7. $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \Diamond, \cap \rangle$ is an IF$(cl, \cap)$-Min-TS.

Theorem 8. $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \Box, \cap \rangle$ is an IF$(in, \cap)$-Min-TS.

Finally, we can mention that the two modal operators used in the discussed eight structures are interior idempotent, because for each IFS $A \in \mathcal{P}(E^*)$:

$$\begin{aligned} \Box \Diamond A &= \Box \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\} \\ &= \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\} \\ &= \Diamond A. \end{aligned}$$

and

$$\begin{aligned} \Diamond \Box A &= \Diamond \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} \\ &= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} \\ &= \Box A. \end{aligned}$$

2.3 IFMTSs with intuitionistic fuzzy topological operators $\mathcal{C}_{33}$ and $\mathcal{I}_{33}$

Following [10], we will give definitions of following two intuitionistic fuzzy operations (see, e.g., [2]) and for the first time we will study some of their properties:

$$\begin{aligned} A \cap_{33} B &= \{\langle x, \min(\mu_A(x), \mu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\}; \\ A \cup_{33} B &= \{\langle x, 1 - \min(\nu_A(x), \nu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}. \end{aligned}$$

These operations do not have a direct analogue with those operations from the fuzzy set theory. The geometrical interpretations of these new operations are given on Figures 1 and 2.

By analogy with constructing the first two intuitionistic fuzzy topological operators $\mathcal{C}$ and $\mathcal{I}$ on the basis of the standard operations $\cup$ and $\cap$ (see [1, 2]), in [10] we constructed two intuitionistic fuzzy topological operators by

$$\begin{aligned} \mathcal{C}_{33}(A) &= \{\langle x, 1 - \inf_{y \in E} \nu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\}; \\ \mathcal{I}_{33}(A) &= \{\langle x, \inf_{y \in E} \mu_A(y), 1 - \inf_{y \in E} \mu_A(y)\rangle | x \in E\}, \end{aligned}$$

for each IFS A.

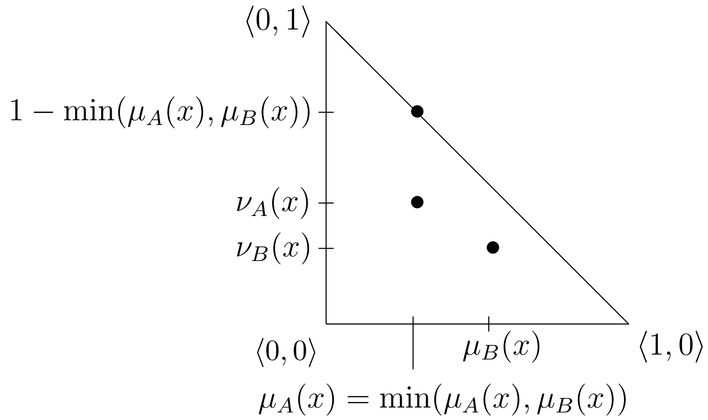

Figure 1. A geometrical interpretation of the operation $\cap_{33}$

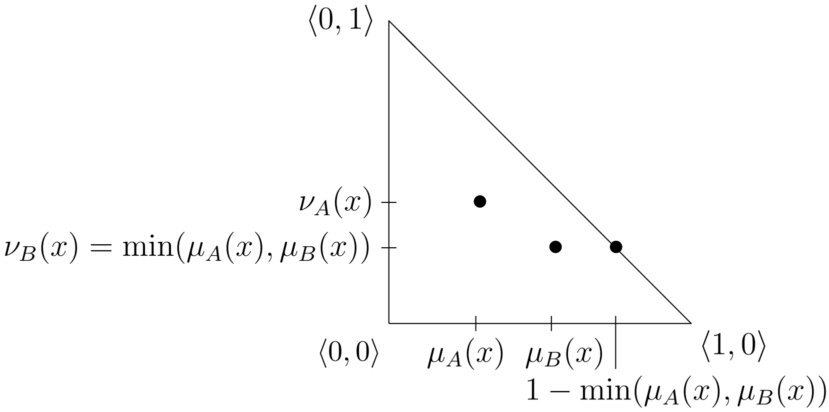

Figure 2. A geometrical interpretation of the operation $\cup_{33}$

The geometrical interpretations of the two intuitionistic fuzzy topological operators are given on Figures 3 and 4.

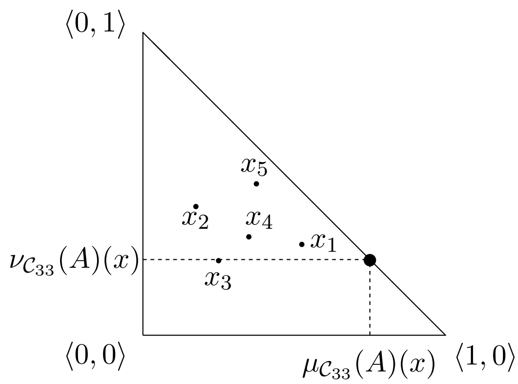

Figure 3. Geometrical interpretation of the topological operator $\mathcal{C}_{33}$

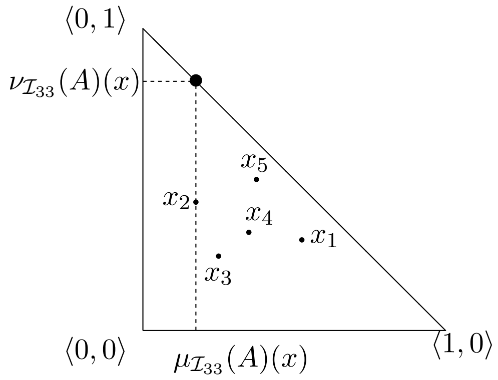

Figure 4. Geometrical interpretation of the topological operator $\mathcal{I}_{33}$

In [10] it is proved for each IFS A that:

$$\neg\mathcal{C}_{33}(\neg A) = \mathcal{I}_{33}(A),$$
$$\neg\mathcal{I}_{33}(\neg A) = \mathcal{C}_{33}(A),$$
$$\mathcal{I}_{33}(A) \subseteq \mathcal{I}(A) \subseteq \mathcal{C}(A) \subseteq \mathcal{C}_{33}(A).$$

Having in mind the results from [10], we formulate, using the new notation, the following two theorems, that are proved there.

Theorem 9. [10] $\langle \mathcal{P}(E^*), \mathcal{C}_{33}, \cup_{33}, \Diamond, \cup_{33}\rangle$ is a IF($cl, \cup_{33}$)-Mcl-TS.

We must mention that the object $\langle \mathcal{P}(E^*), \mathcal{C}_{33}, \cup_{33}, \Diamond, \cap_{33}, \rangle$ is not an IF($cl, \cap_{33}$)-Mcl-TS because for the condition C1 for the modal operator, we obtain:

$$\begin{aligned}
\Diamond(A \cap_{33} B) &= \Diamond(\{\langle x, \min(\mu_A(x), \mu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\});\\
&= \{\langle x, \min(\mu_A(x), \mu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\};\\
&\subseteq \{\langle x, \min(1-\nu_A(x), 1-\nu_B(x)), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\};\\
&\subseteq \{\langle x, \min(1-\nu_A(x), 1-\nu_B(x)), 1 - \min(1-\nu_A(x), 1-\nu_B(x))\rangle | x \in E\};\\
&= \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\} \cap_{33} \{\langle x, 1-\nu_B(x), \nu_B(x)\rangle | x \in E\}\\
&= \Diamond A \cap_{33} \Diamond B.
\end{aligned}$$

Therefore, this object is an IF feeble MTS.

Theorem 10. [13] $\langle \mathcal{P}(E^*), \mathcal{I}_{33}, \cap_{33}, \square, \cap_{33}\rangle$ is an IF($in, \cap_{33}$)-Min-TS.

Now, we can mention that the object $\langle \mathcal{P}(E^*), \mathcal{I}_{33}, \cap_{33}, \square, \cup_{33}\rangle$ is not an IF($cl, \cup_{33}$)-Mcl-TS because for the condition I1 for the modal operator, we obtain:

$$\begin{aligned}
\square(A \cup_{33} B) &= \square(\{\langle x, 1 - \min(\nu_A(x), \nu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\});\\
&= \{\langle x, 1 - \min(\nu_A(x), \nu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\};\\
&\supseteq \{\langle x, 1 - \min(1-\mu_A(x), 1-\mu_B(x)), \min(1-\mu_A(x), 1-\mu_B(x))\rangle | x \in E\};\\
&= \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in E\} \cup_{33} \{\langle x, \mu_B(x), 1-\mu_B(x)\rangle | x \in E\}\\
&= \square A \cup_{33} \square B.
\end{aligned}$$

In the same way we can see that the structures $\langle \mathcal{P}(E^*), \mathcal{C}_{33}, \cup_{33}, \diamondsuit, \cup_{33}\rangle$, $\langle \mathcal{P}(E^*), \mathcal{C}_{33}, \cup_{33}, \diamondsuit, \cap_{33}\rangle$, $\langle \mathcal{P}(E^*), \mathcal{I}_{33}, \cap_{33}, \diamondsuit, \cup_{33}\rangle$, and $\langle \mathcal{P}(E^*), \mathcal{I}_{33}, \cap_{33}, \diamondsuit, \cap_{33}\rangle$ are not IFMTSs. Now, the reason is on the remark in Subsection 2.1 for the expressions containing simultaneously operations "max" and "min".

2.4 A correction of the MTS definition

In the papers over MTS, published during two years, the author used the conditions C1–C4 and I1–I4 in the present or another form, but in the present sense. Finally, when he started writing the present paper, he realized that the conditions C4 and I4 for the case of the cl- and in-modal operators, respectively, are not necessary and therefore, their check is not obligatory. Moreover, in some cases they are not valid. Really, in the standard modal logic, the first three C- and I-conditions exist, while conditions C4 and I4 – not. In my first paper (see, [3]), I included these conditions for the modal operators only by analogy with the respective topological operators and because for the standard intuituionistic fuzzy modal operator $\Box$ and $\diamondsuit$, these conditions were valid. But, as we will see below, they are not valid for other forms of intuituionistic fuzzy modal operators.

2.5 IFMTSs with intuitionistic fuzzy modal operators $\boxtimes$ and $\diamondplus$

In the beginning, following [7], let us define the two modal operators:

$$\begin{aligned} \boxtimes A &= \{\langle x, 0, \nu_A(x)\rangle | x \in E\}; \\ \diamondplus A &= \{\langle x, \mu_A(x), 0\rangle | x \in E\}. \end{aligned}$$

Their geometrical interpretations are shown on Figure 5.

For these operators we can check directly the validity of the following assertions for each IFS A:

(a) $\boxtimes A \subseteq A \subseteq \diamondplus A$,

(b) $\neg \boxtimes \neg A = \diamondplus A$,

(c) $\neg \diamondplus \neg A = \boxtimes A$,

(d) $\boxtimes \boxtimes A = \boxtimes A$,

(e) $\diamondplus \diamondplus A = \diamondplus A$,

(f) $\boxtimes \diamondplus A = O^* = \diamondplus \boxtimes A$,

(g) $\boxtimes A \subseteq A \subseteq \diamondsuit A$,

(h) $\Box A \subseteq A \subseteq \diamondplus A$,

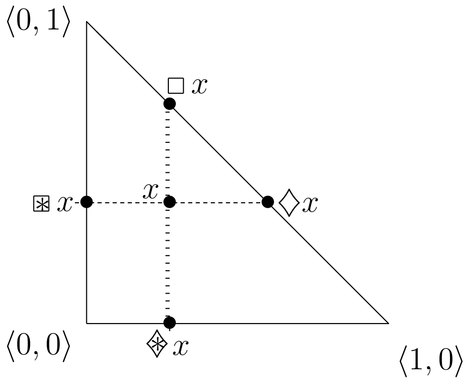

Figure 5. The geometrical interpretation of an elements $\boxast x \in E$ and $\diamondplus x \in E$

Also, we can see that for each IFS A:

$$O^* = \Box \boxast A \subseteq \boxast \Box A \subseteq \boxast \Diamond A \subseteq U^* \subseteq \diamondplus \Box A \subseteq \diamondplus \Diamond A \subseteq \Diamond \diamondplus A = E^*.$$

Now, following [7], we formulate the following assertions (in the notation) that are proved in [7].

Theorem 11. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \diamondplus, \cap\rangle$ is an IF$(cl, \cap)$-Mcl-TS.

As we mentioned in [7], condition C4 is not valid for the operator $\diamondplus$, because

$$\begin{aligned} \diamondplus O^* &= \diamondplus \{\langle x, 0, 1\rangle | x \in E\} \\ &= \{\langle x, 0, 0\rangle | x \in E\} \\ &= U^*, \end{aligned}$$

but already, this condition is superfluous for the modal operator.

Theorem 12. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \boxast, \cup\rangle$ is an IF$(in, \cup)$-Min-TS.

Again, condition I4 is not valid for operator $\boxast$, because

$$\begin{aligned} \boxast E^* &= \boxast \{\langle x, 1, 0\rangle | x \in E\} \\ &= \{\langle x, 0, 0\rangle | x \in E\} \\ &= U^*. \end{aligned}$$

Theorem 13. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxast, \cap\rangle$ is an IF$(in, \cap)$-Mcl-TS.

Theorem 14. For each universe E, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \diamondplus, \cup\rangle$ is an IF$(cl, \cup)$-Min-TS.

By analogy with the last four theorems, here we can formulate for a first time the following assertions, too, that are proved by the same manner.

Theorem 15. $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxast, \cup\rangle$ is an IF$(in, \cup)$-Mcl-TS.

Theorem 16. $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \diamondplus, \cup\rangle$ is an IF$(cl, \cup)$-Mcl-TS.

Theorem 17. $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \diamondplus, \cap\rangle$ is an IF$(cl, \cap)$-Min-TS.

Theorem 18. $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \boxast, \cap\rangle$ is an IF$(in, \cap)$-Min-TS.

For comparison to [7], here we can mention that the two modal operators used in the discussed eight structures are non-idempotent, because for each IFS $A \in \mathcal{P}(E^*)$:

$$\begin{aligned}
\boxast \Diamond\!\!\!\!\ast\, A &= \boxast \{\langle x, \mu_A(x), 0\rangle | x \in E\}\\
&= \{\langle x, 0, 0\rangle | x \in E\}\\
&= U^*.
\end{aligned}$$

and

$$\begin{aligned}
\Diamond\!\!\!\!\ast\, \boxast A &= \Diamond\!\!\!\!\ast\, \{\langle x, 0, \nu_A(x)\rangle | x \in E\}\\
&= \{\langle x, 0, 0\rangle | x \in E\}\\
&= U^*.
\end{aligned}$$

Similarly, we can use the same concepts for the topological operators: the pairs of operators $(\mathcal{C}, \mathcal{I})$ and $(\mathcal{C}_{33}, \mathcal{I}_{33})$ are interior idempotent ones, because, for example

$$\begin{aligned}
\mathcal{C}_{33}(\mathcal{I}_{33}(A)) &= \mathcal{C}_{33}(\{\langle x, \inf_{y\in E} \mu_A(x), 1 - \inf_{y\in E} \mu_A(x)\rangle | x \in E\})\\
&= \{\langle x, \inf_{z\in E} \inf_{y\in E} \mu_A(x), 1 - \inf_{z\in E} \inf_{y\in E} \mu_A(x)\rangle | x \in E\}\\
&= \{\langle x, \inf_{y\in E} \mu_A(x), 1 - \inf_{y\in E} \mu_A(x)\rangle | x \in E\}\\
&= \mathcal{I}_{33}(A)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{33}(\mathcal{C}_{33}(A)) &= \mathcal{I}_{33}(\{\langle x, 1 - \inf_{y\in E} \nu_A(x), \inf_{y\in E} \nu_A(x)\rangle | x \in E\})\\
&= \{\langle x, 1 - \inf_{z\in E} \inf_{y\in E} \nu_A(y), \inf_{z\in E} \inf_{y\in E} \nu_A(y)\rangle | x \in E\}\\
&= \{\langle x, 1 - \inf_{y\in E} \nu_A(y), \inf_{y\in E} \nu_A(y)\rangle | x \in E\}\\
&= \mathcal{I}_{33}(A).
\end{aligned}$$

This property for pair $(\mathcal{C}, \mathcal{I})$ is checked, e.g., in [1, 2].

2.6 IFMTSs with intuitionistic fuzzy modal operators $H^{\#}_{\alpha}$ and $J^{\#}_{\alpha}$

Following [13], we define the following two (particular) modal operators

$$\begin{aligned}
H^{\#}_{\alpha}(A) &= \{\langle x, \alpha.\mu_A(x), \nu_A(x)\rangle | x \in E\},\\
J^{\#}_{\alpha}(A) &= \{\langle x, \mu_A(x), \alpha\nu_A(x)\rangle | x \in E\},
\end{aligned}$$

where A is an IFS and $\alpha \in [0, 1]$.

We see immediately that for each IFS A and for each $\alpha \in [0, 1]$:

$$H^{\#}_{\alpha}(A) = H_{\alpha,0}(A),$$
$$J^{\#}_{\alpha}(A) = J_{0,\alpha}(A),$$

i.e., the new operators are particular cases of operators $H_{\alpha,\beta}$ and $J_{\alpha,\beta}$ described in details in [2].

The geometrical interpretation of both operators is shown on Figure 6, where $H^{\#}_{\alpha}(x)$ and $J^{\#}_{\beta}(x)$ are the results of the applying of the two operators over element $x \in E$.

For both operators we see that the following equalities are valid for each IFS A and for each $\alpha \in [0, 1]$:

$$\begin{aligned}\neg J^{\#}_{\alpha}(\neg A) &= \neg J^{\#}_{\alpha}(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}) \\ &= \neg\{\langle x, \nu_A(x), \alpha\mu_A(x)\rangle | x \in E\}) \\ &= \{\langle x, \alpha\mu_A(x), \nu_A(x)\rangle | x \in E\}) \\ &= H^{\#}_{\alpha}(A)\end{aligned}$$

and by analogy,

$$\neg H^{\#}_{\alpha}(\neg A) = J^{\#}_{\alpha}(A).$$

Moreover, for each IFS A and for each $\alpha \in [0, 1]$:

$$H^{\#}_{\alpha}(A) = G_{\alpha,1}(A),$$

$$J^{\#}_{\alpha}(A) = G_{1,\alpha}(A).$$

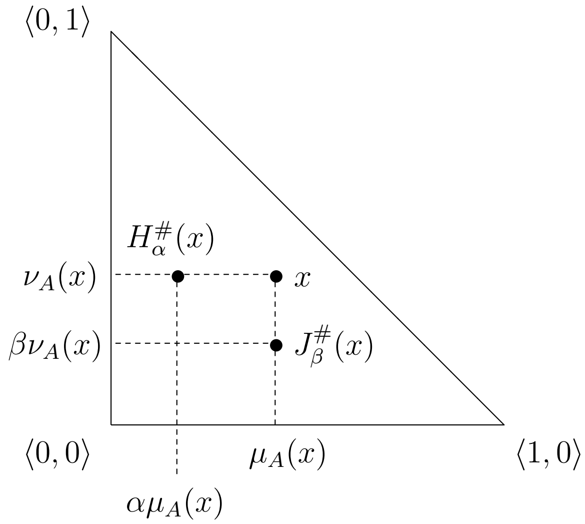

Figure 6. The geometrical interpretation of $H^{\#}_{\alpha}(x)$ and $J^{\#}_{\beta}(x)$.

Now, using the operators $H^{\#}_{\alpha}$ and $J^{\#}_{\alpha}$ and following [13], we formulate (in a new form) the following four assertions that are proved in [13].

Theorem 19. For each universe E and for each real number $\alpha \in [0, 1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, J^{\#}_{\alpha}, \cap\rangle$ is an IF$(cl, \cap)$-Mcl-TS.

Corollary 1. For each universe E and for each real number $\alpha \in [0, 1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, G_{1,\alpha}, \cap\rangle$ is a IF$(cl, \cap)$-Mcl-TS (for operator $G_{\alpha,\beta}$ see [2]).

Theorem 20. For each universe E and for each real number $\alpha \in [0, 1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, H^{\#}_{\alpha}, \cup\rangle$ is an IF$(in, \cup)$-Min-TS.

Corollary 2. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, G_{\alpha,1}, \cup \rangle$ is an IF($in, \cup$)-Min-TS.

Theorem 21. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, H^{\#}_{\alpha}, \cap \rangle$ is an IF($in, \cap$)-Mcl-TS.

Corollary 3. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cap, G_{\alpha,1}, \cap \rangle$ is an IF($in, \cap$)-Mcl-TS.

Theorem 22. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, J^{\#}_{\alpha}, \cup \rangle$ is an IF($cl, \cup$)-Min-TS.

Corollary 4. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, G_{1,\alpha}, \cup \rangle$ is an IF($cl, \cup$)-Min-TS.

By analogy, we can formulate (for the first time) and prove the following eight assertions, too.

Theorem 23. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, J^{\#}_{\alpha}, \cup \rangle$ is an IF($cl, \cup$)-Mcl-TS.

Corollary 5. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, G_{1,\alpha}, \cup \rangle$ is an IF($cl, \cup$)-Mcl-TS.

Theorem 24. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cap, H^{\#}_{\alpha}, \cup \rangle$ is an IF($cl, \cup$)-Mcl-TS.

Corollary 6. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cap, G_{\alpha,1}, \cup \rangle$ is an IF($cl, \cup$)-Mcl-TS.

Theorem 25. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, H^{\#}_{\alpha}, \cap \rangle$ is an IF($in, \cup$)-Min-TS.

Corollary 7. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, G_{\alpha,1}, \cap \rangle$ is an IF($in, \cup$)-Min-TS.

Theorem 26. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, J^{\#}_{\alpha}, \cap \rangle$ is an IF($cl, \cap$)-Min-TS.

Corollary 8. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, G_{1,\alpha}, \cap \rangle$ is an IF($cl, \cap$)-Min-TS.

2.7 IFMTSs with intuitionistic fuzzy modal operators $\boxtimes_{\alpha,\beta}$ and $\boxplus_{\alpha,\beta}$

Below, following [11], we will use the two modal operators of second type in the forms:

$$\boxplus_{\alpha,\beta} A \equiv \boxplus_{\alpha,\beta,1-\beta} A = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x) + 1 - \beta\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta} A \equiv \boxtimes_{\alpha,\beta,1-\alpha} A = \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta \in [0,1]$. When we use the first operator, we must have in mind that the inequality

$$\max(\alpha, \beta) + 1 - \beta \leq 1,$$

must hold, i.e.,

$$\alpha \leq \beta,$$

while, when we use the second operator, the inequality

$$\max(\alpha, \beta) + 1 - \alpha \leq 1,$$

must hold, i.e.

$$\beta \leq \alpha.$$

Now, we give new formulation of the assertions from [11]. Their proofs are in the cited paper [11].

Theorem 27. For each universe E and for every two $\alpha, \beta \in [0, 1], \alpha \geq \beta$: $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxtimes_{\alpha,\beta}, \cap \rangle$ is an IF$(cl, \cap)$-Mcl-TS.

Theorem 28. For each universe E and for every two $\alpha, \beta \in [0, 1], \alpha \leq \beta$: $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \boxplus_{\alpha,\beta}, \cup \rangle$ is an IF$(in, \cup)$-Min-TS.

Theorem 29. For each universe E and for every two $\alpha, \beta \in [0, 1], \alpha \leq \beta$: $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxplus_{\alpha,\beta}, \cap \rangle$ is an IF$(in, \cap)$-Mcl-TS.

Theorem 30. For each universe E and for every two $\alpha, \beta \in [0, 1], \alpha \geq \beta$: $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \boxtimes_{\alpha,\beta}, \cup \rangle$ is an IF$(cl, \cup)$-Min-TS.

2.8 IFMTSs with intuitionistic fuzzy modal operators $\Box_\alpha$ and $\Diamond_\alpha$

First, we modify the simplest IF-modal operators to the forms

$$\Box_\alpha A = \{\langle x, \alpha\mu_A(x), 1 - \mu_A(x)\rangle | x \in E\},$$
$$\Diamond_\alpha A = \{\langle x, 1 - \nu_A(x), \alpha\nu_A(x)\rangle | x \in E\},$$

where $\alpha \in [0, 1]$. For them, we immediately see that

$$\Box_\alpha A = \neg\Diamond_\alpha\neg A,$$
$$\Diamond_\alpha A = \neg\Box_\alpha\neg A.$$

Their geometrical interpretations are shown in Figure 7.

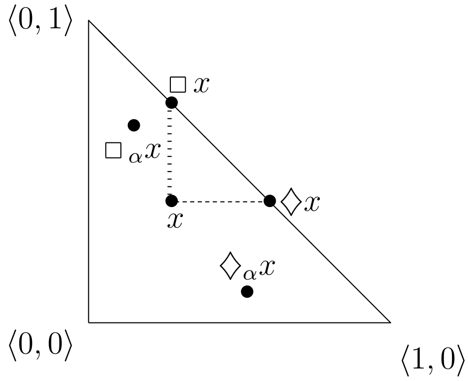

Figure 7. The geometrical interpretation of an elements $\Box_\alpha x \in E$ and $\Diamond_\alpha x \in E$

Now, using the definitions of the modal operators $\Box_\alpha$ and $\diamondsuit_{al}$, we formulate and prove the following four assertions.

Theorem 31. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \diamondsuit_\alpha, \cap \rangle$ is an IF$(cl, \cap)$-Mcl-TS.

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ and $\alpha \in [0,1]$, be given. The checks of conditions C1–C4 for the topologiocal operator are given in the proof of Theorem 2. The remaining checks are the following.

C1.
$$\begin{aligned}
\diamondsuit_\alpha(A \cap B) &= \diamondsuit_\alpha(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}) \\
&= \{\langle x, 1 - \alpha \max(\nu_A(x), \nu_B(x)), \alpha \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \min(1 - \alpha\nu_A(x), 1 - \alpha\nu_B(x)), \max(\alpha\nu_A(x), \alpha\nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, 1 - \alpha\nu_A(x), \alpha\nu_A(x)\rangle | x \in E\} \cap \{\langle x, 1 - \alpha\nu_B(x), \alpha\nu_B(x)\rangle | x \in E\} \\
&= \diamondsuit_\alpha(A) \cap \diamondsuit_\alpha(B);
\end{aligned}$$

C2.
$$\begin{aligned}
A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, 1 - \nu_A(x), \alpha\nu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, 1 - \alpha\nu_A(x), \alpha\nu_A(x)\rangle | x \in E\} \\
&= \diamondsuit_\alpha(A);
\end{aligned}$$

C3. Let $\alpha, \beta \in [0,1]$. Then

$$\begin{aligned}
\diamondsuit_\alpha(\diamondsuit_\beta(A) &= \diamondsuit_\alpha(\{\langle x, 1 - \beta\nu_A(x), \beta\nu_A(x)\rangle | x \in E\}) \\
&= \{\langle x, 1 - \alpha\beta\nu_A(x), \alpha\beta\nu_A(x)\rangle | x \in E\} \\
&= \diamondsuit_{\alpha\beta} A;
\end{aligned}$$

(*)
$$\begin{aligned}
\diamondsuit_\alpha(\mathcal{C}(A)) &= \diamondsuit_\alpha(\{\langle x, \sup_{y \in E} \mu_A(y), \inf_{y \in E} \nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, 1 - \alpha \inf_{y \in E} \mu_A(y), \alpha \inf_{y \in E} \nu_A(y)\rangle | x \in E\} \\
&= \{\langle x, \sup_{y \in E} 1 - \alpha\nu_A(y), \inf_{y \in E} \alpha\nu_A(y)\rangle | x \in E\} \\
&= \mathcal{C}(\{\langle x, 1 - \alpha\nu_A(y), \alpha\nu_A(y)\rangle | x \in E\}) \\
&= \mathcal{C}(\diamondsuit_\alpha(A).
\end{aligned}$$

This completes the proof. □

Theorem 32. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \diamondsuit_\alpha, \cup \rangle$ is an IF$(cl, \cup)$-Mcl-TS.

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ and $\alpha \in [0,1]$, be given. Having in mind the above proof, we must check only the following equality.

C1.

$$\begin{aligned}
\Diamond_\alpha(A \cup B) &= \Diamond_\alpha(\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}) \\
&= \{\langle x, 1 - \alpha \min(\nu_A(x), \nu_B(x)), \alpha \min(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \max(1 - \alpha\nu_A(x), 1 - \alpha\nu_B(x)), \min(\alpha\nu_A(x), \alpha\nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, 1 - \alpha\nu_A(x), \alpha\nu_A(x)\rangle | x \in E\} \cup \{\langle x, 1 - \alpha\nu_B(x), \alpha\nu_B(x)\rangle | x \in E\} \\
&= \Diamond_\alpha(A) \cup \Diamond_\alpha(B),
\end{aligned}$$

which proves the assertion. □

Theorem 33. For each universe E and for each real number $\alpha \in [0, 1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \Box_\alpha, \cup\rangle$ is an IF$(in, \cup)$-Min-TS.

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. The checks of conditions I1 – I4 for the topological operator are given in the proof of Theorem 3. So, we will check only the validity of conditions I1 – I3 and (*) for the modal operator.

I1.

$$\begin{aligned}
\Box_\alpha(A \cup B) &= \Box_\alpha(\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}) \\
&= \{\langle x, \alpha \max(\mu_A(x), \mu_B(x)), 1 - \alpha \max(\mu_A(x), \mu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \max(\alpha\mu_A(x), \alpha\mu_B(x)), \min(1 - \alpha\mu_A(x), 1 - \alpha\mu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \alpha\mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} \cup \{\langle x, \alpha\mu_B(x), 1 - \alpha\mu_B(x)\rangle | x \in E\} \\
&= \Box_\alpha(A) \cup \Box_\alpha(B);
\end{aligned}$$

I2.

$$\begin{aligned}
\Box_\alpha(A) &= \{\langle x, \alpha\mu_A(x), 1 - \alpha\mu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\
&= A;
\end{aligned}$$

I3. Let $\alpha, \beta \in [0, 1]$. Then

$$\begin{aligned}
\Box_\alpha(\Box_\beta(A) &= \Box_\alpha(\{\langle x, \beta\mu_A(x), 1 - \beta\mu_A(x)\rangle | x \in E\}) \\
&= \{\langle x, \alpha\beta\mu_A(x), 1 - \alpha\beta\mu_A(x)\rangle | x \in E\} \\
&= \Box_{\alpha\beta} A;
\end{aligned}$$

(*)

$$\begin{aligned}
\Box_\alpha(\mathcal{I}(A)) &= \Box_\alpha(\{\langle x, \inf_{y \in E} \mu_A(y), \sup_{y \in E} \nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, \alpha \inf_{y \in E} \mu_A(y), 1 - \alpha \inf_{y \in E} \mu_A(y)\rangle | x \in E\} \\
&= \{\langle x, \inf_{y \in E} \alpha\mu_A(y), \sup_{y \in E} 1 - \alpha\mu_A(y)\rangle | x \in E\} \\
&= \mathcal{I}(\{\langle x, \alpha\mu_A(x), 1 - \alpha\mu_A(x)\rangle | x \in E\}) \\
&= \mathcal{I}(\Box_\alpha(A).
\end{aligned}$$

This completes the proof. □

In the same manner we prove the following assertions.

Theorem 34. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \square_\alpha, \cap \rangle$ is an IF$(in, \cap)$-Min-TS.

Theorem 35. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \square_\alpha, \cap \rangle$ is an IF$(in, \cap)$-Mcl-TS.

Theorem 36. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \diamondsuit_\alpha, \cup \rangle$ is an IF$(cl, \cup)$-Min-TS.

Theorem 37. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \square_\alpha, \cup \rangle$ is an IF$(in, \cup)$-Mcl-TS.

Theorem 38. For each universe E and for each real number $\alpha \in [0,1]$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \diamondsuit_\alpha, \cap \rangle$ is an IF$(cl, \cup)$-Min-TS.

3 Concluding remarks

We mention that it is interesting to check that the following equalities are valid

$$\begin{aligned}
\mathcal{C}(U^*) &= U^*,\\
\mathcal{I}(U^*) &= U^*,\\
\mathcal{C}_{33}(U^*) &= E^*,,\\
\mathcal{I}_{33}(U^*) &= O^*,\\
\square U^* &= O^*,\\
\diamondsuit U^* &= E^*,\\
\boxtimes U^* &= U^*,\\
\diamondsuit U^* &= U^*,\\
\boxplus_{\alpha,\beta} &= \{\langle 0, \beta\rangle | x \in E\},\\
\boxtimes_{\alpha,\beta} &= \{\langle \beta, 0\rangle | x \in E\},\\
\square_\alpha U^* &= O^*,\\
\diamondsuit_\alpha U^* &= E^*.
\end{aligned}$$

Therefore, we cannot add a condition (by analogy with C4 and I4) giving a relationship between the topological and modal operators and IFS U^*, in contrast with the relationship between the topological operators and IFSs E^* and O^*.

In the present paper, we extended the results from the previous one. In future, we will discuss some other extensions of the concept of a MTS, illustrating them with temporal IFSs, IFSs over which level-operators are defined and others.

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