Notes on Intuitionistic Fuzzy Sets
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On a study on intuitionistic fuzzy *r*-normal spaces

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Abstract: The purpose of this paper is to give some new inferences of intuitionistic fuzzy normal spaces based on the concept of the most studied topics as fuzzy topological spaces. After that, the authors embed an implication among these notions and show that all these conceptions are good extensions of normal spaces. Moreover, the image and the pre-image of intuitionistic fuzzy normal space are also intuitionistic fuzzy normal space.
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1 Introduction

The fuzzy sets are indeed suitable to model vagueness. However, they cannot model uncertainty precisely because there is no means to attribute reliability of information to the membership degrees. The vast existence of indecision in day-to-day life necessitated researchers to develop some mathematical frameworks that can handle ambivalence more accurately than fuzzy sets.

For this, the theory of Intuitionistic Fuzzy Sets (IFS) is very convenient in different decision-making problems such as logic programming, medical diagnostics, electoral systems, career determination, appointment procedures, and pattern recognition [13]. Moreover, in literature, there are many distance measures defined over IFSs such as Hamming distance, normalized Hamming distance, Euclidean distance, and so on [18].

The concept of a fuzzy set was introduced by Zadeh [26] in 1965. Since then, the notions of Fuzzy sets naturally play a very significant role in the study of its related areas such as L-fuzzy sets, Fuzzy logic, Fuzzy control, Fuzzy groups, Fuzzy rings, Fuzzy vector spaces, and so on. After that, using these ideas C. L. Chang [9] first defined fuzzy topology in 1968. Furthermore, Atanassov [4] introduced the notion of an IFS in 1983 which takes into account both the degrees of membership and non-membership subject to the condition that their sum less than or equal to 1 as well as he and his colleagues have published a great amount of research on the subject
[5–7]. Later, the concept of an intuitionistic fuzzy topological space and some of their main properties were introduced by D. Çoker and his coworkers [8, 10–12]. Moreover, separation axioms of fuzzy topological spaces and intuitionistic fuzzy topological spaces were explored by many fuzzy topologists [3, 14, 15, 16, 17, 20–23], especially Al-Qubati *et. al.* [2] introduced and studied new types of b-separation axioms (bTi-space, for i = 0, 1, 2) in intuitionistic fuzzy topological spaces and Al-Qubati [1] studies the classes of normal spaces, namely β-normal spaces, β^*-normal spaces, β^*-generalized normal spaces and π generalized β^*-normal spaces in intuitionistic fuzzy topological spaces. In this paper, we define the inference of intuitionistic fuzzy r-normal spaces in nine different ways by using Intuitionistic fuzzy sets and investigate their properties.

Throughout this paper, X and Y will be nonempty sets, $\emptyset$ will be the empty set, r and s will be any positive numbers with $0 < s < r < 1$. Also, $\lambda, u, \dots$ will be fuzzy sets in X in the sense of Zadeh, $\mathcal{A}, \mathcal{B}, \dots$ will be the sets of intuitionistic type, $A, B, \dots$ will be IFSs, T will be a general topology, t will be a fuzzy topology, $\mathcal{T}$ will be the intuitionistic topology and τ, δ will be the intuitionistic fuzzy topologies, I = [0, 1], and the functions $\mu_A: X \to I$ and $\nu_A: X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of none membership (namely $\nu_A(x)$), respectively.

The remaining part of the paper is organized in the following way: Section 2 provides some basic definitions of a set of intuitionistic fuzzy type, intuitionistic fuzzy sets, and its operations, Intuitionistic Topological Space (ITS), Intuitionistic Fuzzy Topological space (IFTS), and its mappings and Intuitionistic Fuzzy Normal Space (IFNS) though Section 3 cover up the nine new notions of Intuitionistic Fuzzy r-Normal (or in short IFr-N) spaces, these notions form an implication among them, and also these concepts hold the various features and properties. Section 4 discusses a new concept of intuitionistic fuzzy normal spaces and queries some of its properties. Finally, Section 5 gives the conclusion of our work and enforces the importance of further studies.

2 Preliminaries

We recall some basic definitions and known results of a set of intuitionistic fuzzy type, intuitionistic fuzzy sets, intuitionistic topology, intuitionistic fuzzy topology, intuitionistic fuzzy normal topology, characteristics function, and intuitionistic fuzzy mapping.

Definition 2.1 ([10]). *A set of intuitionistic fuzzy type $\mathcal{A}$ is an object having the form $\mathcal{A} = \langle X, \mathcal{A}_1, \mathcal{A}_2 \rangle$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are subsets of a nonempty set X satisfying $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. The set $\mathcal{A}_1$ is called the set of members of $\mathcal{A}$, while $\mathcal{A}_2$ is called the set of non-members of $\mathcal{A}$.*

Throughout this paper, we use the simpler notation $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ for a set of intuitionistic fuzzy type. Note that, every subset $\mathcal{A}$ of a nonempty set X may obviously be regarded as a set of intuitionistic fuzzy type having the form $\mathcal{A} = (\mathcal{A}, \mathcal{A}^c)$, where $\mathcal{A}^c = X - \mathcal{A}$.

Definition 2.2 ([10]). *Let the sets of intuitionistic fuzzy type $\mathcal{A}$ and $\mathcal{B}$ in X be have the forms $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ and $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$, respectively. Furthermore, let $\{\mathcal{A}_j, j \in J\}$ be an arbitrary family of a set of intuitionistic fuzzy type in X, where $\mathcal{A}_j = \left(\mathcal{A}_j^{(1)}, \mathcal{A}_j^{(2)}\right)$. Then*

(a) $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A}_1 \subseteq \mathcal{B}_1$ and $\mathcal{A}_2 \supseteq \mathcal{B}_2$.
(b) $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.
(c) $\mathcal{A}^c = (\mathcal{A}_2, \mathcal{A}_1)$ denotes the complement of $\mathcal{A}$.
(d) $\cap \mathcal{A}_j = \left(\cap \mathcal{A}_j^{(1)}, \cup \mathcal{A}_j^{(2)}\right)$.
(e) $\cup \mathcal{A}_j = \left(\cup \mathcal{A}_j^{(1)}, \cap \mathcal{A}_j^{(2)}\right)$.
(f) $\emptyset_\sim = (\emptyset, X)$ and $X_\sim = (X, \emptyset)$.

Definition 2.3 ([4, 5]). *Let X be a non empty set. An intuitionistic fuzzy set A in X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and every $x \in X$ satisfying the requirement* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Throughout this paper, we use the simpler notation $A = (\mu_A, \nu_A)$ instead of $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ for intuitionistic fuzzy sets.

Definition 2.4. *Let X be a nonempty set and $A \subseteq X$, then the set A may be regarded as a fuzzy set in X by its characteristic function $1_A : X \to \{0,1\}$ which is defined by*

$$1_A(x) = \begin{cases} 1, & if\ x \in A. \\ 0, & if\ x \notin A\ or\ x \in A^c. \end{cases}$$

Obviously, every fuzzy set λ in X is an intuitionistic fuzzy set of the form $(\lambda, 1 - \lambda) = (\lambda, \lambda^c)$ and every intuitionistic set $A = (A_1, A_2)$ in X is an intuitionistic fuzzy set of the form $\left(1_{A_1}, 1_{A_2}\right)$. Again, we know that a fuzzy set λ in X may be regarded as an intuitionistic fuzzy set by $(\lambda, 1 - \lambda) = (\lambda, \lambda^c)$. So every subset A of X may be regarded as an intuitionistic fuzzy set by $(1_A, 1 - 1_A) = (1_A, 1_{A^c})$.

Definition 2.5 ([4, 5])**.** *Let X be a nonempty set and A, B be intuitionistic fuzzy sets on X with degrees of membership and non-membership given by (μ_A, ν_A) and (μ_B, ν_B), respectively, then*

(a) $A \subseteq B$ *if* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for all* $x \in X$.
(b) $A = B$ *if* $A \subseteq B$ *and* $B \subseteq A$.
(c) $\bar{A} = (\nu_A, \mu_A)$.
(d) $A \cap B = (\mu_A \cap \mu_B,\ \nu_A \cup \nu_B)$.
(e) $A \cup B = (\mu_A \cup \mu_B,\ \nu_A \cap \nu_B)$.

Definition 2.6 ([5])**.** *Let* $\left\{A_j = (\mu_{A_j}, \nu_{A_j}), j \in J\right\}$ *be an arbitrary family of IFSs in* X. *Then*

(a) $\cap A_j = \left(\cap \mu_{A_j},\ \cup \nu_{A_j}\right)$.
(b) $\cup A_j = \left(\cup \mu_{A_j},\ \cap \nu_{A_j}\right)$.
(c) $0_\sim = (\underline{0}, \underline{1})$, $1_\sim = (\underline{1}, \underline{0})$.

Definition 2.7 ([12])**.** *Let X be a nonempty set. A family $\mathcal{T}$ of some sets of intuitionistic type in X is called an intuitionistic topology (IT for short) on X if the following conditions hold*

(1) $\emptyset_\sim, X_\sim \in \mathcal{T}$.
(2) $\mathcal{A} \cap \mathcal{B} \in \mathcal{T}$ *for all* $\mathcal{A}, \mathcal{B} \in \mathcal{T}$.
(3) $\cup \mathcal{A}_j \in \mathcal{T}$ *for any arbitrary family* $\{\mathcal{A}_j \in \mathcal{T}, j \in J\}$.

Then the pair $(X, \mathcal{T})$ is called an intuitionistic topological space (ITS, for short), members of $\mathcal{T}$ are called intuitionistic open sets (IOS, for short) in X and their complements are called intuitionistic closed sets (ICS, for short) in X.

Definition 2.8 ([11])**.** *An intuitionistic fuzzy topology (IFT for short) on a nonempty set X is a family τ of IFSs in X satisfying the following axioms*

(1) $0_\sim, 1_\sim \in \tau$.
(2) $A \cap B \in \tau$, *for all* $A, B \in \tau$.
(3) $\cup A_j \in \tau$ *for any arbitrary family* $\{A_j \in \tau,\ j \in J\}$.

Definition 2.9 ([11])**.** *Let (X, τ) be an IFTS and A be an IFS in X. Then the closure and interior of A are defined by*

$$cl(A) = \cap\{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\},$$

$$int(A) = \cup\{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}.$$

Definition 2.10 ([5])**.** *Let X and Y be two nonempty sets and $f: X \to Y$ be a function. If $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ and $B = \{(y, \mu_B(y), \nu_B(y)) : y \in Y\}$ are IFSs in X and Y, respectively, then the pre-image of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by $f^{-1}(B) = \left\{\left(x,\ (f^{-1}(\mu_B))(x), (f^{-1}(\nu_B))(x)\right) : x \in X\right\} = \{(x,\ \mu_B(f(x)), \nu_B(f(x))) : x \in X\}$ and the image of A under f, denoted by $f(A)$, is the IFS in Y defined by $f(A) = \left\{\left(y,\ (f(\mu_A))(y), (f(\nu_A))(y)\right) : y \in Y\right\}$, where for each $y \in Y$*

$$(f(\mu_A))(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & if\ \ f^{-1}(y) \neq \emptyset \\ 0, & otherwise \end{cases}$$

$$(f(\nu_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \nu_A(x), & if\ f^{-1}(y) \neq \emptyset \\ 1, & otherwise. \end{cases}$$

Definition 2.11 ([11]). *Let (X, τ) and (Y, δ) be IFTSs. A function $f: X \to Y$ is called continuous if $f^{-1}(B)$ is IFOS for all IFOS B or equivalently $f^{-1}(B)$ is IFCS for all IFCS B.*

Definition 2.12 ([11]). *Let (X, τ) and (Y, δ) be IFTSs and $f: X \to Y$ is a function. f is called open if the image of an open set is open and f is called closed if the image of a closed set is closed.*

Definition 2.13 ([19]). *A topological space (X, T) is called normal if for all closed sets F_1 and F_2 with $F_1 \cap F_2 = \emptyset$, there exist $G, H \in T$ such that $F_1 \subset G$, $F_2 \subset H$ and $G \cap H = \emptyset$.*

Definition 2.14 ([14]). *A fuzzy topological space (X, t) is called normal if for all closed fuzzy sets m and open fuzzy sets u with $m \subset u$, there exists an open fuzzy set v such that $m \subset v \subset \overline{v} \subset u$, where $\overline{v}$ is the closer of v.*

Definition 2.15 ([24]). *An intuitionistic topological space $(X, \mathcal{T})$ is called normal if for all closed sets $\mathcal{F}$ and $\mathcal{G}$ with $\mathcal{F} \cap \mathcal{G} = \emptyset_{\sim}$, there exist $\mathcal{A}, \mathcal{B} \in \mathcal{T}$ such that $\mathcal{F} \subset \mathcal{A}$, $\mathcal{G} \subset \mathcal{B}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset_{\sim}$.*

Definition 2.16 ([1]). *An intuitionistic fuzzy topological space (X, τ) is said to be intuitionistic fuzzy β-normal space (for short IFβ-N) if for every pair of disjoint intuitionistic fuzzy closed sets A and B, there exist two disjoint intuitionistic fuzzy β open sets (IFβOSs) U and V such that $A \subseteq U$, $B \subseteq V$.*

3 Definitions and properties of intuitionistic fuzzy *r*-normal spaces

In this section, we introduce the nine new notions of intuitionistic fuzzy *r*-normal (for short IFr-N) spaces and form an implication among them. Also, we discuss the various features and properties of these concepts.

Definition 3.1. *Let r be any positive number and $r \in (0,1)$. Then an intuitionistic fuzzy topological space (X, τ) is called*

(a) IFr-N(i) if for all IFCSs $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ with $(\mu_F \cap \mu_G)(x) = 0$, $r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$, then there exist IFOSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $F \subset A$ and $G \subset B$ with $(\mu_A \cap \mu_B)(y) = 0, r < (\nu_A \cup \nu_B)(y)$ for all $y \in X$.

(b) IFr-N(ii) if for all IFCSs $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ with $(\mu_F \cap \mu_G)(x) = 0$, $r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$, then there exist IFOSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $F \subset A$ and $G \subset B$ with $(\mu_A \cap \mu_B)(y) < r < (\nu_A \cup \nu_B)(y)$ for all $y \in X$.

(c) IFr-N(iii) if for all IFCSs $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ with $(\mu_F \cap \mu_G)(x) = 0$, $r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$, then there exist IFOSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $F \subset A$ and $G \subset B$ with $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$ for all $y \in X$.

(d) IFr-N(iv) if for all IFCSs $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ with $(\mu_F \cap \mu_G)(x) < r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$, then there exist IFOSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $F \subset A$ and $G \subset B$ with $(\mu_A \cap \mu_B)(y) = 0, r < (\nu_A \cup \nu_B)(y)$ for all $y \in X$.

(e) *IFr-N(v) if for all IFCSs* $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ *with* $(\mu_F \cap \mu_G)(x) < r < (\nu_F \cup \nu_G)(x)$ *for all* $x \in X$, *then there exist IFOSs* $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ *such that* $F \subset A$ *and* $G \subset B$ *with* $(\mu_A \cap \mu_B)(y) < r < (\nu_A \cup \nu_B)(y)$ *for all* $y \in X$.

(f) *IFr-N(vi) if for all IFCSs* $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ *with* $(\mu_F \cap \mu_G)(x) < r < (\nu_F \cup \nu_G)(x)$ *for all* $x \in X$, *then there exist IFOSs* $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ *such that* $F \subset A$ *and* $G \subset B$ *with* $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$ *for all* $y \in X$.

(g) *IFr-N(vii) if for all IFCSs* $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ *with* $(\mu_F \cap \mu_G) \subset (\nu_F \cup \nu_G)$, *then there exist IFOSs* $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ *such that* $F \subset A$ *and* $G \subset B$ *with* $(\mu_A \cap \mu_B)(y) = 0, r < (\nu_A \cup \nu_B)(y)$ *for all* $y \in X$.

(h) *IFr-N(viii) if for all IFCSs* $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ *with* $(\mu_F \cap \mu_G) \subset (\nu_F \cup \nu_G)$, *then there exist IFOSs* $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ *such that* $F \subset A$ *and* $G \subset B$ *with* $(\mu_A \cap \mu_B)(y) < r < (\nu_A \cup \nu_B)(y)$ *for all* $y \in X$.

(i) *IFr-N(ix) if for all IFCSs* $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ *with* $(\mu_F \cap \mu_G) \subset (\nu_F \cup \nu_G)$, *then there exist IFOSs* $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ *such that* $F \subset A$ *and* $G \subset B$ *with* $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$ *for all* $y \in X$.

Theorem 3.1. Let (X, τ) be an IFTS. Then the above notions form the following implications

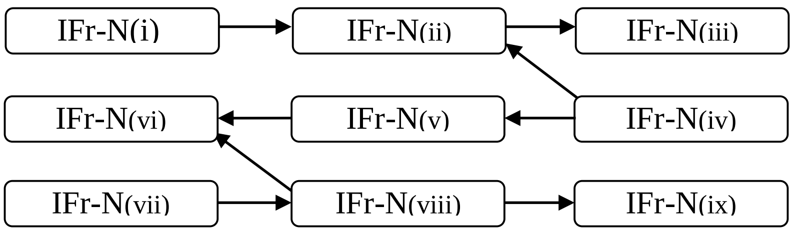

Figure 1. Implications among the intuitionistic fuzzy *r*-normal concepts

Proof: Suppose that (X, τ) is IFr-N(i). Let $F = (\mu_F, \nu_F)$ and $G = (\mu_G, \nu_G)$ are closed sets in (X, τ) with $(\mu_F \cap \mu_G)(x) = 0$, $r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$. Since (X, τ) is IFr-N(i), there exist $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $F \subset A$ and $G \subset B$ with

$$(\mu_A \cap \mu_B)(y) = 0, r < (\nu_A \cup \nu_B)(y) \text{ for all } y \in X \tag{1}$$

$$\Longrightarrow (\mu_A \cap \mu_B)(y) < r < (\nu_A \cup \nu_B)(y) \text{ for all } y \in X \tag{2}$$

$$\Longrightarrow (\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B). \tag{3}$$

From (1), (2), and (3) we see that IFr-N(i) ⇒ IFr-N(ii) ⇒ IFr-N(iii).

Again suppose that (X, τ) is IFr-N(iv). Let $F = (\mu_F, \nu_F)$ and $G = (\mu_G, \nu_G)$ are closed sets in (X, τ) with $(\mu_F \cap \mu_G)(x) < r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$. Since (X, τ) is IFr-N(iv), there exist $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $F \subset A$ and $G \subset B$ with

$$(\mu_A \cap \mu_B)(y) = 0, r < (\nu_A \cup \nu_B)(y) \text{ for all } y \in X \tag{4}$$

$$\Longrightarrow (\mu_A \cap \mu_B)(y) < r < (\nu_A \cup \nu_B)(y) \text{ for all } y \in X \tag{5}$$

$$\Longrightarrow (\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B). \tag{6}$$

From (4), (5), and (6) we obtain IFr-N(iv) ⇒ IFr-N(v) ⇒ IFr-N(vi).

Moreover, let us assume that (X,τ) is IFr-N(vii). Suppose that $F=(\mu_F,\nu_F)$ and $G=(\mu_G,\nu_G)$ are closed sets in (X,τ) with $(\mu_F\cap\mu_G)\subset(\nu_F\cup\nu_G)$. Since (X,τ) is IFr-N(vii), there exist $A=(\mu_A,\nu_A), B=(\mu_B,\nu_B)\in\tau$ such that $F\subset A$ and $G\subset B$ with

$$(\mu_A\cap\mu_B)(y)=0, r<(\nu_A\cup\nu_B)(y) \text{ for all } y\in X \tag{7}$$

$$\Longrightarrow (\mu_A\cap\mu_B)(y)<r<(\nu_A\cup\nu_B)(y) \text{ for all } y\in X \tag{8}$$

$$\Longrightarrow (\mu_A\cap\mu_B)\subset(\nu_A\cup\nu_B). \tag{9}$$

From (7), (8), and (9) we show that IFr-N(vii) ⇒ IFr-N(viii) ⇒ IFr-N(ix).

Also, we consider that (X,τ) is IFr-N(iv). Let $F=(\mu_F,\nu_F)$ and $G=(\mu_G,\nu_G)$ are closed sets in (X,τ) with $(\mu_F\cap\mu_G)(x)=0$, $r<(\nu_F\cup\nu_G)(x)$ for all $x\in X$. But $(\mu_F\cap\mu_G)(x)=0$, $r<(\nu_F\cup\nu_G)(x)$ implies $(\mu_F\cap\mu_G)(x)<r<(\nu_F\cup\nu_G)(x)$ for all $x\in X$. Since (X,τ) is IFr-N(iv), there exists $A=(\mu_A,\nu_A), B=(\mu_B,\nu_B)\in\tau$ such that $F\subset A$ and $G\subset B$ with $(\mu_A\cap\mu_B)(y)=0$, $r<(\nu_A\cup\nu_B)(y)$ for all $y\in X$. But $(\mu_A\cap\mu_B)(y)=0$, $r<(\nu_A\cup\nu_B)(y)$ for all $y\in X$ implies $(\mu_A\cap\mu_B)(y)<r<(\nu_A\cup\nu_B)(y)$ for all $y\in X$. Therefore (X,τ) is IFr-N(ii). And finally, in the similar way we can show that IFr-N(viii) ⇒ IFr-N(ix). □

Theorem 3.2. *Let (X,T) be a topological space and (X,τ) be its corresponding IFTS where $\tau=\left\{\left(1_{A_j},\ 1^c_{A_j}\right), j\in J : A_j\in T\right\}$. Then (X,T) is normal if and only if (X,τ) is IFr-N(k) for $k=i,ii,iii,iv,v,vi,vii,viii,ix$.*

Proof: We shall prove the theorem for the case $k=i$. Suppose that (X,T) is normal space. Let $(1_F,1_{F^c})$ and $(1_G,1_{G^c})$ are closed sets in (X,τ) with $(1_F\cap 1_G)(x)=0, r<(1_{F^c}\cup 1_{G^c})(x)$ for all $x\in X$. Now $(1_F\cap 1_G)\,(x)=0$ for all $x\in X$ implies $F\cap G=\emptyset$ and clearly F,G are closed in (X,T) by the definition of τ. Since (X,T) is normal, then there exist $A,B\in T$ such that $F\subset A$, $G\subset B$ and $A\cap B=\emptyset$. By the definition of τ, it is clear that $(1_A,1_{A^c}),(1_B,1_{B^c})\in\tau$. Also clearly $(1_F,1_{F^c})\subset(1_A,1_{A^c})$ and $(1_G,1_{G^c})\subset(1_B,1_{B^c})$ as $F\subset A$ and $G\subset B$. Since $A\cap B=\emptyset$, thence $(1_A\cap 1_B)(y)=0$, and $(1_{A^c}\cup 1_{B^c})(y)=1$ for all $y\in X$. That implies $(1_A\cap 1_B)(y)=0, r<(1_{A^c}\cup 1_{B^c})(y)$ for all $y\in X$. Therefore (X,τ) is IFr-N(i).

Conversely, suppose that (X,τ) is IFN-r(i). Let F,G are closed in (X,T) with $F\cap G=\emptyset$. Clearly $(1_F,1_{F^c})$, $(1_G,1_{G^c})$ are closed in (X,τ) with $(1_F\cap 1_G)(x)=0$, $(1_{F^c}\cup 1_{G^c})(x)$=1 for all $x\in X$. That implies $(1_F\cap 1_G)(x)=0$, r $<(1_{F^c}\cup 1_{G^c})(x)$ for all $x\in X$. Since (X,τ) is IFr-N(i), there exist $(1_A,1_{A^c}),(1_B,1_{B^c})\in\tau$ such that $(1_F,1_{F^c})\subset(1_A,1_{A^c})$ and $(1_G,1_{G^c})\subset(1_B,1_{B^c})$ with $(1_A\cap 1_B)(y)=0$, $r<(1_{A^c}\cup 1_{B^c})(y)$ for all $y\in X$. By the definition of τ, it is clear that $A,B\in T$ and clearly $(1_A\cap 1_B)(y)=0$ for all $y\in X$ as 1_A, 1_B are characteristic functions. That implies $A\cap B=\emptyset$. Therefore (X,T) is normal. □

Theorem 3.3. *Let $(X,\mathcal{T})$ be an intuitionistic topological space and (X,τ) be its corresponding IFTS where $\tau=\left\{\left(1_{A_{j1}},\ 1_{A_{j2}}\right), j\in J :\ A_j=\left(A_{j1},A_{j2}\right)\in\mathcal{T}\right\}$. Then $(X,\mathcal{T})$ is normal if and only if (X,τ) is IFr-N(k) for $k=i,ii,iii,iv,v,vi,vii,viii,ix$.*

Proof: The proof of all implications is similar. As an example, we shall prove $(X,\mathcal{T})$ is normal ⇔ (X,τ) is IFr-N(iv). Suppose $(X,\mathcal{T})$ is normal. Let $\left(1_{F_1},1_{F_2}\right)$ and $\left(1_{G_1},1_{G_2}\right)$ be closed in (X,τ) with $\left(1_{F_1}\cap 1_{G_1}\right)(x)<r<\left(1_{F_2}\cup 1_{G_2}\right)(x)$ for all $x\in X$. Since 1_{F_1}, 1_{F_2}, $1_{G_1}$and 1_{G_2} are characteristic functions and $r\in(0,1)$ thence $\left(1_{F_1}\cap 1_{G_1}\right)(x)=0$ and $\left(1_{F_2}\cup 1_{G_2}\right)(x)=1$ for all $x \in X$. By the definition of τ it is clear that $F = (F_1, F_2)$, $G = (G_1, G_2)$ are closed in $(X, \mathcal{T})$. Now $F_1 \cap G_1 = \emptyset$ and $F_2 \cup G_2 = X$ as $(1_{F_1} \cap 1_{G_1})(x) = 0$ and $(1_{F_2} \cup 1_{G_2})(x) = 1$ for all $x \in X$. That implies, $F \cap G = (\emptyset, X) = \emptyset_{\sim}$. Since $(X, \mathcal{T})$ is normal, there exist $A = (A_1, A_2)$, $B = (B_1, B_2) \in \mathcal{T}$ such that $F \subset A$, $G \subset B$ with $A \cap B = \emptyset_{\sim}$. Now $F \subset A$ and $G \subset B$ implies $F_1 \subset A_1$, $F_2 \supset A_2$ and $G_1 \subset B_1$, $G_2 \supset B_2$. Also $A \cap B = \emptyset_{\sim}$ implies $A_1 \cap B_1 = \emptyset$ and $A_2 \cup B_2 = X$. By the definition of τ, $(1_{A_1}, 1_{A_2}), (1_{B_1}, 1_{B_2}) \in \tau$. Clearly $(1_{A_1} \cap 1_{B_1})(x) = 0$ and $(1_{A_2} \cup 1_{B_2})(x) = 1$ for all $x \in X$ as $A_1 \cap B_1 = \emptyset$ and $A_2 \cup B_2 = X$. That implies $(1_{A_1} \cap 1_{B_1})(x) = 0$, $r < (1_{A_2} \cup 1_{B_2})(x)$ for all $x \in X$. Also, it is clear that $(1_{F_1}, 1_{F_2}) \subset (1_{A_1}, 1_{A_2})$ and $(1_{G_1}, 1_{G_2}) \subset (1_{B_1}, 1_{B_2})$ as $F_1 \subset A_1$, $F_2 \supset A_2$ and $G_1 \subset B_1$, $G_2 \supset B_2$. Therefore (X, τ) is IFr-N(iv).

Conversely, suppose that (X, τ) is IFr-N(iv). Let $F = (F_1, F_2)$, $G = (G_1, G_2)$ be closed in $(X, \mathcal{T})$ with $F \cap G = (\emptyset, X) = \emptyset_{\sim}$. By the definition of τ it is clear that $(1_{F_1}, 1_{F_2})$ and $(1_{G_1}, 1_{G_2})$ are closed in (X, τ). Also $(1_{F_1} \cap 1_{G_1})(x) = 0$ and $(1_{F_2} \cup 1_{G_2})(x) = 1$ for all $x \in X$. That implies $(1_{F_1} \cap 1_{G_1})(x) < r < (1_{F_2} \cup 1_{G_2})(x)$ for all $x \in X$ as $r \in (0,1)$. Since (X, τ) is IFr-N(iv), there exist $(1_{A_1}, 1_{A_2}), (1_{B_1}, 1_{B_2}) \in \tau$ such that $(1_{F_1}, 1_{F_2}) \subset (1_{A_1}, 1_{A_2})$ and $(1_{G_1}, 1_{G_2}) \subset (1_{B_1}, 1_{B_2})$ with $(1_{A_1} \cap 1_{B_1})(x) = 0$, $r < (1_{A_2} \cup 1_{B_2})(x)$ for all $x \in X$. Henceforth $(1_{A_1} \cap 1_{B_1})(x) = 0$, $(1_{A_2} \cup 1_{B_2})(x) = 1$ as 1_{A_2} and 1_{B_2} are characteristic functions and $r \in (0,1)$. By the definition of τ, $(A_1, A_2), (B_1, B_2) \in \mathcal{T}$. Now $(A_1, A_2) \cap (B_1, B_2) = (A_1 \cap B_1, A_2 \cup B_2)$=$(\emptyset, X) = \emptyset_{\sim}$[$\because (1_{A_1} \cap 1_{B_1})(x) = 0$ and $(1_{A_2} \cup 1_{B_2})(x) = 1$ for all $x \in X$]. Again $(1_{F_1}, 1_{F_2}) \subset (1_{A_1}, 1_{A_2}) \Rightarrow F_1 \subset A_1$ and $F_2 \supset A_2 \Rightarrow (F_1, F_2) \subset (A_1, A_2)$. In the same way, we have $(G_1, G_2) \subset (B_1, B_2)$. Therefore $(X, \mathcal{T})$ is normal. □

Theorem 3.4. *Let (X, τ) be an IFTS, and r, s be any positive numbers with $0 < s < r < 1$, then the following relations are true:*

(a) if (X, τ) is IFs-N(iii), then (X, τ) is IFr-N(iii), and

(b) if (X, τ) is IFr-N(vii), then (X, τ) is IFs-N (vii).

Proof: Let us assume that (X, τ) is IFs-N(iii) and $s < r$. Let $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ are closed sets in (X, τ) with $(\mu_F \cap \mu_G)(x) = 0$, $r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$. Now $(\mu_F \cap \mu_G)(x) = 0$, $r < (\nu_F \cup \nu_G)(x)$ implies $(\mu_F \cap \mu_G)(x) = 0$, $s < (\nu_F \cup \nu_G)(x)$ for all $x \in X$ as $s < r$. Since (X, τ) is IFs-N(iii), there exist $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $F \subset A$ and $G \subset B$ with $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$. That is if for any closed sets $F = (\mu_F, \nu_F)$, $G = (\mu_G, \nu_G)$ in (X, τ) with $(\mu_F \cap \mu_G)(x) = 0, r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$, then there exist IFOSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $F \subset A$ and $G \subset B$ with $(\mu_A \cap \mu_B) \subset (\nu_A \cup \nu_B)$. Therefore, (X, τ) is IFr-N(iii). In the same manner, we show that (X, τ) is IFr-N(vii) $\Rightarrow (X, \tau)$ is IFs-N(vii). □

Theorem 3.5. *Let (X, τ) and (Y, δ) be IFTSs and $f: X \to Y$ be one-to-one, onto, closed and continuous mapping. Then (Y, δ) is IFr-N(k) implies that (X, τ) is IFr-N(k) for $k = i, ii, iii, iv, v, vi, vii, viii, ix$.*

Proof: We shall prove the theorem for the case $k = i$. Let us consider that (Y, δ) is IFr-N(i). Let $F = (\mu_F, \nu_F)$ and $G = (\mu_G, \nu_G)$ be closed in (X, τ) with $(\mu_F \cap \mu_G)(x) = 0$, $r < (\nu_F \cup \nu_G)(x)$ for all $x \in X$. Now $f(F) = (f(\mu_F), f(\nu_F)$ and $f(G) = (f(\mu_G), f(\nu_G)$ are closed in (Y,δ) as f is closed. Since f is one-to-one and onto, therefore, for any $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$. That implies $f^{-1}(y) = \{x\}$. Now for any $y \in Y$, we have $\big(f(F) \cap f(G)\big)(y) = \Big(\big(f(\mu_F) \cap f(\mu_G)\big)(y), (f(\nu_F) \cup f(\nu_G))(y)\Big)$.

But $\big(f(\mu_F) \cap f(\mu_G)\big)(y) = \min(f(\mu_F)(y), f(\mu_G)(y))$

$$= \min\Big(\sup_{p\in f^{-1}(y)} \mu_F(p), \sup_{p\in f^{-1}(y)} \mu_G(p)\Big)$$

$$= \min\big(\mu_F(x),\ \mu_G(x)\big) \text{ as } f^{-1}(y) = \{x\}$$

$$= (\mu_F \cap \mu_G)(x) = 0.$$

And $\big(f(\nu_F) \cup f(\nu_G)\big)(y) = \max\big(f(\nu_F)(y), f(\nu\mu_G)(y)\big)$

$$= \max\Big(\inf_{p\in f^{-1}(y)} \nu_F(p), \inf_{p\in f^{-1}(y)} \nu_G(p)\Big)$$

$$= \max\big(\nu_F(x),\ \nu_G(x)\big) \text{ as } f^{-1}(y) = \{x\}$$

$$= (\nu_F \cup \nu_G)(x) > r.$$

Therefore, $\big(f(\mu_F) \cap f(\mu_G)\big)(y) = 0$, $r < \big(f(\nu_F) \cup f(\nu_G)\big)(y)$ for all $y \in Y$. But (Y,δ) is IFr-N(i), so there exist $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \delta$ such that $f(F) \subset A$ and $f(G) \subset B$ with $(\mu_A \cap \mu_B)(y) = 0$, $r < (\nu_A \cup \nu_B)(y)$ for all $y \in Y$. Now, $f(F) \subset A \Rightarrow f^{-1}(f(F)) \subset f^{-1}(A)$. But $F = f^{-1}(f(F))$ as f is one-to-one. So that $F \subset f^{-1}(A)$ and similarly, $G \subset f^{-1}(B)$. Also $f^{-1}(A), f^{-1}(B) \in \tau$, as f is continuous. Now, for any $x \in X$, we have

$(f^{-1}(A) \cap f^{-1}(B))(x) = ((f^{-1}(\mu_A), f^{-1}(\nu_A)) \cap (f^{-1}(\mu_B), f^{-1}(\nu_B)))(x)$

$$=\Big(\big(f^{-1}(\mu_A) \cap f^{-1}(\mu_B)\big)(x), (f^{-1}(\nu_A) \cup f^{-1}(\nu_B))(x)\Big).$$

But then,

$\big(f^{-1}(\mu_A) \cap f^{-1}(\mu_B)\big)(x) = \min\Big(\big(f^{-1}(\mu_A)\big)(x), \big(f^{-1}(\mu_B)\big)(x)\Big)$

$$= \min\big(\mu_A(f(x)), \mu_B(f(x))\big) = (\mu_A \cap \mu_B)(f(x)) = 0$$

as $(\mu_A \cap \mu_B)(y) = 0$ for all $y \in Y$. And

$\big(f^{-1}(\nu_A) \cup f^{-1}(\nu_B)\big)(x) = \max\Big(\big(f^{-1}(\nu_A)\big)(x), \big(f^{-1}(\nu_B)\big)(x)\Big)$

$$= \max\big(\nu_A(f(x)), \nu_B(f(x))\big) = (\nu_A \cup \nu_B)\big(f(x)\big) > r$$

as $r < (\nu_A \cup \nu_B)(y)$ for all $y \in Y$.

Therefore, $\big(f^{-1}(\mu_A) \cap f^{-1}(\mu_B)\big)(x) = 0, r < \big(f^{-1}(\nu_A) \cup f^{-1}(\nu_B)\big)(x),\ \forall\, x \in X$. So (X,τ) is IFr-N (i). $\square$

Theorem 3.6. *If (X,τ) and (Y,δ) be IFTSs and $f: X \to Y$ is one-to-one, onto, continuous and open mapping. Then (X,τ) is IFr-N(k) implies that (Y,δ) is IFr-N(k) for $k = i, ii, iii, iv, v, vi, vii, viii, ix$.*

Proof: We shall prove the theorem for $k = ii$. Suppose that (X,τ) is IFr-N(ii). Let $F = (\mu_F, \nu_F)$ and $G = (\mu_G, \nu_G)$ are closed in (Y, δ) with $(\mu_F \cap \mu_G)(y) = 0$, $r < (\nu_F \cup \nu_G)(y)$ for all $y \in Y$. Since f is continuous, so $f^{-1}(F)$ and $f^{-1}(G)$ are closed in (X,τ). Now for any $x \in X$, we have,
$(f^{-1}(F) \cap f^{-1}(G))(x)$=$((f^{-1}(\mu_F), f^{-1}(\nu_F)) \cap (f^{-1}(\mu_G), f^{-1}(\nu_G)))(x)$

$$=\left(f^{-1}(\mu_F) \cap f^{-1}\left(\mu_g\right), f^{-1}(\nu_F) \cup f^{-1}(\nu_G)\right)(x)$$

$$=\left(\left(f^{-1}(\mu_F) \cap f^{-1}(\mu_G)\right)(x), (f^{-1}(\nu_F) \cup f^{-1}(\nu_G))(x)\right).$$

But then,

$$\left(f^{-1}(\mu_F) \cap f^{-1}(\mu_G)\right)(x) = \min\left(\left(f^{-1}(\mu_F)\right)(x), \left(f^{-1}(\mu_G)\right)(x)\right)$$

$$= \min\left(\mu_F\left(f(x)\right), \mu_G\left(f(x)\right)\right) = (\mu_F \cap \mu_G)\left(f(x)\right) = 0$$

as $(\mu_F \cap \mu_G)(y) = 0$ for all $y \in Y$. And

$$\left(f^{-1}(\nu_F) \cup f^{-1}(\nu_B)\right)(x) = \max\left(\left(f^{-1}(\nu_F)\right)(x), \left(f^{-1}(\nu_G)\right)(x)\right)$$

$$= \max\left(\nu_F(f(x)), \nu_G(f(x))\right) = (\nu_F \cup \nu_G)\left(f(x)\right) > r$$

as $r < (\nu_F \cup \nu_G)(y)$ for all $y \in Y$.

Therefore, $\left(f^{-1}(\mu_F) \cap f^{-1}(\mu_G)\right)(x) = 0$, $r < \left(f^{-1}(\nu_F) \cup f^{-1}(\nu_B)\right)(x)$ for all $x \in X$. But since (X,τ) is IFr-N(ii), so there exists $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $f^{-1}(F) \subset A$ and $f^{-1}(G) \subset B$ with $(\mu_A \cap \mu_B)(x) < r < (\nu_A \cup \nu_B)(x)$ for all $x \in X$. Since f is open, therefore $f(A), f(B) \in \delta$. Also, $f^{-1}(F) \subset A \Rightarrow f\left(f^{-1}(F)\right) \subset f(A) \Rightarrow F \subset f(A)$ as f is onto. Similarly, $G \subset f(B)$. Since f is one-to-one and onto, then for any $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y \Longrightarrow f^{-1}(y) = x$. Now, for any $y \in Y$, we have $\left(f(A) \cap f(B)\right)(y) = \left(\left(f(\mu_A) \cap f(\mu_B)\right)(y), (f(\nu_A) \cup f(\nu_B))(y)\right)$. But

$$\left(f(\mu_A) \cap f(\mu_B)\right)(y) = \min\left(f(\mu_A)(y), f(\mu_B)(y)\right) = \min\left(\sup_{p \in f^{-1}(y)} \mu_A(p), \sup_{p \in f^{-1}(y)} \mu_B(p)\right)$$

$= \min\left(\mu_A(x),\ \mu_B(x)\right)$ as $f^{-1}(y) = \{x\}$

$= (\mu_A \cap\ \mu_B)(x) < r$ as $(\mu_A \cap \mu_B)(y) < r$ for all $x \in X$. And

$$\left(f(\nu_A) \cup f(\nu_A)\right)(y) = \max\left(f(\nu_A)(y), f(\nu_B)(y)\right)=\max\left(\inf_{p \in f^{-1}(y)} \nu_A(p), \inf_{p \in f^{-1}(y)} \nu_B(p)\right)$$

$= \max\left(\nu_A(x),\ \nu_B(x)\right)$ as $f^{-1}(y) = \{x\}$

$= (\nu_A \cup\ \nu_B)(x) > r$ as $r < (\nu_A \cup \nu_B)(x)$ for all $x \in X$. Therefore,

$$\left(f(\mu_A) \cap f(\mu_B)\right)(y) < r < \left(f(\nu_A) \cup f(\nu_A)\right)(y)$$

for all $y \in Y$. So (Y, δ) is IFr-N(ii). □

4 General type intuitionistic fuzzy normal spaces

In this section, we yield a new concept of general type of intuitionistic fuzzy normal spaces and query some of its properties.

Definition 4.1. *An intuitionistic fuzzy topological space (X,τ) is called IF-Normal if for all IFCS F and IFOS A with $F \subset A$ there exist an IFOS V such that $F \subset V \subset cl(V) \subset A$.*

Theorem 4.1. *Let (X,T) be a topological space and (X,τ) be its corresponding IFTS where $\tau = \left\{\left(1_{A_j}, 1_{A_j}^c\right), j \in J : A_j \in T\right\}$. Then (X,T) is normal if and only if (X,τ) is IF-normal.*

Proof: Suppose that (X,T) is normal. Let $(1_F, 1_{F^c})$ is an IFCS and $(1_A, 1_{A^c})$ is an IFOS with $(1_F, 1_{F^c}) \subset (1_A, 1_{A^c})$. Then by the definition of τ, it is clear that F is closed in (X,T), A is open in (X,T) and $F \subset A$. Therefore A^c is closed in (X,T) and $F \cap A^c = \emptyset$. Since (X,T) is normal, there exist $G, H \in T$ such that $F \subset G$, $A^c \subset H$ with $G \cap H = \emptyset$. Now it is clear that $(1_G, 1_{G^c}), (1_H, 1_{H^c}) \in \tau$ and $(1_G, 1_{G^c}) \cap (1_H, 1_{H^c}) = (\underline{0}, \underline{1})$. Since $(1_G, 1_{G^c}) \cap (1_H, 1_{H^c}) = (\underline{0}, \underline{1})$, thence $(1_G, 1_{G^c}) \subset (1_H, 1_{H^c})^c = (1_{H^c}, 1_H)$. Again since $A^c \subset H$, thence $H^c \subset A$. This implies that $(1_{H^c}, 1_H) \subset (1_A, 1_{A^c})$. Also $F \subset G$ implies $(1_F, 1_{F^c}) \subset (1_G, 1_{G^c})$. Therefore $(1_F, 1_{F^c}) \subset (1_G, 1_{G^c}) \subset (1_{H^c}, 1_H) \subset (1_A, 1_{A^c})$. Since G, H are open in (X,T), thence $(1_G, 1_{G^c})$ is open and $(1_{H^c}, 1_H)$ is closed in (X,τ). Therefore $(1_G, 1_{G^c}) \subset cl(1_G, 1_{G^c}) \subseteq (1_{H^c}, 1_H)$. So we can write $(1_F, 1_{F^c}) \subset (1_G, 1_{G^c}) \subset cl(1_G, 1_{G^c}) \subset (1_A, 1_{A^c})$. Therefore (X,τ) is IF-normal.

Conversely, suppose that (X,τ) is IF-normal. Let F is closed in (X,T), A is open in (X,T) with $F \subset A$. Then $(1_F, 1_{F^c})$ is closed and $(1_A, 1_{A^c})$ is open in (X,τ) with $(1_F, 1_{F^c}) \subset (1_A, 1_{A^c})$. Since (X,τ) is IF-Normal, there exist an IFOS $(1_V, 1_{V^c})$ such that $(1_F, 1_{F^c}) \subset (1_V, 1_{V^c}) \subset cl(1_V, 1_{V^c}) \subset (1_A, 1_{A^c})$. Let $cl(1_V, 1_{V^c}) = (1_H, 1_{H^c})$. Therefore $(1_H, 1_{H^c})$ is closed and $(1_F, 1_{F^c}) \subset (1_V, 1_{V^c}) \subset (1_H, 1_{H^c}) \subset (1_A, 1_{A^c})$. So $F \subset V \subset H \subset A$ where V is open and H is closed in (X,T). Since $V \subset H$ and H is closed, thence $V \subset cl(V) \subseteq H$. So we can write $F \subset V \subset cl(V) \subset A$. Therefore (X,T) is normal. □

Theorem 4.2. *Let (X,t) be a fuzzy topological space and (X,τ) be its corresponding IFTS where $\tau = \{(\lambda, \lambda^c) : \lambda \in t\}$. Then (X,t) is normal if and only if (X,τ) is IF-normal.*

Proof: Let us consider that (X,t) be fuzzy normal. Let $F = (f, f^c)$ is IFCS and $A = (a, a^c)$ is IFOS in (X,τ) with $F \subset A$. Then f is closed, a is open and $f \subset a$ in (X,t). Since (X,t) is fuzzy normal, there exists an open set $v \in t$ such that $f \subset v \subset \overline{v} \subset a$. Since $\overline{v}$ is closed thence $\overline{v}^c$ is open. By the definition of τ, it is clear that $V = (v, v^c)$ and $\left(\overline{v}^c, \left(\overline{v}^c\right)^c\right) = (\overline{v}^c, \overline{v})$ are open in (X,τ) as v and $\overline{v}^c$ are open in (X,t). Since $(\overline{v}^c, \overline{v})$ is open then the complement of $(\overline{v}^c, \overline{v}) = (\overline{v}, \overline{v}^c)$ is closed in (X,τ). Also, it is clear that $(f, f^c) \subset (v, v^c) \subset (\overline{v}, \overline{v}^c) \subset (a, a^c)$. That implies, $F \subset V \subset \left(\overline{v}, \overline{v}^c\right) \subset A$. Since $(\overline{v}, \overline{v}^c)$ is closed and $V \subset (\overline{v}, \overline{v}^c)$, thence $cl(V) \subset (\overline{v}, \overline{v}^c)$. Therefore, $F \subset V \subset cl(V) \subset A$. Hence (X,τ) is intuitionistic fuzzy normal.

Conversely, suppose that (X,τ) is intuitionistic fuzzy normal. Let m be closed, u be open in (X,t) with $m \subset u$. Since m is closed thence m^c is open in (X,t). By the definition of τ, $(m^c, (m^c)^c) = (m^c, m)$ and $U = (u, u^c)$ are open in (X,τ). Since (m^c, m) is open, then the complement of $(m^c, m) = (m, m^c) = M$ is closed in (X,τ). Again since $m \subset u$, thence $M = (m, m^c) \subset (u, u^c) = U$. That implies, $M \subset U$. Since (X,τ) is normal, there exists an

IFOS
$V = (v, v^c) \in \tau$ such that $M \subset V \subset cl(V) \subset U \Rightarrow (m, m^c) \subset (v, v^c) \subset (w, w^c) \subset (u, u^c)$, where $cl(V) = (w, w^c)$. Therefore, $m \subset v \subset w \subset u$. Clearly v is open. Now $cl(V) = (w, w^c)$ is closed, so (w^c, w) is open in (X, τ). By the definition of τ, w^c is open in (X, t). So $(w^c)^c = w$ is closed in (X, t). Since w is closed and $v \subset w$, thence $\overline{v} \subset w$. Therefore, it is clear that $m \subset v \subset \overline{v} \subset w \subset u \Rightarrow m \subset v \subset \overline{v} \subset u$. Hence, (X, t) is fuzzy normal. □

Theorem 4.3. *Let (X, τ) and (Y, δ) be IFTSs and $f: X \to Y$ be an open, closed, one-to-one and continuous mapping. Then (Y, δ) is IF-normal if and only if (X, τ) is IF-normal.*

Proof: Suppose that (Y, δ) is IF-normal. Let F is IFCS and A is IFOS in (X, τ) with $\subset A$, Now $f(F)$ is IFCS and $f(A)$ is IFOS in (Y, δ) as f is an open and closed function. Again $P \subset Q \Rightarrow f(P) \subset f(Q)$, therefore $f(F) \subset f(A)$. Since (Y, δ) is IF-normal, there exist an open set $B \in \delta$ such that $f(F) \subset B \subset cl(B) \subset f(A)$. But $P \subset Q \Rightarrow f^{-1}(P) \subset f^{-1}(Q)$. Therefore, we have $f^{-1}(f(F)) \subset f^{-1}(B) \subset f^{-1}(cl(B)) \subset f^{-1}(f(A))$. Since f is one-to-one thence $f^{-1}\big(f(F)\big) = F$. Therefore $F \subset f^{-1}(B) \subset f^{-1}(cl(B)) \subset A$. Since f is a continuous function, B is IFOS and $cl(B)$ is IFCS in (Y, δ), therefore $f^{-1}(B)$ is IFOS and $f^{-1}(cl(B))$ is IFCS in (X, τ). Also by the definition of closer $f^{-1}(B) \subset cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$. Therefore, we have shown that $F \subset f^{-1}(B) \subset cl(f^{-1}(B)) \subset A$. So (X, τ) is IF-normal.

Conversely, suppose that (X, τ) is IF-normal. Let F be IFCS and A be IFOS in (Y, δ) with $F \subset A$. Then $f^{-1}(F)$ is IFCS and $f^{-1}(A)$ is IFOS in (X, τ) as f is continuous. Again $P \subset Q \Rightarrow f^{-1}(P) \subset f^{-1}(Q)$. Therefore $f^{-1}(F) \subset f^{-1}(A)$. But (X, τ) is IF-normal, so there exists IFOS, $B \in \tau$ such that $f^{-1}(F) \subset B \subset cl(B) \subset f^{-1}(A)$. But $P \subset Q \Rightarrow f(P) \subset f(Q)$. Therefore, $f(f^{-1}(F)) \subset f(B) \subset f(cl(B)) \subset f(f^{-1}(A))$. Also, $f(f^{-1}(P)) = P$ as f is onto. Therefore, $F \subset f(B) \subset f(cl(B)) \subset A$. Now, since f is open and B is IFOS in (X, τ), thence $f(B)$ is IFOS in (Y, δ). Again since f is closed and cl(B) is IFCS in (X, τ), thence $f(cl(B))$ is IFCS in (Y, δ). Hence, by the definition of the closure we have $f(B) \subset cl(\, f(B)) \subset f(cl(B))$. So we have $F \subset f(B) \subset cl(\, f(B)) \subset A$. Therefore, (Y, δ) is IF-normal. □

5 Conclusion

In this article, we have given nine new ideas of intuitionistic fuzzy r-normal space and established some relationships among them. We have observed that Theorem 3.5 and Theorem 3.6 represent that our notions convey topological property in the sense of intuitionistic fuzzy normal space. Also, Theorem 4.1 and Theorem 4.2 demonstrate that it is a 'Good extension' of normal spaces and fuzzy normal spaces. Moreover, by careful investigation, these nine conjectures are more general than Al-Qubati and his coworker [1, 2].

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