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A VARIANT OF CRAIG'S INTERPOLATION THEOREM FOR INTUITIONISTIC FUZZY FORMULAE. Part 1

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What is formulated and proved is an analogue of the W. Craig's interpolation theorem [1,2] for the case of Intuitionistic Fuzzy Propositional Calculus (IFPC) formulae [3].

To each proposition p in IFPC (see [3]) we can assign a "truth degree" $\mu(p) \in [0,1]$ and a "falsity degree" $\nu(p) \in [0,1]$, such that

$$\mu(p) + \nu(p) \leq 1.$$

Let this assignment be provided by an evaluation function V defined over a set of propositions S in such a way that:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

When the values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be extended also for the operation "$\rightarrow$" through the definition :

$$V(p \rightarrow q) = \langle max(\nu(p), \mu(q)), min(\mu(p), \nu(q)) \rangle$$

and let

$$V(p) \rightarrow V(q) = V(p \rightarrow q).$$

For the needs of the discussion below we shall define the notion of Intuitionistic Fuzzy Tautology (IFT) through:

"A is an IFT" iff if $V(A) = \langle a, b \rangle$, then $a \geq b$.

All the above notions for propositions are extended for the case of formulae analogically.

Let $\mathcal{F}$ be a set of formulae, with the property that for all $\langle a, b \rangle \in [0,1] \times [0,1]$ such that $a + b \leq 1$, there exists a formula $f \in \mathcal{F}$ such that $V(F) = \langle a, b \rangle$.

Theorem: Let F and G be different formulae and let $F \rightarrow G$ be an IFT. Then there exists a formula H different than F and G, such that $F \rightarrow H$ and $H \rightarrow G$ are IFTs.

Proof: Let

$$V(F) = \langle \mu_F, \nu_F \rangle,$$
$$V(G) = \langle \mu_G, \nu_G \rangle.$$

Then

$$V(F \rightarrow G) = \langle max(\nu_F, \mu_G), min(\mu_F, \nu_G) \rangle$$

and by condition,

$$max(\nu_G, \mu_G) \geq min(\mu_F, \nu_G).$$

Let

$$V(H) = \langle \mu_H, \nu_H \rangle.$$

Then

$$V(F \to H) = \langle max(\nu_F, \mu_H), min(\mu_F, \nu_H)\rangle,$$
$$V(H \to G) = \langle max(\nu_H, \mu_G), min(\mu_H, \nu_G)\rangle.$$

There are three cases.

Case 1: $\mu_F \geq \mu_G$ and $\nu_F \leq \nu_G$. Then we put, e.g.,

$$\mu_H = \frac{\mu_F + \mu_G}{2},$$
$$\nu_H = \frac{\nu_F + \nu_G}{2}.$$

Therefore,

$$\mu_F \geq \mu_H \geq \mu_G,$$
$$\nu_F \leq \nu_H \leq \nu_G,$$

and

$$max(\nu_F, \mu_H) \geq max(\nu_F, \mu_G) \geq min(\mu_F, \nu_G) \geq min(\mu_F, \nu_H),$$
$$max(\nu_H, \mu_G) \geq max(\nu_F, \mu_G) \geq min(\mu_F, \nu_G) \geq min(\mu_H, \nu_G).$$

Case 2: $\mu_F < \mu_G$. Then we put,

$$\mu_H = \mu_F,$$
$$\nu_H = \nu_G.$$

Therefore,

$$max(\nu_F, \mu_H) - min(\mu_F, \nu_H) \geq \mu_H - \mu_F = 0$$
$$max(\nu_H, \mu_G) - min(\mu_H, \nu_G) \geq \mu_G - \mu_H > \mu_F - \mu_H = 0.$$

Case 3: $\nu_F > \nu_G$. Then we put,

$$\mu_H = \mu_G,$$
$$\nu_H = \nu_F.$$

Therefore,

$$max(\nu_F, \mu_H) - min(\mu_F, \nu_H) \geq \nu_F - \nu_H = 0$$
$$max(\nu_H, \mu_G) - min(\mu_H, \nu_G) \geq \nu_H - \nu_G = \nu_F - \mu_G > 0.$$

Hence in all cases $F \to H$ and $H \to G$ are IFTs. Finally, we choose this formula H for which $V(H) = \langle \mu_H, \nu_H \rangle$, for the above constructed values of μ_H and ν_H.

References:

[1] Craig W. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. *J. Symbolic Logic*, Vol. 22, 1957, 269-285.

[2] Barwise J. (Ed.) Handbook of Mathematical Logic, North-Holland, Amsterdam, 1977.

[3] Atanassov, K. Intuitionistic Fuzzy Sets. Springer Physica-Verlag, Heidelberg, 1999.