Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy sets and some of their so-called extensions

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Abstract: In the present paper we investigate the relationship between intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and some proposed extensions of fuzzy sets, intuitionistic fuzzy sets, and sets that have similar structure. In many cases we establish that the proposed concepts claiming to be extensions of the intuitionistic fuzzy sets are either incorrect in their definitions or offer nothing new compared to the already existing notions. Finally, we formulate four requirements that must be observed so that a new concept claiming to extend an existing one can be considered legitimate and viable, expanding the boundaries of the existing mathematical theory in a meaningful way.

Keywords: Fuzzy sets, Intutionistic fuzzy sets, Extensions.
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1 Introduction

Around 23 centuries ago, Aristotle introduced the concepts of propositions and predicates, evaluated as true or false, and formulated the Law of the Excluded Middle (LEM), asserting that no third truth value exists. However, in the 20th century, Brouwer challenged LEM and proposed intuitionism, a new form of mathematics where LEM does not hold true. Supporting this, Łukasiewicz introduced 3-valued logic, allowing intermediate values like $\frac{1}{2}$ to express partial truth, later generalizing it to n-valued logic.

In 1965, Zadeh extended this concept further by proposing fuzzy logic, where truth values span the full interval $[0, 1]$, though some criticized it due to the possibility for interference of irrational values. In practice, people tend to use rational numbers, and fuzzy sets quickly found real-world applications, such as in control systems and robotics. Further extensions followed, like Goguen's L-fuzzy sets and Pawlak's rough sets, all capturing degrees of truth or membership.

In 1983, intuitionistic fuzzy sets (IFS) were introduced, which use two values: degree of membership μ and non-membership ν, constrained by the requirement $\mu + \nu \leq 1$, allowing for an additional degree of indeterminacy $\pi = 1 - \mu - \nu$. This framework builds on Brouwer's ideas of intuitionism and enables richer interpretations than traditional fuzzy sets. While FSs are often application-driven, IFS have spurred more theoretical development, including unique modal and topological operators and multiple geometric interpretations.

As IFSs gained popularity, numerous modifications emerged — some valid extensions, others merely cosmetic. A key issue is that many so-called generalizations (e.g., changing the value interval) are fully representable within standard IFSs. This paper clarifies the theory, identifies which modifications are true extensions, and maps out a hierarchy of genuine IFS extensions, providing both structure and direction for future research.

Any proposed extension of a mathematical object must be properly defined, demonstrably extend the original concept, and offer new, distinct properties. Unfortunately, recent papers claim to extend IFS while merely reformulating them — such as by replacing the interval $[0, 1]$ with $[a, b]$ — without recognizing the simple bijective transformation between them, making these "extensions" mathematically redundant.

This paper addresses such issues by clarifying the core theory of IFSs and examining how various proposed ideas fit—or fail to fit—within its framework. It also surveys current developments, including interval-valued and higher-type IFS, and outlines future research directions. The primary aim is to demonstrate that many new constructs are in fact representable within standard IFS or Interval-Valued IFS (IVIFS), and to distinguish them from genuine extensions, which are also explored and organized into a coherent hierarchy.

2 IFSs and sets related to them

In this paper, we will use the definitions from [4], where the basic results over IFSs are given. Here, we will discuss some ideas and concepts which have certain correspondence with IFSs.

The first such concept of **"indefinite set"** was introduced by A. Narinyani in 1980 [23]. Unfortunately, his original paper is not readily available but a later publication of his [24], provides sufficient (even if a bit less detailed) description of the concept. In essence there are introduced two sets X^+ for the elements who are at the current stage known to belong to a certain not completely identified ("indefinite" or "subdefinite") set X, those elements that are at the current stage known not to belong X^- and X^c as the cardinal of X. Note, that although there is certain resemblance in the fact that there is a membership and non-membership assigned, Narinyani was interested in a crisp result at the end and considered these sets as an intermediary step in obtaining more efficient rules for manipulating models involving such not completely known sets.

Another concept sharing some kinship with FSs and IFSs are the so-called **grey sets** [13]. They are another way of extending the characteristic function of a set this time using the so-called grey numbers.

In some sense, the **Complex Atanassov's IFSs** of A. M. Alkouri and A. R. Salleh (see [1,29]), are real extensions, because the evaluation interval $[0, 1]$ now is extended to the set of complex numbers. On the other hand, these sets are special type of IFS for which

$$\mu_A(x) = r_A(x).e^{i\omega_{\mu A}(x)}, \quad \nu_A(x) = k_A(x).e^{i\omega_{\nu A}(x)}.$$

where i is the imaginary unit and $r_A(x), k_A(x), \omega_{\mu A}(x), \omega_{\nu A}(x)$ are real numbers, such that $r_A(x) + k_A(x) \leq 1$.

Further we will discuss some concepts that while termed as "extensions" by their authors are not really extensions of IFSs.

A **vague set** [15] is a set with two types of membership functions: truth-membership function $f_t(x)$ and false-membership function $f_v(x)$, with the constraint $t_v(x) + f_v(x) \leq 1$. (cf. IFSs restirction of $\mu_A(x)+\nu_A(x) \leq 1$). P. Burillo and H. Bustince in [10] demonstrated that vague sets coincide with IFSs. Indeed, putting $\mu(x) = t_v(x)$ and $\nu(x) = f_v(x)$, we recover all the inclusions, complements, etc. of the vague sets by using the ones identified for IFSs. It is worth noting that the approach of Gau and Buehrer was aimed at representing the uncertainty about membership as a possible interval range where the currently unknown real membership, for which only upper and lower bounds are known, should fall. In this sense their idea is similar to Narinyani's approach and also to **Interval Valued Fuzzy Sets** (IVFS) – another concept that is equipollent with the IFSs as proved in [7].

Another venue for "generalization" of the notion of IFS is the notion introduced by I. Despi et al [14,22]. There they propose several "generalizations" of the IFSs, which differ mainly in the way the membership and non-membership functions are represented and the constraints posed over them. In [30] it is shown that all these cases may be considered as purely cosmetic change of variables – and thus coincide with IFSs.

The **Variable Fuzzy Set** (VFS), introduced in [27], is a very particular case of IFSs in which $\mu + \nu = 1$. Therefore, these sets coincide with the standard fuzzy sets, while the authors assert that they had introduced "new and effective IFS theory".

In [2], a "new approach" towards IFS is proposed by C. Annamalai, under which the degree $\pi_A(x)$ is also included, i.e.,

$$A^* = \{\langle x, \mu_A(x), \nu_A(x), \pi_A(x)\rangle \mid x \in E\},$$

where $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$ for each x, but this does not lead to anything new. Moreover, based on the same idea, some five years earlier, in [31], Y. Yang and F. Chiclana proposed a new spherical geometric interpretation of the IFS.

In [25], S. Rizvi, H. J. Naqvi and D. Nadeem define **rough intuitionistic fuzzy sets**, but they themselves show that these are IFS. It is unclear why they have given them a new name. The same situation is true for the sets introduced in [19] by S. Indira and R. R. Rajeswari under the name **star intuitionistic sets**. Apart from omitting the word "fuzzy" after "intuitionistic" and using incorrect denotations, the "proof" that star intiuitionistic sets are IFS is also wrong.

Among the many "extensions" of IFS, there are many extravagant objects. For instance in the paper [20], there is practically no definition for the proposed by the authors concept of **"intuitionistic evidence sets"**, and just explanations, which make it clear that it concerns a particular case of IFS, for which the universe is a set of subsets of a fixed set, and the functions μ and ν are given by the functions of A. P. Dempster and G. Shafer from [12] and [26]. Obviously, the title of the paper and the name of the newly introduced concept are incorrect because they do not contain the world "fuzzy".

Another extravagant idea has been described in the paper [16] of T. Gerstenkorn and J. Manko. There a new name – **bifuzzy set** is proposed for IFS – even though IFS had already been defined 12 years prior to it.

Interval-valued intuitionistic fuzzy set (IVIFS) A over E is an object of the form:

$$A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\},$$

where

$$M_A(x) \subset [0, 1] \text{ and } N_A(x) \subset [0, 1]$$

are intervals and for all $x \in E$: $\sup M_A(x) + \sup N_A(x) \leq 1$.

Obviously, each IVFS A can be represented by an IVIFS as

$$A = \{\langle x, M_A(x), [1 - \sup M_A(x), 1 - \inf M_A(x)]\rangle \mid x \in E\}.$$

Here we will discuss some concepts related to IVIFS. Further we represent briefly the relevant results from [9].

In [17, 18], the concept of an **Inconsistent Intuitionistic Fuzzy Set** (IIFS) was introduced in the following manner:

$$A^i = \{\langle x, \mu^i_A(x), \nu^i_A(x), \iota^i_A(x)\rangle \mid x \in E\},$$

where

$$0 \leq \mu^i_A(x) + \nu^i_A(x) + \iota^i_A(x) \leq 1$$

is called IFS and functions $\mu^i_A : E \to [0, 1]$ and $\nu^i_A : E \to [0, 1]$ represent, respectively, the *degree of membership (validity, etc.)* and *non-membership (non-validity, etc.) of element $x \in E$ to a fixed set $A \subseteq E$.*

We can define again function $\pi_A : E \to [0, 1]$ by means of

$$\pi^i_A(x) = 1 - \mu^i_A(x) - \nu^i_A(x) - \iota^i_A(x)$$

and it corresponds to *degree of indeterminacy (uncertainty, etc.).*

We now outline four IVIFS-interpretations of a fixed IIFS A^i.

Putting:

$$\begin{aligned}
\inf M_A(x) &\stackrel{\text{def}}{=} \mu^i_A(x),\\
\inf N_A(x) &\stackrel{\text{def}}{=} \nu^i_A(x),\\
\sup M_A(x) &\stackrel{\text{def}}{=} \frac{\mu^i_A(x)}{\mu^i_A(x)+\nu^i_A(x)}(1-\iota^i_A(x)),\\
\sup N_A(x) &\stackrel{\text{def}}{=} \frac{\nu^i_A(x)}{\mu^i_A(x)+\nu^i_A(x)}(1-\iota^i_A(x)).
\end{aligned}$$

Evidently,

$$\sup M_A(x) + \sup N_A(x) = 1 - \iota_A(x) \leq 1.$$

Therefore, one IVIFS-interpretation of the IIFS A^i, presented visually in Figure 1 is the following:

$$A = \left\{ \left\langle x, \left[\mu^i_A(x), \frac{\mu^i_A(x)\cdot(1-\iota^i_A(x))}{\mu^i_A(x)+\nu^i_A(x)}\right], \left[\nu^i_A(x), \frac{\nu^i_A(x)\cdot(1-\iota^i_A(x))}{\mu^i_A(x)+\nu^i_A(x)}\right] \right\rangle \mid x \in E \right\}.$$

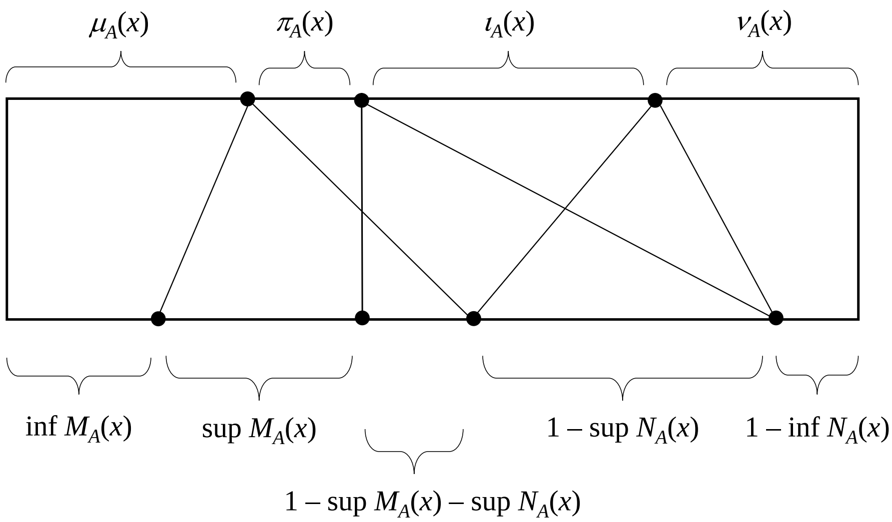

Figure 1. Geometric correspondence for the first case

If we put:

$$\begin{aligned}
\inf M_A(x) &\stackrel{\text{def}}{=} \mu^i_A(x),\\
\inf N_A(x) &\stackrel{\text{def}}{=} \nu^i_A(x),\\
\sup M_A(x) &\stackrel{\text{def}}{=} \frac{\mu^i_A(x)}{\mu^i_A(x)+\nu^i_A(x)}(1-\pi^i_A(x)),\\
\sup N_A(x) &\stackrel{\text{def}}{=} \frac{\nu^i_A(x)}{\mu^i_A(x)+\nu^i_A(x)}(1-\pi^i_A(x)).
\end{aligned}$$

Hence,

$$\sup M_A(x) + \sup N_A(x) = 1 - \pi_A(x) \leq 1.$$

Therefore, a second IVIFS-interpretation of the IIFS A^i presented visually in Figure 2 is the following:

$$A = \left\{\left\langle x, \left[\mu^i_A(x), \frac{\mu^i_A(x)\cdot(1-\pi^i_A(x))}{\mu^i_A(x)+\nu^i_A(x)}\right], \left[\nu^i_A(x), \frac{\nu^i_A(x)\cdot(1-\pi^i_A(x))}{\mu^i_A(x)+\nu^i_A(x)}\right]\right\rangle \mid x \in E\right\}.$$

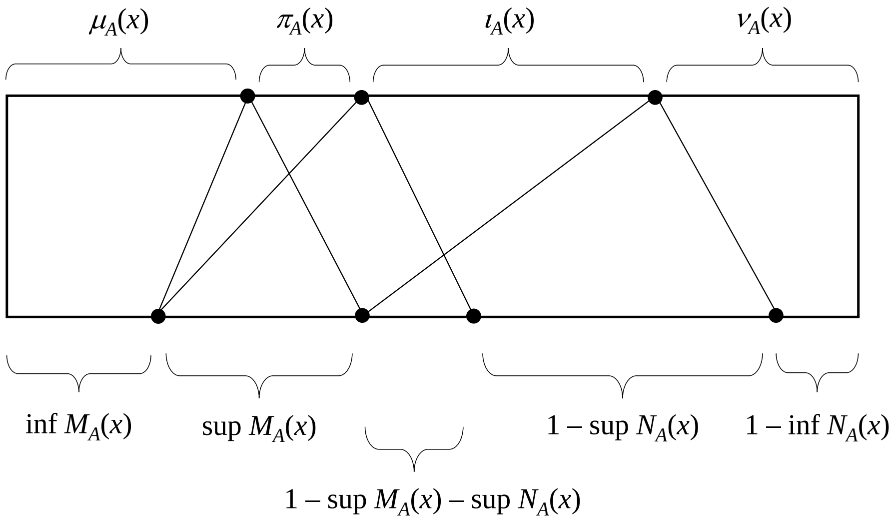

Figure 2. Geometric correspondence for the second case

Putting:

$$\begin{aligned}
\inf M_A(x) &\stackrel{\text{def}}{=} \mu^i_A(x),\\
\inf N_A(x) &\stackrel{\text{def}}{=} \nu^i_A(x),\\
\sup M_A(x) &\stackrel{\text{def}}{=} \mu^i_A(x) + \iota^i_A(x),\\
\sup N_A(x) &\stackrel{\text{def}}{=} \nu^i_A(x) + \pi^i_A(x).
\end{aligned}$$

Hence,

$$\sup M_A(x) + \sup N_A(x) = 1.$$

Therefore, a third IVIFS-interpretation of the IIFS A^i, presented visually in Figure 3 is the following:

$$A = \left\{\left\langle x, \left[\mu^i_A(x), \mu^i_A(x) + \iota^i_A(x)\right], \left[\nu^i_A(x), \nu^i_A(x) + \pi^i_A(x)\right]\right\rangle \mid x \in E\right\}.$$

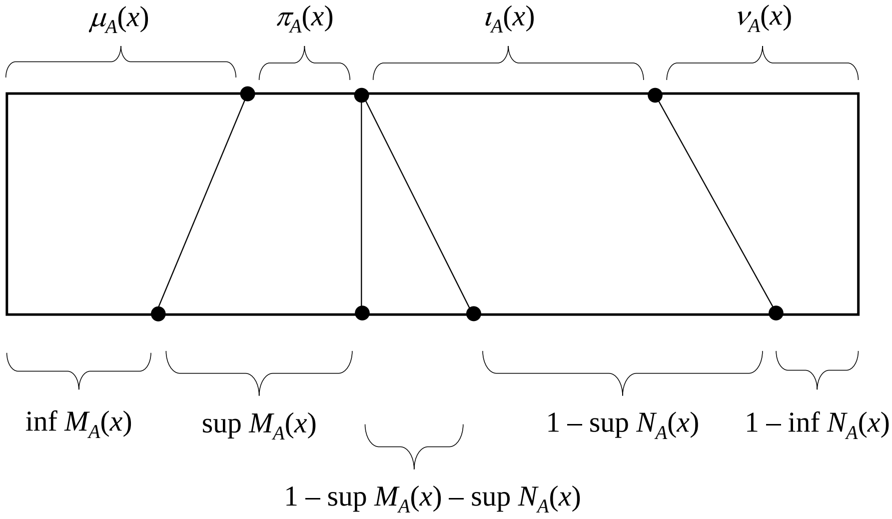

Figure 3. Geometric correspondence for the third case

Putting:

$$\begin{aligned}
\inf M_A(x) &\stackrel{\text{def}}{=} \mu_A^i(x),\\
\inf N_A(x) &\stackrel{\text{def}}{=} \nu_A^i(x),\\
\sup M_A(x) &\stackrel{\text{def}}{=} \mu_A^i(x) + \pi_A^i(x),\\
\sup N_A(x) &\stackrel{\text{def}}{=} \nu_A^i(x) + \iota_A^i(x).
\end{aligned}$$

Obviously,

$$\sup M_A(x) + \sup N_A(x) = 1.$$

Therefore, a fourth IVIFS-interpretation of the IIFS A^i presented visually in Figure 4 is the following:

$$A = \left\{ \left\langle x, \left[\mu_A^i(x), \mu_A^i(x) + \pi_A^i(x)\right], \left[\nu_A^i(x), \nu_A^i(x) + \iota_A^i(x)\right] \right\rangle \mid x \in E \right\}.$$

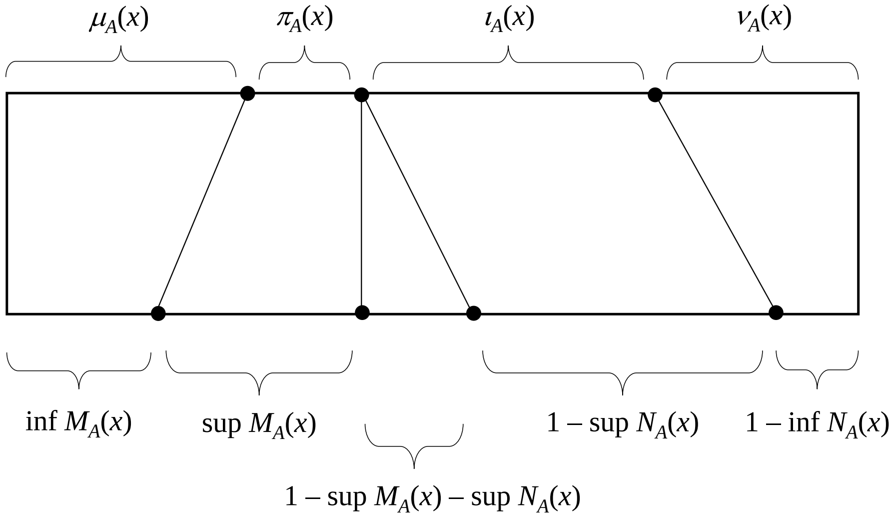

Figure 4. Geometric correspondence for the third case

The concept of **Picture Fuzzy Set** (PFS) was introduced by B. C. Cuong in 2013 (see [11]). A PFS A^* is an object of the following form:

$$A^* = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle | x \in E\}$$

where the functions $\mu_A : E \to [0,1]$ and $\eta_A(x) : E \to [0,1]$ and $\nu_A : E \to [0,1]$ are called *"degree of positive membership"*, *"degree of neutral membership"* and *"degree of negative membership"* of x in A^*, respectively, and for all $x \in E$,

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1.$$

The degree of *"refusal of membership"* is therefore defined as:

$$r_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x).$$

It is easy to see that beyond the semantics, the structure of PFS is identical to IIFS, hence, the constructions proposed above also work in this case by replacing ι with η in all the relations above. Therefore, PFS are completely representable by IVIFS.

PFS may also be represented by a couple of IFSs (using the fact that any IVIFS can be represented by a couple of IFSs and applying the above correspondence).

In [28], the concept of a **Neutrosophic Fuzzy Set** (NFS) is introduced, as follows

$$A^n = \{\langle x, \mu_A^n(x), \nu_A^n(x), \pi_A^n(x)\rangle \mid x \in E\},$$

where $\mu_A^n(x), \nu_A^n(x), \pi_A^n(x) \in [0,1]$ and have the same sense as IFS.

Let

$$\sup_{y\in E} \mu_A^n(y) + \sup_{y\in E} \nu_A^n(y) + \sup_{y\in E} \pi_A^n(y) \neq 0.$$

Then we define:

$$\mu_A^i(x) = \frac{\mu_A^n(x)}{\sup_{y\in E} \mu_A^n(y) + \sup_{y\in E} \nu_A^n(y) + \sup_{y\in E} \pi_A^n(y)},$$

$$\nu_A^i(x) = \frac{\nu_A^n(x)}{\sup_{y\in E} \mu_A^n(y) + \sup_{y\in E} \nu_A^n(y) + \sup_{y\in E} \pi_A^n(y)},$$

$$\pi_A^i(x) = \frac{\pi_A^n(x)}{\sup_{y\in E} \mu_A^n(y) + \sup_{y\in E} \nu_A^n(y) + \sup_{y\in E} \pi_A^n(y)},$$

$$\iota_A^i(x) = 1 - \mu_A^i(x) - \nu_A^i(x) - \pi_A^i(x).$$

If

$$\sup_{y\in E} \mu_A^n(y) + \sup_{y\in E} \nu_A^n(y) + \sup_{y\in E} \pi_A^n(y) = 0,$$

then we define:

$$\mu_A^i(x) = 0, \;\; \nu_A^i(x) = 0, \;\; \pi_A^i(x) = 0, \;\; \iota_A^i(x) = 1.$$

Therefore, the NFS can be represented by an IIFS. Using the above four IVIFS-interpretations of IIFS, we can construct four IVIFS-interpretations of a NFS.

If for the element $x \in E : \mu_A^n(x) + \nu_A^n(x) + \pi_A^n(x) \neq 0$, then we define:

$$\mu_A(x) = \frac{\mu_A^n(x)}{\mu_A^n(x) + \nu_A^n(x) + \pi_A^n(x)},$$

$$\nu_A(x) = \frac{\nu_A^n(x)}{\mu_A^n(x) + \nu_A^n(x) + \pi_A^n(x)},$$

$$\pi_A(x) = \frac{\pi_A^n(x)}{\mu_A^n(x) + \nu_A^n(x) + \pi_A^n(x)}.$$

If $\mu_A^n(x) + \nu_A^n(x) + \pi_A^n(x) = 0$, then we define:

$$\mu_A(x) = 0, \;\; \nu_A(x) = 0, \;\; \pi_A(x) = 1.$$

Therefore, $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$, i.e.,

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$$

is an IFS.

Therefore, the NFS can be represented by an IFS.

The concept of a **Bipolar IFS** (BIFS) is defined in [21] as follows (we cite the definition exactly): *Let X be a non empty set. A BIFS is*

$$B = \{(x, \mu^P(x), \mu^N(x), \nu^P(x), \nu^N(x)) \mid x \in X\},$$

where $\mu^P : X \to [0,1], \mu^N : X \to [-1,0], \nu^P : X \to [0,1], \nu^N : X \to [-1,0]$ are the mappings such that $0 \leq \mu^P(x) + \nu^P(x) \leq 1, -1 \leq \mu^N(x) + \nu^N(x) \leq 0$. We use the positive membership degree ($\mu^P(x)$) to denote the satisfaction degree of an element x to the property corresponding to a bipolar intuitionistic fuzzy set B and the negative membership degree ($\mu^N(x)$) to denote the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar intuitionistic fuzzy set. Similarly, we use the positive non-membership degree ($\nu^P(x)$) to denote the satisfaction degree of an element x to the property corresponding to a bipolar intuitionistic fuzzy set and the negative non-membership degree ($\nu^N(x)$) to denote the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar intuitionistic fuzzy set.

Repeating the procedure from [3] we can construct a bijective function.

Let

$$L^* = \{\langle a, b\rangle | a, b \in [0,1] \;\&\; a + b \leq 1\}$$

(see [5, 8]). Let us define function $F : [0,1] \times [-1,0] \to L^*$ by

$$F(\mu^P(x), \mu^N(x)) = \left\langle \frac{\mu^P(x)}{1 - \mu^N(x)}, \frac{-\mu^N(x)}{1 + \mu^P(x)} \right\rangle.$$

Obviously, the denominators are greater than or equal to 1 and F is a continuous function, for which the equalities

$$\begin{aligned} F(0,-1) &= \langle 0, 1\rangle, \\ F(0,0) &= \langle 0, 0\rangle, \\ F(1,0) &= \langle 1, 0\rangle \\ F(1,-1) &= \langle \tfrac{1}{2}, \tfrac{1}{2}\rangle \end{aligned}$$

hold. For $\mu^P(x) \in [0,1]$ and $\mu^N(x) \in [-1,0]$ we directly see that

$$0 \le \frac{\mu^P(x)}{1-\mu^N(x)} \le 1, \quad 0 \le \frac{-\mu^N(x)}{1+\mu^P(x)} \le 1.$$

Let

$$Y \equiv \frac{\mu^P(x)}{1-\mu^N(x)} + \frac{-\mu^N(x)}{1+\mu^P(x)}.$$

Therefore $Y \ge 0$ and

$$Y = \frac{\mu^P(x)+\mu^P(x)^2-\mu^N(x)+\mu^N(x)^2}{1-\mu^N(x)+\mu^P(x)-\mu^P(x)\mu^N(x)}.$$

Let

$$\begin{aligned}
Z &\equiv 1-\mu^N(x)+\mu^P(x)-\mu^P(x)\mu^N(x)-(\mu^P(x)+\mu^P(x)^2-\mu^N(x)+\mu^N(x)^2)\\
&= 1-\mu^P(x)\mu^N(x)-\mu^P(x)^2-\mu^N(x)^2\\
&= 1+\mu^P(x)(-\mu^N(x))-\mu^P(x)^2-(-\mu^N(x))^2\\
&\ge 1+\mu^P(x)(-\mu^N(x))-\mu^P(x)-\mu^N(x)\\
&= (1-\mu^P(x))(1-\mu^N(x)) \ge 0.
\end{aligned}$$

Therefore, $Z \ge 0$ and hence $Y \le 1$, i.e., $\left\langle \frac{\mu^P(x)}{1-\mu^N(x)}, \frac{-\mu^N(x)}{1+\mu^P(x)} \right\rangle$ is an intuitionistic fuzzy pair (see [6]) and F is a surjective function.

Let $a, c \in [0,1], b, d \in [-1,0]$ and $F(a,b) = F(c,d)$. Then

$$\begin{cases} \dfrac{a}{1-b} = \dfrac{c}{1-d} \\ \dfrac{b}{1+a} = \dfrac{d}{1+c} \end{cases}$$

Therefore,

$$\begin{aligned}
a &= \frac{c(1-b)}{1-d}\\
&= \frac{c(1-\frac{d(1+a)}{1+c})}{1-d}\\
&= \frac{c(1+c-d-ad)}{(1+c)(1-d)},
\end{aligned}$$

i.e.,

$$a(1+c-d-cd)+acd = a(1+c-d) = c(1+c-d)$$

and, because $1+c-d > 0$, it follows that $a = c$. By analogy, we can see that $b = d$, i.e., the function F is injective.

Hence, F is a bijection.

In the same way, we can proceed with the third and fourth components of BIFS B.

Let

$$U \equiv \frac{\mu^P(x)}{1-\mu^N(x)} + \frac{\nu^P(x)}{1-\nu^N(x)}.$$

Therefore $U \geq 0$ and

$$U = \frac{\mu^P(x) - \mu^P(x)\nu^N(x) + \nu^P(x) - \mu^N(x)\nu^P(x)}{1 - \mu^N(x) - \nu^N(x) + \mu^N(x)\nu^N(x)}.$$

Let

$$V \equiv 1 - \mu^N(x) - \nu^N(x) + \mu^N(x)\nu^N(x) - \mu^P(x) + \mu^P(x)\nu^N(x) - \nu^P(x) + \mu^N(x)\nu^P(x).$$

Now, we obtain

$$\begin{aligned} V &= 1 - \mu^P(x) - \nu^P(x) - \nu^N(x)(1 - \mu^P(x)) - \mu^N(x)(1 - \nu^P(x) - \nu^N(x)) \\ &\geq 1 - \mu^P(x) - \nu^P(x) \geq 0. \end{aligned}$$

Therefore, $V \geq 0$ and hence $U \leq 1$. Hence, for the universe X we can construct the set

$$A = \left\{ \left\langle x, \left[\frac{\mu^P(x)}{1 - \mu^N(x)}, \frac{-\mu^N(x)}{1 + \mu^P(x)}\right], \left[\frac{\nu^P(x)}{1 - \nu^N(x)}, \frac{-\nu^N(x)}{1 + \nu^P(x)}\right] \right\rangle | x \in X \right\}$$

and, as we can see, it is an IVIFS.

Therefore to an arbitrary BIFS we can assign an IVIFS.

We will mention that other authors, based on [21] define this set by

$$G = \{g, (\mu^+(g), \mu^-(g)), (\sigma^+(g), \sigma^-(g)) | g \in P\},$$

which, obviously is mathematically incorrect (and due to this reason we would not cite their paper).

3 Instead of a conclusion: A piece of advice to those willing to define new modifications and extensions of the IFS

In mathematics, progress often comes from extending old concepts into new contexts. Creating new modifications and extensions of IFS is a very important activity that expands the boundaries of IFS theory. But in order to propose a new modification or extension of an existing object (in particular an IFS), at least four steps and requirements must be observed, so that the extension be regarded as legitimate and viable, expanding in meaningful ways the boundaries of the mathematical theory.

These four central requirements can be identified as: Correctness, Conservativity, Fruitfulness, and Naturalness.

1. **Correctness.** The most basic requirement is that the new concept be mathematically correct. This means the definition must be well-posed, logically consistent, and precise. Ambiguities or contradictions undermine the very possibility of doing rigorous mathematics with the new idea. Correctness ensures the new concept is more than an informal intuition—it is something that can sustain rigorous reasoning, theorems, and proofs.

2. **Conservativity.** A good extension should reduce to the old concept in the cases where the old concept already applies. This property is sometimes called conservativity or the reduction principle, and guarantees continuity of meaning across old and new domains, that is new terms must align with prior ones in boundary cases. This ensures that the old theory is not discarded or contradicted but it is conserved and embedded within a richer one.

3. **Fruitfulness.** An extension is worthwhile only if it brings something new—new results, new insights, or new applications, that has arisen from clear motivations. Otherwise, it risks being an unnecessary duplication. Fruitfulness shows that the extensions are not cosmetic: they solve real problems that the original concept could not address, and ensures that the new concept is not a superficial change but a meaningful enrichment of the mathematical landscape with the simplest and most canonical formalization.

4. **Naturalness.** Among possible extensions, mathematicians often prefer the one that appears most natural and elegant. This usually means it is motivated by a clear principle, preserves desirable properties, and integrates seamlessly into the existing theory. Naturalness often gives a sense of inevitability: the new concept feels like the obvious continuation rather than an artificial construction. In many cases, multiple extensions are possible, but only one is regarded as "canonical" because it best balances simplicity, generality, and harmony with established mathematics. Naturalness is therefore partly aesthetic, but also pragmatic: it guides mathematicians toward definitions that will be most useful and enduring.

Mathematics advances not just by invention but by carefully judged extensions of existing ideas. The case of set theory—classical, fuzzy, and intuitionistic fuzzy—shows how the four listed above requirements have guided such progress. Unfortunately, many modifications and extensions of the IFSs do not adhere to these principles, partially or even fully. In the future, in addition to the concepts discussed in the paper, that claim to extend IFSs, some other ones will be described, as well as their possible representation through IFSs or their (proven actual) extensions.

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References

[1] Alkouri, A. M., & Salleh, A. R. (2012). Complex intuitionistic fuzzy sets. *Proceedings of the 2nd International Conference on Fundamental and Applied Sciences 2012* - Kuala Lumpur, AIP Conference Vol. 1482, 464–470.

[2] Annamalai, C. (2014). Intuitionistic fuzzy sets: New approach and applications. *International Journal of Research in Computer and Communication Technology*, 3(3), 283–285.

[3] Atanassov, K. (1987). Generalized index matrices. *Comptes Rendus de l'Academie Bulgare des Sciences*, 40(11), 15–18.

[4] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*. Springer, Berlin.

[5] Atanassov, K. (1989). More on intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 33(1), 37–45.

[6] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013). On intuitionistic fuzzy pairs. *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[7] Atanassov, K., & Gargov, G. (1989). Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 31(3), 343–349.

[8] Deschrijver G., & Kerre, E. (2003). On the composition of intuitionistic fuzzy relations. *Fuzzy Sets and Systems*, 136 (3), 333–361.

[9] Atanassov, K., & Vassilev, P. (2019). Intuitionistic fuzzy sets and other fuzzy sets extensions representable by them. *Journal of Intelligent & Fuzzy Systems*. 38(1), 525–530.

[10] Bustince, H., & Burillo, P. (1996). Vague sets are intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 79(3), 403–405.

[11] Cuong, B. C., & Kreinovich, V. (2013). *Picture fuzzy sets-a new concept for computational intelligence problems*. *Proceedings of the Third World Congress on Information and Communication Technologies WICT'2013*, Hanoi, Vietnam, 15–18 December 2013, 1–6.

[12] Dempster, A. P. (1967). Upper and lower probabilities induced by a multivalued mapping. *The Annals of Mathematical Statistics*, 38(2), 325–339.

[13] Deng, J. L. (1982). Control problems of grey systems. *Systems & Control Letters*, 1(5), 288–294.

[14] Despi, I., Opris, D., & Yalcin, E. (2013). Generalised Atanassov intuitionistic fuzzy sets. *Proceedings of the Fifth International Conference on Information, Process, and Knowledge Management eKNOW*, 24 February - 1 March 2013, Nice, France, 51–56.

[15] Gau, W.-L., & Buehrer, D. J. (1993). Vague sets. *IEEE Transactions on Systems, Man, and Cybernetics*, 23, 610–614.

[16] Gerstenkorn, T., & Manko, J. (1995). Bifuzzy probabilistic sets. *Fuzzy Sets and Systems*, 71, 207–214.

[17] Hinde, C., & Patching, R. (2008). Inconsistent intuitionistic fuzzy sets. *Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics*, 1, 133–153.

[18] Hinde, C., Patching, R., & McCoy, S. (2008). Inconsistent intuitionistic fuzzy sets and mass assignment. *Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics*, 1, 155–174.

[19] Indira, S., & Rajeswari, R. R. (2014). A study on star intuitionistic sets. *International Journal of Mathematics and Statistics Invention*, 2(4), 51–63.

[20] Li, Y., & Deng, Y. Deng (2019). Intuitionistic evidence sets. *IEEE Access*, 7, 106417–106426.

[21] Ezhilmaran, D., & Sankar, K. (2015). Morphism of bipolar intuitionistic fuzzy graphs. *Journal of Discrete Mathematical Sciences & Cryptography*, 18(5), 605–621.

[22] Mondal, T. K., & Samanta, S. (2002). Generalized intuitionistic fuzzy sets. *Journal of Fuzzy Mathematics*, 10(4), 839–862.

[23] Narinyani, A. (1980). *Subdefinite sets – a new type of data for knowledge representation*. Preprint 232, Computer Center of the USSR Academy of sciences, Novosibirsk, 1980 (in Russian).

[24] Narinyani, A. (1986). Subdefiniteness in knowledge representation and processing systems. *Transactions of USSR Acad. of Sciences, Technical Cybernetics*, 5, 3–28.

[25] Rizvi, S., Naqvi, H. J., & Nadeem, D. (2002). Rough intuitionistic fuzzy set. *Proceedings of the 6th Joint Conference on Information Sciences (JCIS)*, Durham, NC, 101–104.

[26] Shafer, G. (1976). *A Mathematical Theory of Evidence, Vol. 1*. Princeton University Press, Princeton, NJ, USA.

[27] Shouyu, C., Yu, G. (2006). Variable fuzzy sets and its application in comprehensive risk evaluation for flood-control engineering system. *Fuzzy Optimization and Decision Making*, 5, 153–162.

[28] Smarandache, F. (1998). *Neutrosophy. Neutrosophic Probability, Set, and Logic*. American Research Press, Rehoboth, USA.

[29] Ulazeez, A., Alkouri, M., & Salleh A. R. (2013). Some operations on complex Atanassov's intuitionistic fuzzy sets. *AIP Conference Proceedings*, Vol. 1571, 987–993.

[30] Vassilev, P., & Atanassov, K. (2020). Generalised Atanassov intuitionistic sets are actually intuitionistic fuzzy sets. In: *Castillo, O., Melin, P., & Kacprzyk, J. (eds.). Intuitionistic and Type-2 Fuzzy Logic Enhancements in Neural and Optimization Algorithms: Theory and Applications*. Studies in Computational Intelligence, Vol. 862, pp. 107–117, Springer, Cham.

[31] Yang, Y., & Chiclana, F. (2009). Intuitionistic fuzzy sets: Spherical representation and distances. *International Journal of Intelligent Systems*, 24, 399–420.