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On the intuitionistic fuzzy form of the classical implication $(A \to B) \vee (B \to A)$

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Abstract: The intuitionistic fuzzy implications that satisfy the well-known logical tautology $(A \to B) \vee (B \to A)$ are described. Some of the intuitionistic fuzzy implications satisfy the exprssion as intuitionistic fuzzy tautology and a part of them – as a tautology in intuitionistic fuzzy sense.

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1 Introduction

One of the well-known logical tautologies is

$$(A \to B) \vee (B \to A). \qquad (*)$$

Here, we discuss its validity for the different cases of intuitionistic fuzzy implications. In [1], 138 of them were given, but after publishing of this book, their number had increased and in [2], 152 implications over intuitionistic fuzzy pairs were given. Three of them were introduced by P. Dworniczak in [3, 4, 5].

Below, we determine which of these 152 intuitionistic fuzzy implications satisfy (*) as a tautologies, which – as intuitionistic fuzzy tautologies and which – do not satisfy (*).

First, we mention that the definition and use of the concept of intuitionistic fuzzy pair is very suitable, because in all cases, as the present one, it can be used as a makeshift of both the elements of an intuitionistic fuzzy sets, and the intuitionistic fuzzy propositions. In [2] it is defined as follows: *"The Intuitionistic Fuzzy Pair (IFP) is an object with the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process and which conponents (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.*

2 Preliminaries

Below, we shall assume that for the two Intuitionistic Fuzzy Pairs (IFPs) A and B the equalities: $V(A) = \langle a, b\rangle, V(B) = \langle c, d\rangle$, $(a, b, c, d, a + b, c + d \in [0, 1])$ hold.

For the needs of the discussion below, following the definition from [1], we shall define the notion of Intuitionistic Fuzzy Tautology (IFT) by:

$$A \text{ is an IFT, if and only if for } V(a) = \langle a, b\rangle \text{ holds: } a \geq b,$$

while A will be a tautology iff $a = 1$ and $b = 0$. As in the case of ordinary logics, A is a tautology, if $V(x) = \langle 1, 0\rangle$.

For two IFPs A and B, the operation "disjunction" ($\vee$) is defined (see [2]) by:

$$V(A \vee B) = \langle \max(a, c), \min(b, d)\rangle.$$

For two IFPs A and B, the relation "equality" is defined (see [2]) by:

$$V(A) = V(B) \text{ if and only if } a = c \text{ and } b = d.$$

As we mentioned above, the list with all 152 implications over IFPs is given in [2]. Here we introduce only three of them that we will use below as illustration:

$$A \to_1 B = \langle \max(b, \min(a, c)), \min(a, d)\rangle,$$

$$A \to_2 B = \langle \overline{\text{sg}}(a - c), d.\text{sg}(a - c)\rangle$$

and

$$A \to_{12} B = \langle \max(b, c), 1 - \max(b, c)\rangle.$$

3 Main results

Here we formulate two theorems. For each of them, two checks (for the first implications in each list) are given: one positive – a case when the respective implication satisfies (*) and one negative – when the implication does not satisfy (*). All other checks are similar.

Theorem 1. Two IFPs A and B satisfy (*) as IFTs for the intuitionistic fuzzy implications $\to_1$, $\to_2$, $\to_3$, $\to_4$, $\to_5$, $\to_6$, $\to_8$, $\to_9$, $\to_{11}$, $\to_{13}$, $\to_{14}$, $\to_{17}$, $\to_{18}$, $\to_{20}$, $\to_{21}$, $\to_{22}$, $\to_{23}$, $\to_{24}$, $\to_{27}$, $\to_{28}$, $\to_{29}$, $\to_{30}$, $\to_{31}$, $\to_{32}$, $\to_{33}$, $\to_{34}$, $\to_{35}$, $\to_{36}$, $\to_{37}$, $\to_{38}$, $\to_{40}$, $\to_{42}$, $\to_{44}$, $\to_{45}$, $\to_{61}$, $\to_{62}$, $\to_{63}$, $\to_{64}$, $\to_{65}$, $\to_{66}$, $\to_{68}$, $\to_{69}$, $\to_{71}$, $\to_{72}$, $\to_{74}$, $\to_{75}$, $\to_{76}$, $\to_{77}$, $\to_{79}$, $\to_{80}$, $\to_{81}$, $\to_{82}$, $\to_{83}$, $\to_{84}$, $\to_{85}$, $\to_{88}$, $\to_{89}$, $\to_{90}$, $\to_{100}$, $\to_{101}$, $\to_{102}$, $\to_{103}$, $\to_{104}$, $\to_{105}$, $\to_{109}$, $\to_{110}$, $\to_{111}$, $\to_{112}$, $\to_{113}$, $\to_{114}$, $\to_{115}$, $\to_{116}$, $\to_{117}$, $\to_{118}$, $\to_{124}$, $\to_{125}$, $\to_{126}$, $\to_{127}$, $\to_{128}$, $\to_{129}$, $\to_{130}$, $\to_{131}$, $\to_{132}$, $\to_{133}$, $\to_{139}$, $\to_{140}$, $\to_{141}$, $\to_{142}$, $\to_{143}$, $\to_{144}$, $\to_{145}$, $\to_{146}$, $\to_{147}$, $\to_{148}$, $\to_{149}$, $\to_{150}$, $\to_{151}$, $\to_{152}$, $\to_{153}$.

Proof. Let the IFPs A and B be given. Then for $\to_1$ we obtain:

$$(A \to_1 B) \vee (B \to_1 A)$$

$$= (\langle a,b\rangle \to_1 \langle c,d\rangle) \vee (\langle c,d\rangle \to_1 \langle a,b\rangle)$$

$$= \langle \max(b, \min(a,c)), \min(a,d)\rangle \vee \langle \max(d, \min(c,a)), \min(c,b)\rangle$$

$$= \langle \max(b,d,\min(a,c)), \min(a,b,c,d)\rangle.$$

Let

$$X \equiv \max(b,d,\min(a,c)) - \min(a,b,c,d).$$

Then

$$X \geq \min(a,c) - \min(a,b,c,d) \geq 0.$$

Therefore, $(A \to_1 B) \vee (B \to_1 A)$ is an IFT.

Analogously, for $\to_{12}$ we obtain:

$$(A \to_{12} B) \vee (B \to_{12} A)$$

$$\langle \max(b,c), 1 - \max(b,c)\rangle \vee \langle \max(d,a), 1 - \max(d,a)\rangle$$

$$\langle \max(a,b,c,d), \min(1 - \max(b,c), 1 - \max(d,a))\rangle$$

$$\langle \max(a,b,c,d), 1 - \max(\max(b,c), \max(d,a))\rangle$$

$$\langle \max(a,b,c,d), 1 - \max(a,b,c,d)\rangle.$$

Obviously, if $a = b = c = d = 0$,

$$\max(a,b,c,d) = 0 < 1 = 1 - \max(a,b,c,d),$$

i.e., $(A \to_{12} B) \vee (B \to_{12} A)$ is not an IFT. □

Theorem 2. Two IFPs A and B satisfy (*) as tautologies for the intuitionistic fuzzy implications $\to_2$, $\to_3$, $\to_8$, $\to_{11}$, $\to_{20}$, $\to_{23}$, $\to_{31}$, $\to_{32}$, $\to_{34}$, $\to_{37}$, $\to_{40}$, $\to_{42}$, $\to_{62}$, $\to_{63}$, $\to_{65}$, $\to_{68}$, $\to_{74}$, $\to_{77}$, $\to_{83}$, $\to_{88}$.

Proof. Let the IFPs A and B be given. Then for $\to_1$ we obtain (as above):

$$(A \to_1 B) \vee (B \to_1 A) = \langle \max(b,d,\min(a,c)), \min(a,b,c,d)\rangle.$$

Obviously, if $a = b = c = d = 0$,

$$(A \to_1 B) \vee (B \to_1 A) = \langle 0,0\rangle \neq \langle 1,0\rangle.$$

Analogously, for $\to_2$ we obtain:

$$(A \to_2 B) \vee (B \to_2 A)$$

$$\langle \overline{\text{sg}}(a-c), d.\text{sg}(a-c)\rangle \vee \langle \overline{\text{sg}}(c-a), b.\text{sg}(c-a)\rangle$$

$$\langle \max(\overline{\text{sg}}(a-c), \overline{\text{sg}}(c-a)), \min(d.\text{sg}(a-c), b.\text{sg}(c-a))\rangle.$$

If $a > c$, then $\overline{\text{sg}}(a-c) = 0, \overline{\text{sg}}(c-a) = 1, \text{sg}(a-c) = 1, \text{sg}(c-a) = 0$ and hence

$$(A \to_2 B) \vee (B \to_2 A) = \langle 1, 0\rangle.$$

If $a = c$, then $\overline{\text{sg}}(a-c) = 1, \overline{\text{sg}}(c-a) = 1, \text{sg}(a-c) = 0, \text{sg}(c-a) = 0$ and hence

$$(A \to_2 B) \vee (B \to_2 A) = \langle 1, 0\rangle.$$

Finally, if $a < c$, then $\overline{\text{sg}}(a-c) = 1, \overline{\text{sg}}(c-a) = 0, \text{sg}(a-c) = 0, \text{sg}(c-a) = 1$ and hence

$$(A \to_2 B) \vee (B \to_2 A) = \langle 1, 0\rangle.$$

Therefore, $(A \to_1 B) \vee (B \to_1 A)$ is a tautology. □

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