Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy quasi-interior ideals of semigroups

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Abstract: In this study, it is purposed to introduced the concept of quasi-interior ideal on intuitionistic fuzzy semigroups. The concept introduced is supported with examples and its basic algebraic properties are examined.
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1 Introduction

The notion of fuzzy set was introduced as an extension of crisp sets by Zadeh [16]. In the following years, many generalizations of fuzzy sets were defined. Atanassov introduced the intuitionistic fuzzy set concept in 1983 [1] as an extension of fuzzy sets by enlarging the truth value set to the lattice $[0, 1] \times [0, 1]$ with $\mu_A(x) + \nu_A(x) \leq 1$ condition. Intuitionistic fuzzy sets have membership degree and non-membership degree for an element in a given set. This theory is widely used as it can obtain more precise results in decision making problems of many areas such as health, finance, geographic marking, etc.

Many algebraic concepts were extended into fuzzy sets and intuitionistic fuzzy sets by several researchers [5–7, 9–12, 14, 15]. This workspace is still up-to-date. The concept of quasi-interior ideal on semigroups was introduced by M. Murali Krishna Rao [13]. In this study, intuitionistic fuzzy quasi-interior ideal on semigroups is studied.

Definition 1 ([1]). *An intuitionistic fuzzy set (shortly, IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A", $\nu_A(x)$, $(\nu_A : X \to [0,1])$ is called the "degree of non-membership of x in A", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Definition 2 ([1]). *An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 3 ([2]). *Let X be universal and $A \in IFS(X)$, then*

1. $\square(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$.
2. $\lozenge(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$.

Definition 4 ([1]). *Let $A \in IFS(X)$ and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Then the set A^c is called the complement of A, where*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

The intersection and the union of two IFSs A and B on X are defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\},$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}.$$

Definition 5 ([3]). *Let X be a semigroup and A, B be intuitionistic fuzzy subsets of X. Then the extension principle would be as following;*

$$A \circ B = \begin{cases} \sup\limits_{z=xy} \{A(x) \wedge B(y)\}, & if z = xy \\ (0,1), & \textit{otherwise.} \end{cases}$$

for $x, y, z \in X$.

Definition 6. *Let M be a non-empty subset of S. The characteristic function of M is an intuitionistic fuzzy subset of S. It is denoted by χ_M and defined as*

$$\chi_M(x) = \begin{cases} (1,0), & \text{if } x \in M, \\ (0,1), & \text{if } x \notin M. \end{cases}$$

A *semigroup* is an algebraic structure consisting of a non-empty set S together with an associative binary operation [4]. A *subsemigroup* of S is a nonempty subset M of S such that $M^2 \subseteq M$. A *subsemigroup* M of S is called an *interior ideal* of S if $SMS \subseteq M$ and called a *quasi-ideal* if $MS \cap SM \subseteq M$.

An element $x \in S$ is called *regular* in S if $x \in xSx$, where $xSx = \{xax : a \in S\}$. A semigroup S is called *regular* if every element is regular.

Definition 7 ([13])**.** *A non-empty subset M of a semigroup S is said to be left (right) quasi-interior ideal of S, if M is a subsemigroup of S and $SMSM \subseteq M$ $(MSMS \subseteq M)$.*

Definition 8 ([13])**.** *A non-empty subset M of a semigroup S is said to be a quasi-interior ideal of S, if M is a sub semigroup of S and M is a left quasi-interior ideal and a right quasi-interior ideal of S.*

Remark 1. *A quasi-interior ideal of a semigroup S need not be an interior ideal of a semigroup S.*

The generalization of semigroup concept to intuitionistic fuzzy sets has a wide range of study. Some fundamental definitions are following:

Definition 9 ([8])**.** *Let S be a semigroup and $A \in IFS(S)$. A is called an intuitionistic fuzzy subsemigroup of S, if $A(xy) \geq \min\{A(x), A(y)\}$, for all $x, y \in S$.*

Definition 10 ([7])**.** *Let S be a semigroup and $A \in IFS(S)$. A is called an intuitionistic fuzzy left (respectively, right) ideal of S, if $A(xy) \geq A(y)$ (resp. $A(xy) \geq A(x)$), for all $x, y \in S$.*

Definition 11 ([13])**.** *Let S be a semigroup. An intuitionistic fuzzy subsemigroup A of S is called an intuitionistic fuzzy interior ideal of S if $A(xay) \geq A(a)$, for all $x, y, a \in S$.*

Definition 12 ([9])**.** *An intuitionistic fuzzy set A in a semigroup S is called an intuitionistic fuzzy quasi-ideal of S, if*

$$\mu_{A\circ\chi_S} \wedge \mu_{\chi_S\circ A} \leq \mu_A \text{ and } \nu_{A\circ\chi_S} \vee \nu_{\chi_S\circ A} \geq \nu_A.$$

2 Main results

In this section, we intorduce the concept of an intuitionistic fuzzy quasi-interior ideal on a semigroup. The main theorems are proved and some properties are studied.

Definition 13. *Let S is a semigroup and $A \in IFS(S)$ is an intuitionistic fuzzy sub semigroup. A is called an intuitionistic fuzzy left (respectively, right) quasi-interior ideal of S, if*

$$\chi_S \circ A \circ \chi_S \circ A \subseteq A \, (\textit{respectively, } A \circ \chi_S \circ A \circ \chi_S \subseteq A).$$

Example 1. *Let $S = \{a, b, c, d, e\}$ be a semigroup with the following operation:*

$\bullet$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

Then $A = \{(a, 0.7, 0.2), (b, 0.5, 0.4), (c, 0.3, 0.6), (d, 0.5, 0.4), (e, 0.3, 0.6)\}$ is an intuitionistic fuzzy quasi-interior ideal of S. But, since $A(de) \not\supseteq A(d)$, it is not an intuitionistic fuzzy ideal of S and it is not intuitionistic fuzzy interior ideal either.

Theorem 1. *Let S be a semigroup and $A \in IFS(S)$. If A is an intuitionistic fuzzy quasi-interior ideal, then $\Box(A)$ and $\Diamond(A)$ are intuitionistic fuzzy quasi-interior ideals.*

Proof. *(i)* Let $x, y \in S$,

$$\begin{aligned} \mu_{\Box(A)}(xy) &= \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \\ &= \min\left\{\mu_{\Box(A)}(x), \mu_{\Box(A)}(y)\right\} \end{aligned}$$

and

$$\begin{aligned} \nu_{\Box(A)}(xy) &= 1 - \mu_A(xy) \leq 1 - \min\{\mu_A(x), \mu_A(y)\} \\ &= \max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\ &= \max\left\{\nu_{\Box(A)}(x), \nu_{\Box(A)}(y)\right\}. \end{aligned}$$

So, $\Box(A)$ is an intuitionistic fuzzy subsemigroup of S.
(ii) Let $x, y, z \in S$,

$$\mu_{\chi_S \circ \Box(A) \circ \chi_S \circ \Box(A)}(z) = \mu_{\chi_S \circ A \circ \chi_S \circ A}(z) \leq \mu_A(z)$$

and for $a, b, e, f \in S$,

$$\begin{aligned} \nu_{\chi_S \circ \Box(A) \circ \chi_S \circ \Box(A)}(z) &= \inf_{z=xy}\left\{\max\left\{\nu_{\chi_S \circ \Box(A)}(x), \nu_{\chi_S \circ \Box(A)}(y)\right\}\right\} \\ &= \inf_{z=xy}\left\{\max\left\{\begin{array}{l} \inf\limits_{x=ab} \max\left\{\nu_{\chi_S}(a), \nu_{\Box(A)}(b)\right\}, \\ \inf\limits_{y=ef} \max\left\{\nu_{\chi_S}(e), \nu_{\Box(A)}(f)\right\} \end{array}\right\}\right\} \\ &= \inf_{z=xy}\left\{\max\left\{\begin{array}{l} \inf\limits_{x=ab} \max\left\{1 - \mu_{\chi_S}(a), 1 - \mu_A(b)\right\}, \\ \inf\limits_{y=ef} \max\left\{1 - \mu_{\chi_S}(e), 1 - \mu_A(f)\right\} \end{array}\right\}\right\} \\ &= 1 - \left(\sup_{z=xy}\left\{\min\left\{\begin{array}{l} \sup\limits_{x=ab} \min\left\{\mu_{\chi_S}(a), \mu_A(b)\right\}, \\ \sup\limits_{y=ef} \min\left\{\mu_{\chi_S}(e), \mu_A(f)\right\} \end{array}\right\}\right\}\right) \\ &\geq 1 - \mu_A(z) = \nu_{\Box(A)}(z). \end{aligned}$$

We obtain that $\square(A)$ is an intuitionistic fuzzy left quasi-interior ideal on semigroup S.

On the other hand, similarly we can show that $\square(A) \circ \chi_S \circ \square(A) \circ \chi_S \subseteq \square(A)$. Also $\lozenge(A)$ can be proved as above. $\square$

Theorem 2. *Let S be a semigroup. If $\{A_i\}_{i\in\Lambda}$ is a family of intuitionistic fuzzy quasi-interior ideals of S, then $\bigcap_{i\in\Lambda} A_i$ is an intuitionistic fuzzy quasi-interior ideal on S.*

Proof. Let $x, y \in S$ and $B = \bigcap_{i\in\Lambda} A_i$,

$$\begin{aligned}
\mu_B(xy) &= \inf_{i\in\Lambda}\mu_{A_i}(xy) \geq \inf_{i\in\Lambda}\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\} \\
&= \min\left\{\inf_{i\in\Lambda}\mu_{A_i}(x), \inf_{i\in\Lambda}\mu_{A_i}(y)\right\} \\
&= \min\{\mu_B(x), \mu_B(y)\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\nu_B(xy) &= \sup_{i\in\Lambda}\nu_{A_i}(xy) \leq \sup_{i\in\Lambda}\{\max\{\nu_{A_i}(x), \nu_{A_i}(y)\}\} \\
&= \max\left\{\sup_{i\in\Lambda}\nu_{A_i}(x), \sup_{i\in\Lambda}\nu_{A_i}(y)\right\} \\
&= \max\{\nu_B(x), \nu_B(y)\}
\end{aligned}$$

Hence, $B = \bigcap_{i\in\Lambda} A_i$ is an intuitionistic fuzzy subsemigroup of S.

Now, let $x, y, z \in S$, then,

$$\begin{aligned}
\mu_{\chi_S\circ B\circ\chi_S\circ B}(z) &= \sup_{z=xy}\min\left\{\mu_{\chi_S\circ B}(x), \mu_{\chi_S\circ B}(y)\right\} \\
&= \sup_{z=xy}\min\left\{\begin{array}{l}\sup_{x=ab}\min\left(\mu_{\chi_S}(a), \inf_{i\in\Lambda}\mu_{A_i}(b)\right), \\ \sup_{y=ef}\min\left(\mu_{\chi_S}(e), \inf_{i\in\Lambda}\mu_{A_i}(f)\right)\end{array}\right\} \\
&= \sup_{z=xy}\min\left\{\begin{array}{l}\inf_{i\in\Lambda}\left(\sup_{x=ab}\min\left(\mu_{\chi_S}(a), \mu_{A_i}(b)\right)\right), \\ \inf_{i\in\Lambda}\left(\sup_{y=ef}\min\left(\mu_{\chi_S}(e), \mu_{A_i}(f)\right)\right)\end{array}\right\} \\
&= \inf_{i\in\Lambda}\left(\sup_{z=xy}\min\left\{\begin{array}{l}\sup_{x=ab}\min\left(\mu_{\chi_S}(a), \mu_{A_i}(b)\right), \\ \sup_{y=ef}\min\left(\mu_{\chi_S}(e), \mu_{A_i}(f)\right)\end{array}\right\}\right),
\end{aligned}$$

for $a, b, e, f \in S$. Since

$$\sup_{z=xy}\min\left\{\sup_{x=ab}\min\left(\mu_{\chi_S}(a), \mu_{A_i}(b)\right), \sup_{y=ef}\min\left(\mu_{\chi_S}(e), \mu_{A_i}(f)\right)\right\} \leq \mu_{A_i}(z), \forall i \in \Lambda,$$

then $\mu_{\chi_S\circ B\circ\chi_S\circ B}(z) \leq \inf_{i\in\Lambda}\mu_{A_i}(z)$.

Similarly, it can be shown that $\nu_{\chi_S\circ B\circ\chi_S\circ B}(z) \geq \sup_{i\in\Lambda}\nu_{A_i}(z)$.

Therefore, $\bigcap_{i\in\Lambda} A_i$ is an intuitionistic fuzzy left quasi-interior ideal on semigroup S. Also, in a similar way, it can be proven $\bigcap_{i\in\Lambda} A_i \circ \chi_S \circ \bigcap_{i\in\Lambda} A_i \circ \chi_S \subseteq \bigcap_{i\in\Lambda} A_i$. $\square$

Theorem 3. *Let S be a semigroup and I be a non-empty subset of S. I is a quasi-interior ideal of S if and only if χ_I is an intuitionistic fuzzy quasi-interior ideal on S.*

Proof. $\Rightarrow$) If $z \in SISI$ and $z = xayb$ is such that $x, y \in S, a, b \in I$, then $z \in I$ and $\chi_I(z) = (1, 0)$.

$$\begin{aligned}
\mu_{\chi_S \circ \chi_I \circ \chi_S \circ \chi_I}(z) &= \sup_{z=uv} \min \left\{ \mu_{\chi_S \circ \chi_I}(u), \mu_{\chi_S \circ \chi_I}(v) \right\} \\
&= \sup_{z=uv} \min \left\{ \sup_{u=xa} \left(\mu_{\chi_S}(x) \wedge \mu_{\chi_I}(a) \right), \sup_{v=yb} \left(\mu_{\chi_S}(y) \wedge \mu_{\chi_I}(b) \right) \right\} \\
&= \sup_{z=uv} \min \left\{ \sup_{u=xa} \mu_{\chi_I}(a), \sup_{v=yb} \mu_{\chi_I}(b) \right\} = 1
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\chi_S \circ \chi_I \circ \chi_S \circ \chi_I}(z) &= \inf_{z=uv} \max \left\{ \nu_{\chi_S \circ \chi_I}(u), \nu_{\chi_S \circ \chi_I}(v) \right\} \\
&= \inf_{z=uv} \max \left\{ \inf_{u=xa} \left(\nu_{\chi_S}(x) \vee \nu_{\chi_I}(a) \right), \inf_{v=yb} \left(\nu_{\chi_S}(y) \vee \nu_{\chi_I}(b) \right) \right\} \\
&= \inf_{z=uv} \max \left\{ \inf_{u=xa} \nu_{\chi_I}(a), \inf_{v=yb} \nu_{\chi_I}(b) \right\} = 0
\end{aligned}$$

$\Rightarrow \chi_S \circ \chi_I \circ \chi_S \circ \chi_I \subseteq \chi_I$. Similarly we can show that $\chi_I \circ \chi_S \circ \chi_I \circ \chi_S \subseteq \chi_I$.

$\Leftarrow$) Let $z \in SISI$ and $z = xayb$ is such that $x, y \in S, a, b \in I$.

$$\begin{aligned}
\chi_I(z) &\geq \chi_S \circ \chi_I \circ \chi_S \circ \chi_I(z) \\
&= \sup_{z=uv} \min \left\{ \sup_{u=xa} \left(\chi_S(x) \wedge \chi_I(a) \right), \sup_{v=yb} \left(\chi_S(y) \wedge \chi_I(b) \right) \right\} \\
&= (1, 0) \Rightarrow z \in I \Rightarrow SISI \subseteq I
\end{aligned}$$

and also it is clear that $ISIS \subseteq I$. □

Proposition 1. *Let S be a semigroup and A be an intuitionistic fuzzy ideal of S. Then A is an intuitionistic fuzzy quasi-interior ideal on S.*

Proof. For $x \in S$,

$$\begin{aligned}
A \circ \chi_S(x) &= \sup_{x=ab} \min \left\{ A(a), \chi_S(b) \right\} \\
&= \sup_{x=ab} A(a) \\
&\leq \sup_{x=ab} A(a\beta b) = A(x).
\end{aligned}$$

From this inequality we have:

$$\begin{aligned}
A \circ \chi_S \circ A \circ \chi_S(z) &= \sup_{z=xy} \min \left\{ A \circ \chi_S(x), A \circ \chi_S(y) \right\} \\
&\leq \sup_{z=xy} \min \left\{ \sup_{x=ab} A(ab), \sup_{y=cd} A(cd) \right\} \\
&= \sup_{z=xy} \min \left\{ A(x), A(y) \right\} \\
&= A(z).
\end{aligned}$$

Therefore, it is clear that $\chi_S \circ A \circ \chi_S \circ A \subseteq A$.
A is an intuitionistic fuzzy quasi-interior ideal on S. □

The converse of Proposition 1 is not true in general, however, the concepts of intuitionistic fuzzy ideals and intuitionistic fuzzy quasi-interior ideals on regular semigroup coincide.

Proposition 2. *Let S be a regular semigroup and A be an intuitionistic fuzzy quasi-interior ideal of S. Then A is an intuitionistic fuzzy ideal of S.*

Proof. Let $A \circ \chi_S \circ A \circ \chi_S \subseteq A$ and $x, y \in S$. Since S is regular there exists $a \in S$ such that $x = xax$.

$$\begin{aligned} A(xy) &= A(xaxy) \geq A \circ \chi_S \circ A \circ \chi_S (xy) \\ &= \sup_{xy=xaxy} \left(A \circ \chi_S (xa) \wedge A \circ \chi_S (xy)\right) \\ &= \sup_{xy=xaxy} \left(\sup_u \min\{A(x), \chi_S(a)\} \wedge \sup_v \min\{A(x), \chi_S(y)\}\right) \\ &= A(x), \end{aligned}$$

where $u = xa, v = xy$.
Similarly, we can show that $A(xy) \geq A(y)$. Thus A is an intuitionistic fuzzy ideal of S. □

3 Conclusion

In this study, the concept of intuitionistic fuzzy quasi-interior ideal is introduced and has been shown with the counterexample that this concept is different from quasi-ideal and interior ideal. Basic algebraic properties have been proven.

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