Notes on Intuitionistic Fuzzy Sets
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Zariski topology on the spectrum of intuitionistic fuzzy classical primary submodules

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Abstract: In this paper, we define and study the notion of intuitionistic fuzzy classical primary submodules over a unitary R-module M, where R is a commutative ring with unity. This is a generalisation of intuitionistic fuzzy primary ideals and intuitionistic fuzzy classical prime submodules. We further topologize the collection of all intuitionistic fuzzy submodules on an R-module M with a topology having the intuitionistic fuzzy primary Zariski topology on the intuitionistic fuzzy classical primary spectrum $IF_{cp}\mathrm{spec}(M)$ as a subspace topology and investigate the properties of this topological space.
Keywords: Zariski topology, Classical primary submodule, Intuitionistic fuzzy classical primary submodule, Intuitionistic fuzzy classical primary spectrum, Intuitionistic fuzzy primary ideal.
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1 Introduction

Throughout this paper, all rings are commutative with identity, and all modules are unitary. A proper submodule P of an R-module M is called a classical primary submodule if $abm \in P$ for $a, b \in R$, and $m \in M$, implies that $am \in P$ or $b^n m \in P$, for some $n \in \mathbb{N}$. The concept of classical primary submodule, which is a generalization of primary ideals and classical prime submodules in module theory plays crucial role in algebra. The Zariski topology on the prime spectrum of an R-module has been introduced by Lu [9], and this concept was further explored by many authors [4,5,10,20]. In 2021, Goswami and Saikia [7] gave the concept of the spectrum of weakly prime submodules and investigated related properties.

The authors of [14] investigated many properties of the Zarisky topology of intuitionistic fuzzy prime submodules. The concept of intuitionistic fuzzy classical prime submodules, which is a generalisation of intuitionistic fuzzy prime ideals and intuitionistic fuzzy prime submodules, was studied by the author in [12].

In this paper, we focus on the intuitionistic fuzzy classical primary submodules and then introduce a topology on $IF_{cp}\text{spec}(M)$, the collection of all intuitionistic fuzzy classical primary submodules of M, called the intuitionistic fuzzy classical primary Zariski topology. This topology generalises the Zariski topology of intuitionistic fuzzy prime submodules $IF\text{spec}(M)$ [14]. We investigate several properties of this topology on $IF_{cp}\text{spec}(M)$.

2 Preliminaries

Throughout this paper, R is a commutative ring with non-zero identity, and M is an R-module. Let θ denote the zero element of M. In order to make this paper easier to follow, we recall in this section various notions from intuitionistic fuzzy commutative algebra theory that will be used in the sequel, which can be found in [1,3,18].

Given a non-empty set M, an intuitionistic fuzzy subset A of M is defined by a complex function (μ_A, ν_A) from M to $[0,1] \times [0,1]$. We denote by $IFS(M)$, the set of all intuitionistic fuzzy subsets of M. For any two elements A, B in $IFS(M)$ we write $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in M$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$. By an intuitionistic fuzzy point (IFP) $x_{(p,q)}$ of set M, $x \in M$, $p, q \in [0,1]$ such that $p + q \leq 1$ we mean that $x_{(p,q)} \in IFS(M)$ is defined by $x_{(p,q)}(y) = (p, q)$, if $y = x$, otherwise $(0, 1)$. If $A \in IFS(M)$ and $x \in M$ such that $\mu_A(x) \geq p$ and $\nu_A(x) \leq q$, then $x_{(p,q)} \subseteq A$. We designate this fact as $x_{(p,q)} \in A$. The intuitionistic fuzzy characteristic function of M with respect to a subset N is denoted by χ_N and is defined as $\chi_N(y) = (1, 0)$, if $y \in N$, otherwise $(0, 1)$. If $x = \theta$ and $p = 1, q = 0$, then $x_{(p,q)} = \theta_{(1,0)}$ (or $N = \{\theta\}$) is called the intuitionistic fuzzy zero point of M and is denoted by $\chi_{\{\theta\}}$.

Definition 2.1. [15] Let A, B be any two intuitionistic fuzzy ideals of the ring R. Then the intuitionistic fuzzy product AB and intuitionistic intrinsic product $A \circ B$ of A and B are defined as: For all $x \in R$

$$\mu_{AB}(x) = \begin{cases} \sup[\inf_{i=1}^{n}\{\mu_A(a_i) \wedge \mu_B(b_i)\}] & \text{if } x = \Sigma_{i=1}^{n} a_i b_i, a_i, b_i \in R, n \in \mathbb{N} \\ 0, & \text{if } x \text{ is not expressible as } x = \Sigma_{i=1}^{n} a_i b_i \end{cases}$$

and

$$\nu_{AB}(x) = \begin{cases} \inf[\sup_{i=1}^{n}\{\nu_A(a_i) \vee \nu_B(b_i)\}] & \text{if } x = \Sigma_{i=1}^{n} a_i b_i, a_i, b_i \in R, n \in \mathbb{N} \\ 1, & \text{if } x \text{ is not expressible as } x = \Sigma_{i=1}^{n} a_i b_i, \end{cases}$$

$$(\mu_{A\circ B}(x), \nu_{A\circ B}(x)) = \begin{cases} (\sup_{x=yz}(\mu_A(y) \wedge \mu_B(z)), \inf_{x=yz}(\nu_A(y) \vee \nu_B(z)), & \text{if } x = yz \\ (0,1), & \text{otherwise} \end{cases}$$

where as usual supremum and infimum of an empty set are taken to be 0 and 1 respectively. Note that $A \circ B \subseteq AB$.

Definition 2.2. [3] Let $A \in IFS(R)$. Then A is called an intuitionistic fuzzy ideal (IFI) of R if for all $x, y \in R$, the followings are satisfied:

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;
(ii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;
(iii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$;
(iv) $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Observe that $\chi_{\{0\}}, \chi_R$ are intuitionistic fuzzy ideals of R.

Definition 2.3. [3,6,8] Let $A \in IFS(M)$. Then A is called an intuitionistic fuzzy module (IFM) (or an intuitionistic fuzzy submodule (IFSM)) of M if for all $x, y \in M, r \in R$, the followings are satisfied:

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;
(ii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;
(iii) $\mu_A(rx) \geq \mu_A(x)$;
(iv) $\nu_A(rx) \leq \nu_A(x)$;
(v) $\mu_A(\theta) = 1$;
(vi) $\nu_A(\theta) = 0$.

Observe that χ_θ, χ_M are intuitionistic fuzzy modules of M. We abbreviate by $IFM(M)$, the family of all intuitionistic fuzzy R-modules of M, and by $IFI(R)$, the family of all intuitionistic fuzzy ideals of the ring R. Observe that when $R = M$, then A is an intuitionistic fuzzy module of M if and only if A is and intuitionistic fuzzy ideal of R and $\mu_A(\theta) = 1, \nu_A(\theta) = 0$. Further if $A \in IFM(M)$, then the set $A_* = \{x \in M | \mu_A(x) = \mu_A(\theta) \text{ and } \nu_A(x) = \nu_A(\theta)\}$ is a submodule of M.

Lemma 2.4. [15,16] *Let $C \in IFI(R), A, B \in IFM(M)$. Then:*

(i) $CB \subseteq A$ if and only if $C \circ B \subseteq A$.

(ii) $r_{(s,t)} \in IFS(R), x_{(p,q)} \in IFS(M)$ be $IFPs$. Then $r_{(s,t)} \circ x_{(p,q)} = (rx)_{(s\wedge p, t\vee q)}$.

(iii) If $\mu_C(0) = 1, \nu_C(0) = 0$ then $C \circ A \in IFM(M)$.

(iv) Let $r_{(s,t)} \in IFS(R)$ be an IFP. Then for all $w \in M$,

$$\mu_{r_{(s,t)}\circ B}(w) = \begin{cases} \sup[s \wedge \mu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 0, & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

and

$$\nu_{r_{(s,t)}\circ B}(w) = \begin{cases} \inf[t \vee \nu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 1, & \text{if } w \text{ is not expressible as } w = rx. \end{cases}$$

Definition 2.5. [18, 19] The intuitionistic fuzzy prime radical of an intuitionistic fuzzy ideal A of a ring R denoted by $\sqrt{A} = (\mu_{\sqrt{A}}, \nu_{\sqrt{A}})$, is an intuitionistic fuzzy set of R, and is defined as

$\mu_{\sqrt{A}}(r) = \sup_{n\in\mathbb{N}}\{\mu_A(r^n)\}$ and $\nu_{\sqrt{A}}(r) = \inf_{n\in\mathbb{N}}\{\nu_A(r^n)\}$, for all $r \in R, n \in \mathbb{N}$.

Definition 2.6. [2, 11, 17] A non-constant IFI C is termed as an intuitionistic fuzzy prime (respectively, primary) ideal of R if for every $x_{(p,q)}, y_{(r,s)} \in IFP(R)$ such that $x_{(p,q)} \circ y_{(r,s)} \in C$ gives that either $x_{(p,q)} \in C$ or $y_{(r,s)} \in C$ (or respectively, either $x_{(p,q)} \in C$ or $y_{(r,s)} \in \sqrt{C}$).

Definition 2.7. [2, 17] $A \in IFI(R)$ is termed as an intuitionistic fuzzy prime ideal (IFPI) of R if $A \neq \chi_{\{0\}}, \chi_R$ and for any $B, C \in IFI(R)$ so that $B \circ C \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$.

$IF\text{spec}(R)$ denotes the set of all IFPIs of R.

Definition 2.8. [15, 16] For $A, B \in IFS(M)$ and $C \in IFS(R)$, define the residual quotient $(A : B)$ and $(A : C)$ as follows:

(i) $\mu_{(A:B)}(r) = \sup\{s : r_{(s,t)} \circ B \subseteq A\}$; $\nu_{(A:B)}(r) = \inf\{t : r_{(s,t)} \circ B \subseteq A\}, \forall r \in R$.

(ii) $\mu_{(A:C)}(x) = \sup\{\alpha : C \circ x_{(\alpha,\beta)} \subseteq A\}$; $\nu_{(A:C)}(x) = \inf\{\beta : C \circ x_{(\alpha,\beta)} \subseteq A\}, \forall x \in M$.

In [16] it was proved that if $A \in IFM(M)$, $B \in IFS(M)$, $C \in IFI(R)$, then $(A : B) \in IFI(R)$ and $(A : C) \in IFM(M)$.

Theorem 2.9. [15, 16] *If $A, B \in IFS(M), C \in IFS(R)$. Then*

(i) $(A : B) \circ B \subseteq A$ *;*

(ii) $C \circ (A : C) \subseteq A$*;*

(iii) $C \circ B \subseteq A \Leftrightarrow C \subseteq (A : B) \Leftrightarrow B \subseteq (A : C)$.

Definition 2.10. [13, 19] Let M be an R-module and $A \in IFS(M)$. The IFS $\sqrt{A} = (\mu_{\sqrt{A}}, \nu_{\sqrt{A}})$ of R is called the intuitionistic fuzzy radical of A defined by

$\mu_{\sqrt{A}}(r) = \sup_{n\in\mathbb{N}}\{\inf_{m\in M} \mu_A(r^n.m)\}$ and $\nu_{\sqrt{A}}(r) = \inf_{n\in\mathbb{N}}\{\sup_{m\in M} \nu_A(r^n.m)\}$

for all $r \in R, m \in M, n \in \mathbb{N}$.

It is proved that when $A \in IFM(M)$, then the intuitionistic fuzzy radical $\sqrt{A}$ is an IFI of R.

Definition 2.11. [14] Let $A, B \in IFM(M)$ with $A \subseteq B$. Then A is termed as:

(i) an intuitionistic fuzzy prime (IF-prime) submodule of B, if for any IFPs $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ with

$$r_{(s,t)} \circ x_{(p,q)} \in A \Rightarrow x_{(p,q)} \in A \text{ or } r_{(s,t)} \in \sqrt{(A : B)}.$$

In a special case, when $B = \chi_M$, then A is called an IF-prime submodule of M. The set of all IF-prime submodules of M is denoted by $IF\text{spec}(M)$.

(ii) an intuitionistic fuzzy primary (IF-primary) submodule of B, if for any IFPs $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ with

$$r_{(s,t)} \circ x_{(p,q)} \in A \Rightarrow x_{(p,q)} \in A \text{ or } r_{(s,t)} \in (A : B).$$

In a special case, when $B = \chi_M$, then A is called an IF-primary submodule of M.

The following theorem revealed that the two notions intuitionistic fuzzy primary submodule (IF-primary submodule) and intuitionistic fuzzy primary ideal (IF-primary ideal) coincide when R is considered to be a module over itself.

Theorem 2.12. [14] *If $M = R$, then $B \in IFM(M)$, is an IF-primary submodule of M if and only if $B \in IFI(R)$ is an IF-primary ideal.*

Theorem 2.13. [14] *(a) Let N be a primary (prime) submodule of M and $p, q \in (0, 1)$ with $p + q < 1$. If A is an IFS of M defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ p, & \text{otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } y \in N \\ q, & \text{otherwise.} \end{cases}$$

for all $x \in M$. Then A is an IF-primary (prime) submodule of M.

(b) Conversely, any IF-primary (prime) submodule can be obtained as in (a).

Corollary 2.14. [14] *If $A \in IFspec(M)$, then $(A : \chi_M) \in IFspec(R)$.*

Theorem 2.15. [14] *If P is an IF-primary submodule of M, then $\sqrt{(P : \chi_M)}$ is an IF-prime ideal of R.*

Definition 2.16. [12] Let A be an intuitionistic fuzzy submodule of an R-module M. Then A is called an intuitionistic fuzzy classical prime submodule of M if for every $a_{(p,q)}, b_{(t,s)} \in IFP(R)$, $x_{(u,v)} \in IFP(M)$, where $a, b \in R, x \in M$ and $p, q, t, s, u, v \in [0, 1]$ with

$$a_{(p,q)}b_{(t,s)}x_{(u,v)} \in A \Leftrightarrow a_{(p,q)}x_{(u,v)} \in A \text{ or } b_{(t,s)}x_{(u,v)} \in A.$$

3 Topology on intuitionistic fuzzy classical primary submodules

Our definition of an intuitionistic fuzzy classical primary submodule is a generalization of the notion of classical prime and classical primary submodules in module theory.

Definition 3.1. Let $A \in IFM(M)$ be an IFSM of an R-module M. Then A is called an intuitionistic fuzzy classical primary submodule of M if for every $a_{(p,q)}, b_{(t,s)} \in IFP(R)$, $x_{(u,v)} \in IFP(M)$, where $a, b \in R, x \in M$ and $p, q, t, s, u, v \in [0, 1]$ with

$$a_{(p,q)}b_{(t,s)}x_{(u,v)} \in A \Leftrightarrow a_{(p,q)}x_{(u,v)} \in A \text{ or } b^n_{(t,s)}x_{(u,v)} \in A,$$

for some positive integer n.

It is easy to see that every intuitionistic fuzzy classical prime submodule of M is an intuitionistic fuzzy primary submodule but converse need not be true. See the following example.

Example 3.2. Let $\mathbb{Z}$ be the set of all integers. Suppose that $M = R = \mathbb{Z}$ is a commutative ring. Define the IFS A of M as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in 4\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in 4\mathbb{Z} \\ 1, & \text{otherwise.} \end{cases}$$

Then it is easily seen that A is an intuitionistic fuzzy classical primary submodule of an R-module M, which is not an intuitionistic fuzzy classical prime submodule of M (for $2_{(0.7,0.2)}$, $2_{(0.6,0.3)} \in IFP(R)$, $1_{(0.5,0.3)} \in IFP(M)$ we have $2_{(0.7,0.2)}2_{(0.6,0.3)}1_{(0.5,0.3)} = 4_{(0.5,0.3)} \in A$, but $2_{(0.7,0.2)}1_{(0.5,0.3)} = 2_{(0.5,0.3)} \notin A$ and $2_{(0.6,0.3)}1_{(0.5,0.3)} = 2_{(0.5,0.3)} \notin A$.)

In what follows, $IF_{cp}\text{spec}(M)$ denotes the set of all intuitionistic fuzzy classical primary submodules of an R-module M. We call $IF_{cp}\text{spec}(M)$, the intuitionistic fuzzy classical primary spectrum of M. For every intuitionistic fuzzy submodule A of M, the intuitionistic fuzzy classical variety of A is denoted by $\mathbb{V}(A)$, and is defined as the set of all intuitionistic fuzzy classical primary submodule containing A, i.e., $\mathbb{V}(A) = \{B \in IF_{cp}\text{spec}(M) : A \subseteq B\}$.

Theorem 3.3. *For any family of IFSMs $\{A_i : i \in I\}$ of an R-module M, the following properties hold.*

1. $\mathbb{V}(\chi_{\{\theta\}}) = IF_{cp}\text{spec}(M)$ *and* $\mathbb{V}(\chi_M) = \varnothing$.
2. $\bigcap_{i\in I} \mathbb{V}(A_i) = \mathbb{V}(\sum_{i\in I} A_i)$.
3. $\mathbb{V}(A_1) \cup \mathbb{V}(A_2) = \mathbb{V}(A_1 \cap A_2)$.

Proof. **(1)**. Obvious.

(2). Let B be an IFSM of M such that $B \in \bigcap_{i\in I} \mathbb{V}(A_i)$. Then we have $B \in \mathbb{V}(A_i)$ for all $i \in I$, i.e., $A_i \subseteq B$.

$$\begin{aligned} \mu_{\sum_{i\in I} A_i}(x) &= \sup_{x=\sum_{i\in I} x_i} \Big(\inf_{x=\sum_{i\in I} x_i} \mu_{A_i}(x_i)\Big) \\ &\leq \sup_{x=\sum_{i\in I} x_i} \Big(\inf_{x=\sum_{i\in I} x_i} \mu_B(x_i)\Big) \\ &= \sup_{x=\sum_{i\in I} x_i} \mu_B(x_i) \\ &= \mu_B(x). \end{aligned}$$

Similarly, we can show that $\nu_{\sum_{i\in I} A_i}(x) \geq \nu_B(x)$. Thus we have $\sum_{i\in I} A_i \subseteq B$ and so $B \in \mathbb{V}(\sum_{i\in I} A_i)$ which implies that $\bigcap_{i\in I} \mathbb{V}(A_i) \subseteq \mathbb{V}(\sum_{i\in I} A_i)$. On the other hand, let B be an IFSM of M such that $B \in \mathbb{V}(\sum_{i\in I} A_i)$. It is easy to see that $A_i \subseteq \sum_{i\in I} A_i \subseteq B$, i.e., $B \in \mathbb{V}(A_i)$, for all $i \in I$. Therefore $\mathbb{V}(\sum_{i\in I} A_i) \subseteq \bigcap_{i\in I} \mathbb{V}(A_i)$ and hence $\bigcap_{i\in I} \mathbb{V}(A_i) = \mathbb{V}(\sum_{i\in I} A_i)$.

(3). Let B be an IFSM of M such that $B \in \mathbb{V}(A_1) \cup \mathbb{V}(A_2)$. Then we have $A_1 \subseteq B$ or $A_2 \subseteq B$, it follows that $A_1 \cap A_2 \subseteq B$. Thus $B \in \mathbb{V}(A_1 \cap A_2)$ and so $\mathbb{V}(A_1) \cup \mathbb{V}(A_2) \subseteq \mathbb{V}(A_1 \cap A_2)$. On the other hand, let B be an IFSM of M such that $B \in \mathbb{V}(A_i \cap A_2)$. This implies that $A_1 \cap A_2 \subseteq B$, i.e., $A_1 \subseteq B$ or $A_2 \subseteq B$. Also, $B \in \mathbb{V}(A_1) \cap \mathbb{V}(A_2)$. Therefore, we obtain that $\mathbb{V}(A_1 \cap A_2) \subseteq \mathbb{V}(A_1) \cup \mathbb{V}(A_2)$ and hence $\mathbb{V}(A_1) \cup \mathbb{V}(A_2) = \mathbb{V}(A_1 \cap A_2)$. □

Corollary 3.4. *Let A and B be any IFIs of the ring R. Then*

$$\mathbb{V}(A \circ \chi_M) \cup \mathbb{V}(B \circ \chi_M) = \mathbb{V}(A \circ B \circ \chi_M).$$

Set $X = IF_{cp}\text{spec}(M)$. For any IFSM A of an R-module M, define $X(A)$ and τ as follows:

$$X(A) = X - \mathbb{V}(A) \text{ and } \tau = \{X(A) : A \in IFM(M)\}$$

In the next theorem, we will show that the pair (X, τ) is a topological space.

Theorem 3.5. *Let M be an R-module. Then the following statements hold:*

1. *The pair (X, τ) is a topological space.*
2. *X is a T_0 topological space.*

Proof. **(1)**. 1. Since $\mathbb{V}(\chi_{\{\theta\}}) = X$ and $\mathbb{V}(\chi_M) = \varnothing$, we have $X(\chi_{\{\theta\}}) = X - X = \varnothing$ and $X(\chi_M) = X - \varnothing$, i.e., $\varnothing, X \in \tau$.

2. Let A and B be any IFSMs of M. Then by Theorem (3.3)(3), we have

$$\begin{aligned} X(A) \cap X(B) &= (X - \mathbb{V}(A)) \cap (X - \mathbb{V}(B)) \\ &= X - (\mathbb{V}(A) \cup \mathbb{V}(B)) \\ &= X - \mathbb{V}(A \cap B) \\ &= X(A \cap B). \end{aligned}$$

3. For any arbitrary family of IFSMs $\{A_i : i \in I\}$ of M. Then by Theorem (3.3)(2), we have

$$\begin{aligned} \bigcup_{I \in I} X(A_i) &= \bigcup_{i \in I} (X - \mathbb{V}(A_i)) \\ &= X - \bigcap_{i \in I} \mathbb{V}(A_i) \\ &= X - \mathbb{V}(\sum_{i \in I} A_i) \\ &= X(\sum_{i \in I} A_i). \end{aligned}$$

Thus, 2. and 3. show that τ is closed under arbitrary union and finite intersection. Thus the pair (X, τ) satisfies all the axioms of a topological space. Therefore we have (X, τ) is a topological space.

(2). Let A and B be two distinct points of X. If $A \nsubseteq B$, then obviously $B \in X(A)$ and $A \notin X(A)$ showing that X is a T_0 topological space. □

In this case, the topology τ on X is called the **intuitionistic fuzzy primary Zariski topology**. For every IFSM A of M, the set

$$\mathbb{V}^*(A) = \{B \in IF_{cp}\text{spec}(M) : \sqrt{(A : \chi_M)} \subseteq \sqrt{(B : \chi_M)}\}.$$

Then we have the following lemma.

Lemma 3.6. *Let A and B be any IFSMs of an R-module M. If $A \subseteq B$, then*

$$\mathbb{V}^*(B) \subseteq \mathbb{V}^*(A).$$

Proof. Let C be an IFSM of M such that $C \in \mathbb{V}^*(B)$. Then we have $\sqrt{(B : \chi_M)} \subseteq \sqrt{(C : \chi_M)}$. Since $A \subseteq B$, we have $\sqrt{(A : \chi_M)} \subseteq \sqrt{(B : \chi_M)}$, i.e., $\sqrt{(A : \chi_M)} \subseteq \sqrt{(C : \chi_M)}$. Therefore, $C \in \mathbb{V}^*(A)$ and hence $\mathbb{V}^*(B) \subseteq \mathbb{V}^*(A)$. □

Then we have the next results.

Theorem 3.7. *For any family of IFSMs $\{A_i : i \in I\}$ of an R-module M. The following properties hold.*

1. $\mathbb{V}^*(\chi_{\{\theta\}}) = IF_{cp}\mathrm{spec}(M)$ *and* $\mathbb{V}^*(\chi_M) = \varnothing$.
2. $\bigcap_{i\in I} \mathbb{V}^*(A_i) = \mathbb{V}^*(\sum_{i\in I}(A : \chi_M) \circ \chi_M)$.
3. $\mathbb{V}^*(A_1) \cup \mathbb{V}^*(A_2) = \mathbb{V}^*(A_1 \cap A_2)$.

Proof. **(1)**. Obvious.

(2). Let B be an IFSM of M such that $B \in \bigcap_{i\in I} \mathbb{V}^*(A_i)$. Then we have $B \in \mathbb{V}^*(A_i)$, for all $i \in I$, i.e., $\sqrt{(A_i : \chi_M)} \subseteq \sqrt{(B : \chi_M)}$.
Since $(A_i : \chi_M) \circ \chi_M \subseteq \sqrt{(A_i : \chi_M)} \circ \chi_M \subseteq \sqrt{(B : \chi_M)} \circ \chi_M$, it follows that,

$$\begin{aligned} \sqrt{((\sum_{i\in I}(A_i : \chi_M) \circ \chi_M) : \chi_M)} &\subseteq \sqrt{(\sqrt{(B : \chi_M)} \circ \chi_M : \chi_M)} \\ &\subseteq \sqrt{\sqrt{(B : \chi_M)}} \\ &= \sqrt{(B : \chi_M)}. \end{aligned}$$

It is easy to see that $B \in \mathbb{V}^*(\sum_{i\in I}(A_i : \chi_M) \circ \chi_M)$ and so

$$\bigcap_{i\in I} \mathbb{V}^*(A_i) \subseteq \mathbb{V}^*(\sum_{i\in I}(A_i : \chi_M) \circ \chi_M).$$

On the other hand, let B be an IFSM of M such that $B \in \mathbb{V}^*(\sum_{i\in I}(A_i : \chi_M) \circ \chi_M)$. Thus we have

$$\sqrt{((\textstyle\sum_{i\in I}(A_i : \chi_M) \circ \chi_M) : \chi_M)} \subseteq \sqrt{(B : \chi_M)}.$$

Clearly, we have $((A_i : \chi_M) \circ \chi_M) = (A_i : \chi_M)$, for all $i \in I$. Also for each $i \in I$, we obtain that

$$\begin{aligned} \sqrt{(A_i : \chi_M)} &= \sqrt{(((A_i : \chi_M) \circ \chi_M) : \chi_M)} \\ &\subseteq \sqrt{((\sum_{i\in I}(A_i : \chi_M) \circ \chi_M) : \chi_M)} \\ &\subseteq \sqrt{(B : \chi_M)}. \end{aligned}$$

Therefore we obtain that $B \in \bigcap_{i\in I} \mathbb{V}^*(A_i)$ and hence $\mathbb{V}^*(\sum_{i\in I}(A : \chi_M) \circ \chi_M) \subseteq \bigcap_{i\in I} \mathbb{V}^*(A_i)$.

(3). Let B be an IFSM of M such that $B \in \mathbb{V}^*(A_1) \cup \mathbb{V}^*(A_2)$. Then we have

$$\sqrt{(A_1 : \chi_M)} \subseteq \sqrt{(B : \chi_M)} \text{ or } \sqrt{(A_2 : \chi_M)} \subseteq \sqrt{(B : \chi_M)}.$$

If $\sqrt{(A_1 : \chi_M)} \subseteq \sqrt{(B : \chi_M)}$, then $\sqrt{(A_1 \cap A_2 : \chi_M)} \subseteq \sqrt{(A_1 : \chi_M)} \subseteq \sqrt{(B : \chi_M)}$, it follows that, $B \in \mathbb{V}^*(A_1 \cup A_2)$. Similarly, if $\sqrt{(A_2 : \chi_M)} \subseteq \sqrt{(B : \chi_M)}$, then $B \in \mathbb{V}^*(A_1 \cup A_2)$.

On the other hand, let B be an IFSM of M such that $B \in \mathbb{V}^*(A_1 \cup A_2)$. Then $\sqrt{(A_1 \cap A_2 : \chi_M)} \subseteq \sqrt{(B : \chi_M)}$.

Since $A_1 \cap A_2 \subseteq A_1$ and $A_1 \cap A_2 \subseteq A_2$, we have $\sqrt{(A_1 : \chi_M)} \subseteq \sqrt{(A_1 \cap A_2 : \chi_M)}$ and $\sqrt{(A_2 : \chi_M)} \subseteq \sqrt{(A_1 \cap A_2 : \chi_M)}$

$$\Rightarrow \sqrt{(A_1 : \chi_M)} \circ \sqrt{(A_2 : \chi_M)} \subseteq \sqrt{(A_1 \cap A_2 : \chi_M)} \subseteq \sqrt{(B : \chi_M)}.$$

Now since $\sqrt{(B : \chi_M)}$ is an intuitionistic fuzzy prime ideal, it follows that

$$\sqrt{(A_1 : \chi_M)} \subseteq \sqrt{(B : \chi_M)} \text{ or } \sqrt{(A_2 : \chi_M)} \subseteq \sqrt{(B : \chi_M)}.$$

Clearly, we have $B \in \mathbb{V}^*(A_1)$ or $B \in \mathbb{V}^*(A_2)$, i.e., $B \in \mathbb{V}^*(A_1) \cup \mathbb{V}^*(A_2)$. Therefore, $\mathbb{V}^*(A_1 \cap A_2) \subseteq \mathbb{V}^*(A_1) \cup \mathbb{V}^*(A_2)$ and hence $\mathbb{V}^*(A_1 \cap A_2) = \mathbb{V}^*(A_1) \cup \mathbb{V}^*(A_2)$. □

For every IFSM A of an R-module M we define $X^*(A)$ and τ^* as follows:

$$X^*(A) = X - \mathbb{V}^*(A) \text{ and } \tau^* = \{X^*(A) : A \in IFM(M)\}$$

In the next theorem we will show that the pair (X, τ^*) is a topological space.

Theorem 3.8. *Let M be an R-module. Then the following statements hold:*

1. *The pair (X, τ^*) is a topological space.*
2. *X is a T_0 topological space.*

Proof. The proof follows from Theorem (3.5). □

For any R-module M and $A, B \in IFM(M)$ we have the next result.

Proposition 3.9. *Let A and B be any IFSMs of an R-module M. if $\sqrt{(A : \chi_M)} = \sqrt{(B : \chi_M)}$, then $\mathbb{V}^*(A) = \mathbb{V}^*(B)$. Moreover the converse is true if both A and B are intuitionistic fuzzy classical primary submodules of M.*

Proof. Let A and B be any IFSMs of an R-module M such that $\sqrt{(A : \chi_M)} = \sqrt{(B : \chi_M)}$. Next let C be an IFSM of M such that $C \in \mathbb{V}^*(A)$. Then we have $\sqrt{(A : \chi_M)} \subseteq \sqrt{(C : \chi_M)}$, i.e., $\sqrt{(B : \chi_M)} \subseteq \sqrt{(C : \chi_M)}$. Thus $C \in \mathbb{V}^*(B)$ and so $\mathbb{V}^*(A) \subseteq \mathbb{V}^*(B)$. Similarly, we can show that $\mathbb{V}^*(B) \subseteq \mathbb{V}^*(A)$ and so $\mathbb{V}^*(A) = \mathbb{V}^*(B)$.

For the converse, suppose that A and B are intuitionistic fuzzy classical primary submodules of M such that $\mathbb{V}^*(A) = \mathbb{V}^*(B)$. Since $A \in \mathbb{V}^*(A)$ and $B \in \mathbb{V}^*(B)$ and $\mathbb{V}^*(A) = \mathbb{V}^*(B)$. We have $\sqrt{(B : \chi_M)} \subseteq \sqrt{(A : \chi_M)}$ and $\sqrt{(A : \chi_M)} \subseteq \sqrt{(B : \chi_M)}$. Therefore, we obtain that $\sqrt{(A : \chi_M)} = \sqrt{(B : \chi_M)}$. □

For an intuitionistic fuzzy primary ideal $\mathbf{p}$ of ring R, by $IF_{cp}spec_{\mathbf{p}}(M)$ we mean the set of all $A \in IFM(M)$ such that $\sqrt{(A : \chi_M)} = \mathbf{p}$. In other words,

$$IF_{cp}spec_{\mathbf{p}}(M) = \{A \in IF_{cp}\mathrm{spec}(M) : \sqrt{(A : \chi_M)} = \mathbf{p}\}.$$

Theorem 3.10. *Let C and A be any IFI of a ring R and an IFSM of an R-module M. Then the following properties hold.*

1. $\mathbb{V}^*(A) = \bigcup_{\sqrt{(A:\chi_M)}\subseteq \mathbf{p}} IF_{cp}spec_{\mathbf{p}}(M)$.
2. $\mathbb{V}^*(C^m \circ \chi_M) = V(C^n \circ \chi_M)$ *for some positive integers* m, n,
3. $\mathbb{V}(\sqrt{(A:\chi_M)} \circ \chi_M) \subseteq \mathbb{V}^*(A) \subseteq \mathbb{V}^*((A:\chi_M) \circ \chi_M)$.

Proof. **(1)**. Let B be any IFSM of M such that $B \in \mathbb{V}^*(A)$. Then we have $\sqrt{(A:\chi_M)} \subseteq \sqrt{(B:\chi_M)} = \mathbf{p}$ and so $B \in IF_{cp}spec_{\mathbf{p}}(M) \subseteq \bigcup_{\sqrt{(A:\chi_M)}\subseteq \mathbf{p}} IF_{cp}spec_{\mathbf{p}}(M)$.
It is easy to see that $\mathbb{V}^*(A) \subseteq \bigcup_{\sqrt{(A:\chi_M)}\subseteq \mathbf{p}} IF_{cp}spec_{\mathbf{p}}(M)$. On the other hand, let B be an IFSM of M such that $B \in \bigcup_{\sqrt{(A:\chi_M)}\subseteq \mathbf{p}} IF_{cp}spec_{\mathbf{p}}(M)$. Thus there exists an IFPI $\mathbf{p}$ of R such that $\sqrt{(A:\chi_M)} \subseteq \mathbf{p}$ and $B \in IF_{cp}spec_{\mathbf{p}}(M)$. Clearly, we have $\sqrt{(B:\chi_M)} = \mathbf{p}$, i.e., $\sqrt{(A:\chi_M)} \subseteq \sqrt{(B:\chi_M)}$, it follows that, $B \in \mathbb{V}^*(A)$. Therefore we obtain that $\bigcup_{\sqrt{(A:\chi_M)}\subseteq \mathbf{p}} IF_{cp}spec_{\mathbf{p}}(M) \subseteq \mathbb{V}^*(A)$ and hence $\mathbb{V}^*(A) = \bigcup_{\sqrt{(A:\chi_M)}\subseteq \mathbf{p}} IF_{cp}spec_{\mathbf{p}}(M)$.

(2). Let B be any IFSM of M such that $B \in \mathbb{V}^*(C^n \circ \chi_M)$. Then we have $C^n \circ \chi_M \subseteq B$, i.e., $\sqrt{(C^m \circ \chi_M : \chi_M)} \subseteq \sqrt{(B:\chi_M)}$. This implies that $B \in \mathbb{V}^*(C^m \circ \chi_M)$ and so $\mathbb{V}(C^n \circ \chi_M) \subseteq \mathbb{V}^*(C^m \circ \chi_M)$. On the other hand, let B be an IFSM of M such that $B \in \mathbb{V}^*(C^m \circ \chi_M)$. Thus $\sqrt{(C^m \circ \chi_M : \chi_M)} \subseteq \sqrt{(B:\chi_M)}$.
Obviously, $C^m \subseteq (C^m \circ \chi_M : \chi_M)$, we have $C^m \subseteq \sqrt{(B:\chi_M)}$, which implies that, $C^m \circ \chi_M \subseteq B$. It is easy to see that $B \in \mathbb{V}(C^n \circ \chi_M)$.
Therefore $\mathbb{V}^*(C^m \circ \chi_M) \subseteq V(C^n \circ \chi_M)$ and hence $\mathbb{V}^*(C^m \circ \chi_M) = \mathbb{V}(C^n \circ \chi_M)$.

(3). Let B be any IFSM of M such that $B \in \mathbb{V}^*(A)$. Then we have
$\sqrt{(A:\chi_M)} \subseteq \sqrt{(B:\chi_M)}$. Since $(A:\chi_M) \circ \chi_M \subseteq A$, we have

$$\sqrt{(A:\chi_M) \circ \chi_M : \chi_M} \subseteq \sqrt{(A:\chi_M)} \subseteq \sqrt{(B:\chi_M)}$$

This implies that $B \in \mathbb{V}^*((A:\chi_M) \circ \chi_M)$ and so $\mathbb{V}^*(A) \subseteq \mathbb{V}^*((A:\chi_M) \circ \chi_M)$.
Next, let B be an IFSM of M such that $B \in \mathbb{V}(\sqrt{(A:\chi_M)} \circ \chi_M)$. Thus $\sqrt{(A:\chi_M)} \circ \chi_M \subseteq B$. Obviously, $\sqrt{(A:\chi_M)} \subseteq (B:\chi_M)$. Since $(B:\chi_M) \subseteq \sqrt{(B:\chi_M)}$, we have $\sqrt{(A:\chi_M)} \subseteq \sqrt{(B:\chi_M)}$, which implies that, $B \in \mathbb{V}^*(A)$. Therefore $\mathbb{V}(\sqrt{(A:\chi_M)} \circ \chi_M) \subseteq \mathbb{V}^*(A)$ and hence

$$\mathbb{V}(\sqrt{(A:\chi_M)} \circ \chi_M) \subseteq \mathbb{V}^*(A) \subseteq \mathbb{V}^*((A:\chi_M) \circ \chi_M).$$

□

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References

[1] Atanassov, K. T. (1999). *Intuitionistic Fuzzy Sets Theory and Applications*. Studies on Fuzziness and Soft Computing, 35, Physica-Verlag, Heidelberg.

[2] Bakhadach, I., Melliani, S., Oukessou, M., & Chadli, S. L. (2016). Intuitionistic fuzzy ideal and intuitionistic fuzzy prime ideal in a ring. *Notes on Intuitionistic Fuzzy Sets*, 22(2), 59–63.

[3] Basnet, D. K. (2011). *Topics in Intuitionistic Fuzzy Algebra.* Lambert Academic Publishing, ISBN: 978-3-8443-9147-3.

[4] Baziar, M., & Behboodi, M. (2009). Classical primary submodules and decomposition theory of modules. *Journal of Algebra and Its Applications*, 8(3), 351–362.

[5] Behboodi, M., & Noori, M.J. (2009). Zariski-like topology on the classical prime spectrum of a modules. *Bulletin of the Iranian Mathematical Society*, 35(1), 255–271.

[6] Davvaz, B., Dudek, W. A., & Jun, Y. B. (2006). Intuitionistic fuzzy Hv-submodules. *Information Sciences*, 176, 285–300.

[7] Goswami, J., & Saikia, H. K. (2021). On the spectrum of weakly prime submodules. *Thai Journal of Mathematics*, 19(1), 51–58.

[8] Isaac, P., & John, P. P. (2011). On intuitionistic fuzzy submodules of a modules. *International Journal of Mathematical Sciences and Applications*, 1(3), 1447–1454.

[9] Lu, C. P. (1999). The Zariski topology on the prime spectrum of a module. *Houston Journal of Mathematics*, 25(3), 417–425.

[10] McCasland, R. L., Moore, M. E., & Smith, P. F. (1997). On the spectrum of a module over a commutative ring. *Communications in Algebra*, 25(1), 79–103.

[11] Sharma, P. K. (2022). On intuitionistic fuzzy primary ideal of a ring. *Pan-American Journal of Mathematics*, 1, 1–11.

[12] Sharma, P. K. (2023). Intuitionistic L-fuzzy classical prime and intuitionistic L-fuzzy 2-absorbing submodules. *Notes on Intuitionistic Fuzzy Sets*, 29(2), 99–113.

[13] Sharma, P. K., & Kanchan. (2020). On intuitionistic L-fuzzy primary and P-primary submodules. *Malaya Journal of Matematik*, 8(4), 1417–1426.

[14] Sharma, P. K., & Kanchan. (2020). Some topological properties of spectrum of intuitionistic fuzzy prime submodules. *Malaya Journal of Matematik*, 8(4), 1358–1362.

[15] Sharma, P. K., & Kaur, G. (2018). On intuitionistic fuzzy prime submodules. *Notes on Intuitionistic Fuzzy Sets*, 24(4), 97–112.

[16] Sharma, P. K., & Kaur, G. (2017). Residual quotient and annihilator of intuitionistic fuzzy sets of ring and module. *International Journal of Computer Science and Information Technology*, 9(4), 1–15.

[17] Sharma, P. K., & Kaur, G. (2017). Intuitionistic fuzzy prime spectrum of a ring. *CiiT International Journal of Fuzzy Systems*, 9(8), 167–175.

[18] Sharma, P. K., & Kaur, G. (2017). On annihilator of intuitionistic fuzzy subsets of modules. *Proceeding of Third International Conference on Fuzzy Logic Systems (Fuzzy-2017)* Chenai, India, 29-30 July, 2017 , pp. 37–49.

[19] Taherpour, A., Ghalandarzadeh, S., Rad, P. M., & Safari, P. (2022). On radical of intuitionistic fuzzy primary submodule. *Hindawi Journal of Mathematics*, Article ID 1332239.

[20] Toroghy, H. A., & Sarmazdeh, R. O. (2010). On the prime spectrum of a module and Zariski topologies. *Communications in Algebra*, 38, 4461–4475.