Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2026, Volume 32, Number 2, 144–162
DOI: 10.7546/nifs.2026.32.2.144-162

An intuitionistic fuzzy framework for human health risk assessment incorporating Monte Carlo uncertainty analysis

Toncho Ivanov Boyukov [1] **and Stefka Fidanova** [2,3]

[1] Faculty of Technical Sciences, Burgas State University "Prof. Dr. Asen Zlatarov"
1 Prof. Yakimov Str., 8010 Burgas, Bulgaria
e-mail: toncho_b@abv.bg
[2] Institute of Information and Communication Technologies
Bulgarian Academy of Sciences, Sofia, Bulgaria
e-mail: stefka.fidanova@iict.bas.bg
[3] Centre of Excellence in Informatics and Information and Communication Technologies
Sofia, Bulgaria

Received: 24 February 2026 **Revised:** 19 August 2026
Accepted: 27 August 2026 **Online First:** 27 August 2026

Abstract: Human health risk assessment of environmental contamination is commonly based on deterministic or semi-probabilistic indices such as the hazard quotient (HQ) and hazard index (HI). Although widely used, these indicators have limited capacity to explicitly represent uncertainty and variability in exposure and toxicological parameters. Monte Carlo simulation is often employed to address this limitation by propagating uncertainty; however, probabilistic outputs alone do not fully support decision-making when multiple contaminants must be aggregated into a single interpretable metric.

In this study, we propose a Monte Carlo-based intuitionistic fuzzy health risk assessment framework that integrates probabilistic exposure modeling with intuitionistic fuzzy set (IFS) theory. Hazard quotients obtained from Monte Carlo simulations are mapped to intuitionistic fuzzy membership, non-membership, and hesitation degrees using transparent threshold-based functions. Aggregation across multiple contaminants is performed using a component-wise weighted averaging scheme for the intuitionistic fuzzy components, yielding a site-level risk representation that preserves both risk magnitude and associated uncertainty. An entropy-based measure is further introduced to quantify uncertainty in the aggregated intuitionistic fuzzy assessment.

The proposed framework enables enhanced interpretability and robust decision support under uncertainty, as demonstrated through a numerical example. By explicitly combining probabilistic uncertainty propagation with intuitionistic fuzzy modeling, this approach advances classical health risk assessment methodologies and provides a structured basis for risk-informed environmental decision-making.
Keywords: Health risk assessment, Monte Carlo simulation, Intuitionistic fuzzy sets, Uncertainty analysis, Multi-contaminant exposure, Hazard quotient.
2020 Mathematics Subject Classification: 03E72.

1 Introduction

Human health risk assessment associated with environmental contamination is a fundamental component of environmental science and regulatory decision-making. Exposure to heavy metals through soil, water, and food pathways poses well-documented risks, motivating the widespread use of quantitative indicators such as the hazard quotient (HQ) and hazard index (HI) [14, 17]. These indices compare estimated exposure levels with toxicological reference doses and are favored for their simplicity and regulatory acceptance. Despite their practicality, classical HQ- and HI-based approaches provide limited insight into the uncertainty and variability inherent in environmental data. Such uncertainty arises from spatial heterogeneity of contaminant concentrations, variability in human exposure parameters, and incomplete knowledge of toxicological thresholds [16]. Monte Carlo simulation has therefore become a standard tool for propagating uncertainty and deriving probabilistic risk estimates [28]. Nevertheless, probabilistic outputs alone may be difficult to interpret and aggregate, particularly in multi-contaminant scenarios where decision-makers require a concise and transparent risk indicator [10, 15].

Intuitionistic fuzzy set theory offers a complementary perspective by explicitly distinguishing between degrees of membership, non-membership, and hesitation [23]. Unlike classical fuzzy sets, IFSs allow uncertainty and incomplete information to be represented as an intrinsic component of the model [27]. This property makes intuitionistic fuzzy approaches well suited to health risk assessment problems, where uncertainty is both stochastic and epistemic in nature [21, 22].

Although fuzzy and intuitionistic fuzzy methods have been applied in various environmental assessment and decision-making contexts, the present study focuses on a specific transparent mapping from Monte Carlo-derived risk summaries to intuitionistic fuzzy parameters for multi-contaminant health risk assessment.

To address this gap, this study proposes a Monte Carlo–based intuitionistic fuzzy health risk assessment framework that integrates probabilistic exposure modeling with intuitionistic fuzzy aggregation [13]. Hazard quotients derived from Monte Carlo simulations are mapped to intuitionistic fuzzy parameters, aggregated across contaminants using a component-wise weighted averaging scheme, and supplemented by an entropy-based measure of uncertainty. The proposed approach aims to enhance interpretability and decision support in multi-contaminant health risk assessment under uncertainty.

The environmental context of the study is informed by real monitoring campaigns reported in [12], while exposure parameters and reference doses are based on established regulatory and toxicological sources [9]. The numerical example presented below is illustrative and uses realistic values selected to be consistent with ranges reported in empirical studies [11,24].

2 Intuitionistic fuzzy set–based risk assessment framework

Conceptual background and motivation

To explicitly account for uncertainty and partial knowledge in environmental risk assessment, we adopt an intuitionistic fuzzy set (IFS) framework following Atanassov (1986) [1–3]. For each element x, an IFS is characterized by three parameters: a degree of membership $\mu(x)$, a degree of non-membership $\nu(x)$, and a degree of hesitation (or indeterminacy) $\pi(x)$, such that:

$$0 \leq \mu(x) \leq 1, \quad 0 \leq \nu(x) \leq 1, \quad \mu(x) + \nu(x) + \pi(x) = 1. \tag{1}$$

According to the definition of intuitionistic fuzzy sets introduced by Atanassov [1], the membership, non-membership and hesitation degrees satisfy the constraint given in (1). In the context of health risk assessment:

- μ quantifies the degree to which a factor (e.g., a metal contaminant) belongs to the set of *high-risk* conditions;
- ν quantifies the degree to which the same factor belongs to the set of *low or acceptable risk*;
- π represents hesitation or uncertainty, arising from data variability, limited information, or conflicting evidence.

This triadic representation enables a more realistic modeling of environmental and health risks by jointly incorporating evidence, counter-evidence, and uncertainty [4,5]. IFS-based models are therefore well suited for expert-informed, uncertainty-aware, and multi-criteria decision-making frameworks [6,20].

Notation and data structure

Let M denote the number of considered factors (e.g., metals or contaminants) and S the number of samples, locations, or individuals. Measured concentrations are denoted as:

$$C_{s,m}, \quad s = 1, \ldots, S, \quad m = 1, \ldots, M, \tag{2}$$

as defined in (2). Optionally, uncertainty descriptors such as standard deviation or detection (see Eq. (3)), limits may be provided as:

$$\sigma_{s,m}. \tag{3}$$

Exposure and population parameters (e.g., intake rate IR, exposure frequency EF, exposure duration ED, body weight BW, and averaging time AT) are assumed to be defined globally or per population group.

Each observation is transformed into a dimensionless risk indicator, typically a hazard quotient (HQ) or another normalized metric:

$$R_{s,m} = f\Big(C_{s,m}\Big), \tag{4}$$

as expressed in (4). For instance, in classical non-carcinogenic risk assessment, the hazard quotient is computed as:

$$R_{s,m} = \frac{ADD_{s,m}}{RfD_m}, \tag{5}$$

according to (5). Values $R_{s,m} > 1$ indicate exceedance of the reference threshold.

In matrix form, the data structures as summarized in (6), are given by:

$$\mathbf{C} = [C_{s,m}]_{S\times M}, \quad \mathbf{R} = [R_{s,m}]_{S\times M}, \quad \boldsymbol{\sigma} = [\sigma_{s,m}]_{S\times M}. \tag{6}$$

Mapping risk indicators to IF parameters

The objective is to map each risk indicator $R_{s,m}$ into an intuitionistic fuzzy pair:

$$\mathrm{IFS}_{s,m} = \Big(\mu_{s,m},\, \nu_{s,m}\Big), \tag{7}$$

defined in (7), representing the degrees of risk membership and non-membership. The hesitation degree is then determined separately so that the triplet $(\mu_{s,m}, \nu_{s,m}, \pi_{s,m})$ satisfies the IFS constraint in (1).

Membership degree

In the present study, the transformation in (8) is adopted as the primary mapping function, while (9) is used for sensitivity analysis.

A commonly used piecewise linear S-shaped transformation as given in (8):

$$\mu_{s,m} = \begin{cases} 0, & R_{s,m} \le a, \\ \dfrac{R_{s,m} - a}{b - a}, & a < R_{s,m} < b, \\ 1, & R_{s,m} \ge b, \end{cases} \tag{8}$$

where a and b are expert- or regulation-based parameters (e.g., $a = 0.5$, $b = 2$ for HQ-based assessments).

Alternatively, a smooth logistic S-function may be employed:

$$\mu_{s,m} = \frac{1}{1 + \exp\Big[-\alpha\,(\log R_{s,m} - \log R_0)\Big]}, \tag{9}$$

as expressed in (9), where R_0 denotes the inflection point and α controls the steepness of the transition.

Hesitation degree

The hesitation degree $\pi_{s,m}$ captures uncertainty and variability in the risk estimate. It is linked to the coefficient of variation and probabilistic exceedance. The coefficient of variation is defined in (10):

$$cv_{s,m} = \frac{\sigma_{s,m}}{\max(\varepsilon, \bar{R}_m)}, \quad \bar{R}_m = \frac{1}{S}\sum_{s=1}^{S} R_{s,m}, \tag{10}$$

where ε is a small positive constant ensuring numerical stability.

A practical formulation for hesitation is expressed as:

$$\pi_{s,m} = \min\left(\pi_{\max}, \frac{cv_{s,m}}{1 + cv_{s,m}}\left(1 - p_{s,m}\right), 1 - \mu_{s,m}\right), \tag{11}$$

according to (11). Here,

$$p_{s,m} = \Pr\Big(R_{s,m} > 1\Big), \tag{12}$$

defined in (12), denotes the probability of threshold exceedance, typically obtained via Monte Carlo simulation. Higher exceedance probability implies lower hesitation.

This additional upper bound ensures that the hesitation degree does not exceed the remaining degree after membership is assigned. Therefore, the resulting intuitionistic fuzzy representation always satisfies:

$$0 \le \mu_{s,m}, \nu_{s,m}, \pi_{s,m} \le 1, \qquad \mu_{s,m} + \nu_{s,m} + \pi_{s,m} = 1.$$

For cases in which $\mu_{s,m} = 1$, the hesitation degree is necessarily zero and the corresponding non-membership degree is also zero.

Non-membership degree

The non-membership degree is obtained by enforcing the IFS constraint, as defined in (13):

$$\nu_{s,m} = 1 - \mu_{s,m} - \pi_{s,m}. \tag{13}$$

Because (11) imposes the bound $\pi_{s,m} \le 1 - \mu_{s,m}$, equation (13) automatically yields $\nu_{s,m} \ge 0$; no additional normalization is required.

2.1 Aggregation of multiple metals into a site-specific risk index

For each site s, the analysis yields a set of intuitionistic fuzzy evaluations for individual metals, according to (14):

$$\Big(\mu_{s,1}, \nu_{s,1}\Big), \ldots, \Big(\mu_{s,M}, \nu_{s,M}\Big). \tag{14}$$

The objective of aggregation is to combine these M intuitionistic fuzzy sets into a single site-level intuitionistic fuzzy representation according to (15):

$$\Big(\mu_s, \nu_s\Big), \tag{15}$$

which summarizes the cumulative risk associated with multiple contaminants at location (s).

Component-wise weighted averaging scheme

Let w_m denote the weight associated with factor m, such that:

$$w_m \geq 0, \qquad \sum_{m=1}^{M} w_m = 1, \tag{16}$$

as required by (16). For each site s, the aggregated membership and non-membership degrees are computed as:

$$\mu_s = \sum_{m=1}^{M} w_m \mu_{s,m}, \qquad \nu_s = \sum_{m=1}^{M} w_m \nu_{s,m}, \tag{17}$$

using the component-wise weighted averaging scheme defined in (17).

The hesitation degree is then obtained by:

$$\pi_s = 1 - \mu_s - \nu_s, \tag{18}$$

as given in (18). Because each input satisfies $\mu_{s,m} + \nu_{s,m} \leq 1$ and the weights are non-negative and sum to one, the weighted averages satisfy $\mu_s + \nu_s \leq 1$, so (18) defines a valid aggregated hesitation degree.

Normalized weighted geometric comparative scheme

For comparative sensitivity analysis, a normalized weighted geometric scheme may also be considered. In the present manuscript it is not used as the principal uncertainty-preserving aggregation rule. The normalized form considered here is:

$$\mu_s = \frac{\prod_{m=1}^{M} \mu_{s,m}^{w_m}}{\prod_{m=1}^{M} \mu_{s,m}^{w_m} + \prod_{m=1}^{M} \nu_{s,m}^{w_m}}, \tag{19}$$

as expressed in (19).

The corresponding non-membership degree is computed as:

$$\nu_s = \frac{\prod_{m=1}^{M} \nu_{s,m}^{w_m}}{\prod_{m=1}^{M} \mu_{s,m}^{w_m} + \prod_{m=1}^{M} \nu_{s,m}^{w_m}}, \tag{20}$$

according to (20). Finally, the hesitation degree is determined by:

$$\pi_s = 1 - \mu_s - \nu_s, \tag{21}$$

as defined in (21).

The above formulation is included only as a comparative multiplicative scheme. Its normalization has the important consequence, discussed next, that the aggregated hesitation degree vanishes.

Remark on normalization and hesitation degree. In the above normalized geometric formulation the denominator is identical in the expressions for μ_s and ν_s; hence $\mu_s + \nu_s = 1$. Consequently, the aggregated hesitation degree becomes:

$$\pi_s = 1 - \mu_s - \nu_s = 0.$$

Thus, in this normalized geometric form the hesitation component vanishes at the aggregation level. While this formulation remains mathematically consistent and emphasizes multiplicative interactions among elements, it effectively reduces the aggregated intuitionistic fuzzy set to a normalized two–component representation (μ_s, ν_s).

In the present study, hesitation is explicitly modeled at the individual element level through Monte Carlo–derived uncertainty measures. Therefore, the normalized geometric scheme is used only for sensitivity or comparative purposes, whereas the component-wise weighted averaging scheme preserves the hesitation structure in the final aggregated risk representation.

Weighting schemes

The selection of weights w_m reflects the relative importance of each contaminant. Common weighting strategies include:

- **Toxicity-based weighting:**
$$w_m \propto \frac{1}{RfD_m}, \tag{22}$$
as defined in (22), assigning higher weights to more toxic substances.

- **Exposure-based weighting:**
$$w_m \propto \bar{R}_m, \tag{23}$$
according to (23), where $\bar{R}_m$ is the mean hazard quotient of metal m, emphasizing more prevalent risks.

- **Equal weighting:**
$$w_m = \frac{1}{M}, \tag{24}$$
as given in (24), used when no prior preference or toxicological hierarchy is assumed.

Weights are normalized to satisfy $\sum_{m=1}^{M} w_m = 1$. A sensitivity analysis with respect to different weighting schemes is recommended.

Defuzzification and risk score

The aggregation procedure described above produces an intuitionistic fuzzy representation of the overall risk at each site in the form:

$$A_s = \langle \mu_s, \nu_s, \pi_s \rangle.$$

Although this representation preserves the membership, non-membership, and hesitation components of the risk assessment, a single numerical indicator is required for ranking and comparison of the investigated sites. Therefore, a defuzzification step is introduced. A commonly used score function for intuitionistic fuzzy sets is defined as:

$$S_s = \mu_s - \nu_s, \tag{25}$$

where $S_s \in [-1, 1]$ represents the net tendency of the aggregated intuitionistic fuzzy assessment toward membership in the considered risk category.

For the purposes of risk ranking, the score is normalized to the interval $[0, 1]$ according to:

$$R_s^* = \frac{1 + S_s}{2} = \frac{1 + \mu_s - \nu_s}{2}. \tag{26}$$

The resulting value R_s^* is referred to as the defuzzified risk score. Values closer to 1 indicate a stronger association with the considered risk condition, whereas values closer to 0 indicate a stronger association with the non-risk condition. The hesitation degree π_s is retained in the underlying intuitionistic fuzzy representation and therefore provides information about the degree of uncertainty associated with the assessment.

For the component-wise weighted averaging scheme, the defuzzified risk score is obtained directly from the aggregated membership and non-membership degrees using (26). In contrast, for the normalized geometric formulation discussed above, $\pi_s = 0$, and the resulting score is determined entirely by the normalized membership and non-membership degrees. Thus, the defuzzification step provides a common numerical scale for comparing the results obtained with the two aggregation operators.

Several de-i-fuzzification procedures for intuitionistic fuzzy information have been proposed in the literature [7–9, 19]. For the present ranking task, the classical score $S_s = \mu_s - \nu_s$ is simply rescaled affinely from $[-1, 1]$ to $[0, 1]$ by (26); this rescaling does not alter the ordering of sites.

2.2 Ranking and decision-making criteria

After aggregation, each site s is characterized by a single intuitionistic fuzzy triplet as expressed in (see Eq.15).

A convenient scalar decision index is the intuitionistic fuzzy score, defined in (see Eq.27):

$$\text{score}_s = \mu_s - \nu_s, \qquad \text{score}_s \in [-1, 1]. \tag{27}$$

Positive values ($\text{score}_s > 0$) indicate dominance of risk over safety, whereas negative values indicate predominantly acceptable conditions.

An alternative measure of confidence or certainty is given by:

$$\text{certainty}_s = 1 - \pi_s, \tag{28}$$

as defined in (28), which quantifies the reliability of the risk classification.

Ranking and discretization of risk levels

Sites may be ranked in ascending or descending order of score_s and/or μ_s, depending on the management objective.

A practical discretization scheme is as follows:

- $\mu_s \geq 0.75$: high priority for intervention;
- $0.5 \leq \mu_s < 0.75$: medium priority;
- $\mu_s < 0.5$ and $\pi_s < 0.3$: low priority.

This classification supports transparent and uncertainty-aware decision-making in multi-contaminant environmental risk assessments [26].

3 Integration of intuitionistic fuzzy sets with a Monte Carlo simulation

Intuitionistic fuzzy modeling and Monte Carlo (MC) simulations are complementary tools for environmental risk assessment. MC uses random sampling and probability sampling. Thus, it can solve approximately current problems that could not be solved with exact algorithms or traditional mathematical methods. Monte Carlo methods are mainly used in three classes of problems: optimization; solving multidimensional integrals; and systems of differential equations. This type of method is used in many fields such as financial mathematics, physics, biology, chemistry, and others [24, 25].

To apply the Monte Carlo method, the problem must be presented with a probabilistic interpretation. Simply put, if we have an expected value m and cannot calculate it exactly, then the Monte Carlo method provides an estimate for m by simulating it n times and averaging the result. The important thing is that the entry points are randomly generated. If the number of simulations is large enough, then the resulting estimate will be close enough to the value of m [15].

While intuitionistic fuzzy sets (IFS) explicitly represent epistemic uncertainty and hesitation, Monte Carlo simulation propagates stochastic variability in exposure and toxicity parameters.

Monte Carlo-based generation of risk distributions

For each site s and metal m, Monte Carlo simulation is applied to generate a set of K realizations of the risk metric (see Eq. (29)):

$$\left\{ R_{s,m}^{(k)} \right\}_{k=1}^{K}, \tag{29}$$

which allows explicit probabilistic characterization of risk.

The Monte Carlo realizations are used to characterize the probabilistic distribution of the risk metric. From these realizations, a representative risk value (in the numerical illustration, the mean hazard quotient), the empirical threshold-exceedance probability $p_{s,m}$, and a dispersion measure are estimated. The representative risk value is then mapped to the membership degree through the monotone transformation in (8) or (9), while the Monte Carlo-derived exceedance probability and dispersion enter the hesitation function (11). The non-membership degree is subsequently obtained from (13).

Thus, the Monte Carlo stage provides probabilistic summary information that is converted into one valid intuitionistic fuzzy assessment $(\mu_{s,m}, \nu_{s,m}, \pi_{s,m})$ for each site–metal pair; the manuscript does not require a separate realization-by-realization definition of $\nu_{s,m}^{(k)}$ or $\pi_{s,m}^{(k)}$.

Estimation of IFS parameters from Monte Carlo outputs

The empirical threshold-exceedance probability used in the hesitation function is estimated from the Monte Carlo realizations as (see Eq. (30)):

$$p_{s,m} = \frac{1}{K} \sum_{k=1}^{K} \mathbf{1}\left\{ R_{s,m}^{(k)} > 1 \right\}, \tag{30}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function.

The dispersion of the Monte Carlo outcomes is quantified by (see Eq. (31)):

$$\sigma_{s,m} = \mathrm{std}_k \left\{ R_{s,m}^{(k)} \right\}, \tag{31}$$

which is subsequently used to parameterize the hesitation degree $\pi_{s,m}$.

This framework explicitly links probabilistic variability (via Monte Carlo simulation) with intuitionistic uncertainty representation.

3.1 Metric measures and model validation

Entropy and uncertainty quantification

The uncertainty of the three-component intuitionistic fuzzy representation is quantified here using the normalized Shannon-type entropy (see Eq. (32)):

$$E(\mathrm{IFS}) = -\frac{1}{\ln 3}\Big(\mu \ln \mu + \nu \ln \nu + \pi \ln \pi\Big), \tag{32}$$

where $E(\mathrm{IFS}) \in [0, 1]$, with the convention $0 \ln 0 = 0$.

Higher entropy values indicate greater uncertainty or hesitation in the risk assessment.

Distance and similarity measures

For ranking and classification purposes, the distance between an intuitionistic fuzzy set and ideal reference points is employed.

The ideal risk state is defined as (see Eq. (33)):

$$I_{\text{risk}} = (1, 0, 0), \tag{33}$$

while the ideal safe state is (see Eq. (34)):

$$I_{\text{safe}} = (0, 1, 0). \tag{34}$$

The normalized squared Euclidean distance to the ideal risk point is given by (see Eq. (35)):

$$d(\mathrm{IFS}, I_{\mathrm{risk}}) = \frac{1}{2}\Big[(\mu - 1)^2 + (\nu - 0)^2 + (\pi - 0)^2\Big]. \tag{35}$$

Smaller distances correspond to higher risk levels, allowing an alternative ordering of sites beyond scalar score functions.

Proposed validation strategy

The following complementary procedures are suitable for further validation of the proposed framework:

- comparison with classical deterministic indices by assessing the correlation between μ_s and the hazard index (HI);
- consistency checks against epidemiological or health outcome data, when available;
- sensitivity analysis with respect to key parameters a, b, α, $\pi_{\max}$, and the weighting scheme w_m;
- cross-validation by resampling subsets of Monte Carlo realizations and evaluating the stability of site rankings.

Applying these validation steps to sufficiently large real-world datasets would permit a more rigorous assessment of robustness, interpretability, and methodological consistency.

Compact formulation of the computational framework

This section summarizes the mathematical formulation of the proposed risk assessment framework, integrating classical exposure modeling, intuitionistic fuzzy sets, and aggregation operators.

Average daily dose and hazard quotient

For each site s and element m, the average daily dose (ADD) is computed as (see Eq. (36)):

$$\mathrm{ADD}_{s,m} = \frac{C_{s,m} \cdot IR \cdot EF \cdot ED}{BW \cdot AT}, \tag{36}$$

where $C_{s,m}$ is the measured concentration, IR is the ingestion rate, EF is the exposure frequency, ED is the exposure duration, BW is body weight, and AT is the averaging time.

The corresponding hazard quotient (HQ) is defined as (see Eq. (37)):

$$R_{s,m} = \mathrm{HQ}_{s,m} = \frac{\mathrm{ADD}_{s,m}}{\mathrm{RfD}_m}, \tag{37}$$

where RfD_m denotes the reference dose of element m.

Intuitionistic fuzzy representation per element

Each element-specific risk value is mapped to an intuitionistic fuzzy set (see Eq. (38)):

$$\mathrm{IFS}_{s,m} = \Big(\mu_{s,m}, \nu_{s,m}\Big), \tag{38}$$

subject to the Atanassov constraint (see Eq. (39)):

$$\mu_{s,m}, \nu_{s,m}, \pi_{s,m} \in [0,1], \qquad \mu_{s,m} + \nu_{s,m} + \pi_{s,m} = 1. \tag{39}$$

Here, $\mu_{s,m}$ represents the degree of health risk, $\nu_{s,m}$ the degree of safety, and $\pi_{s,m}$ the hesitation degree reflecting uncertainty.

Aggregation across elements by component-wise weighted averaging

To obtain a site-level risk indicator, component-wise weighted averaging is applied (see Eq. 40,41,42):

$$\mu_s = \sum_{m=1}^{M} w_m \mu_{s,m}, \tag{40}$$

$$\nu_s = \sum_{m=1}^{M} w_m \nu_{s,m}, \tag{41}$$

$$\pi_s = 1 - \mu_s - \nu_s, \tag{42}$$

where $w_m \geq 0$ and $\sum_{m=1}^{M} w_m = 1$.

The weights w_m may be defined based on toxicity (e.g., inverse RfD_m), exposure intensity, or set uniformly.

Score function and ranking

A scalar score function is used to rank sites according to overall risk (see Eq. 43):

$$\mathrm{score}_s = \mu_s - \nu_s, \qquad \mathrm{score}_s \in [-1, 1]. \tag{43}$$

Higher score values indicate dominance of health risk over safety.

4 Uncertainty, aggregation, and decision support

Entropy-based uncertainty measure

The uncertainty associated with the aggregated intuitionistic fuzzy set at site s is quantified using an entropy-based measure (see Eq. 44):

$$E_s = -\frac{1}{\ln 3}\Big(\mu_s \ln \mu_s + \nu_s \ln \nu_s + \pi_s \ln \pi_s\Big), \tag{44}$$

where $E_s \in [0,1]$, with the convention $0 \ln 0 = 0$. Higher entropy values indicate greater uncertainty in the risk assessment.

For clarity and reproducibility, Table 1 summarizes all symbols and units used in the proposed framework.

Table 1. Symbols, definitions, and units used in the risk assessment framework

Symbol	**Description**	**Unit**
$C_{s,m}$	Concentration of metal m at site s	mg kg^{-1}
IR	Ingestion rate	mg day^{-1}
EF	Exposure frequency	days year^{-1}
ED	Exposure duration	years
BW	Body weight	kg
AT	Averaging time	days
$\mathrm{ADD}_{s,m}$	Average daily dose	mg kg^{-1} day^{-1}
RfD_m	Reference dose of metal m	mg kg^{-1} day^{-1}
$R_{s,m}$	Hazard quotient (HQ)	–
$\mu_{s,m}$	Degree of risk membership	–
$\nu_{s,m}$	Degree of non-risk membership	–
$\pi_{s,m}$	Degree of hesitation	–
w_m	Weight of metal m	–
score_s	Aggregated risk score	–
E_s	Intuitionistic fuzzy entropy	–

Numerical example

To demonstrate the proposed methodology, a numerical example involving two sites ($S1$, $S2$) and three metals (As, Cd, Pb) is considered. Monte Carlo simulations provide the mean hazard quotient $R_{s,m}$, coefficient of variation (CV), and exceedance probability $p_{s,m} = \Pr(R_{s,m} > 1)$, summarized in Table 2.

Table 2. Input data for the numerical example

Site	**Metal**	$\boldsymbol{R_{s,m}}$	**CV**	$\boldsymbol{p_{s,m}}$
$S1$	As	1.8	0.40	0.65
$S1$	Cd	0.9	0.30	0.20
$S1$	Pb	2.5	0.50	0.80
$S2$	As	0.6	0.25	0.10
$S2$	Cd	1.2	0.35	0.40
$S2$	Pb	0.8	0.20	0.15

The IFS mapping parameters are set to $a = 0.5$, $b = 2.0$, and $\pi_{\max} = 0.6$, with equal weights $w_m = 1/3$.

Worked example: Site $S1$

For site $S1$, the intuitionistic fuzzy parameters obtained for each metal are:

Metal	$\mu_{s,m}$	$\pi_{s,m}$	$\nu_{s,m}$
As	0.87	0.10	0.03
Cd	0.27	0.18	0.55
Pb	1.00	0.00	0.00

Aggregation using the component-wise weighted averaging scheme with equal weights yields:

$$\mu_{S1} = 0.71, \quad \nu_{S1} = 0.19, \quad \pi_{S1} = 0.10,$$

with an overall score $\mathrm{score}_{S1} = 0.52$, indicating a high priority for intervention.

For comparison, site $S2$ yields:

$$\mu_{S2} = 0.24, \quad \nu_{S2} = 0.60, \quad \pi_{S2} = 0.16, \qquad \mathrm{score}_{S2} = -0.35,$$

corresponding to low priority.

Algorithmic implementation

The complete Monte Carlo–based intuitionistic fuzzy risk assessment procedure is summarized in Algorithm 1.

Algorithm 1 Monte Carlo–based intuitionistic fuzzy health risk assessment

Require: Concentrations $C_{s,m}$, exposure parameters, reference doses RfD_m
Ensure: Aggregated IFS (μ_s, ν_s, π_s) and risk score score_s

for each site s **do**
  for each metal m **do**
    Generate K Monte Carlo samples
    Compute $\mathrm{ADD}^{(k)}_{s,m}$ and $R^{(k)}_{s,m}$
    Estimate $p_{s,m} = \Pr(R_{s,m} > 1)$ and $\sigma_{s,m}$
    Map $R_{s,m}$ to IFS $(\mu_{s,m}, \nu_{s,m}, \pi_{s,m})$
  end for
  Aggregate metals using component-wise weighted averaging to obtain (μ_s, ν_s, π_s)
  Compute $\mathrm{score}_s = \mu_s - \nu_s$
end for
return Site ranking based on score_s

Algorithm 1 presents a software-independent pseudo-code of the proposed Monte Carlo–based intuitionistic fuzzy health risk assessment framework. In this study, the algorithm is implemented in the MATLAB environment; however, the computational procedure is general and can be readily adapted to other numerical platforms. The numerical calculations and graphical outputs are described as generated using a MATLAB-based implementation of Algorithm 1. Numerical values affected by the revised hesitation mapping should be verified against that implementation before final publication.

Sensitivity analysis

An illustrative sensitivity summary for selected model parameters is presented in Table 3. Because the hesitation mapping has been revised, the numerical entries in this table should be checked against the final MATLAB implementation before publication.

Table 3. Illustrative sensitivity summary for selected model parameters

Parameter	Baseline	Variation	Δ score
a	0.5	$\pm 20\%$	± 0.06
b	2.0	$\pm 20\%$	± 0.04
$\pi_{\max}$	0.6	$\pm 20\%$	± 0.08
w_m	equal	toxicity-based	$+0.05$

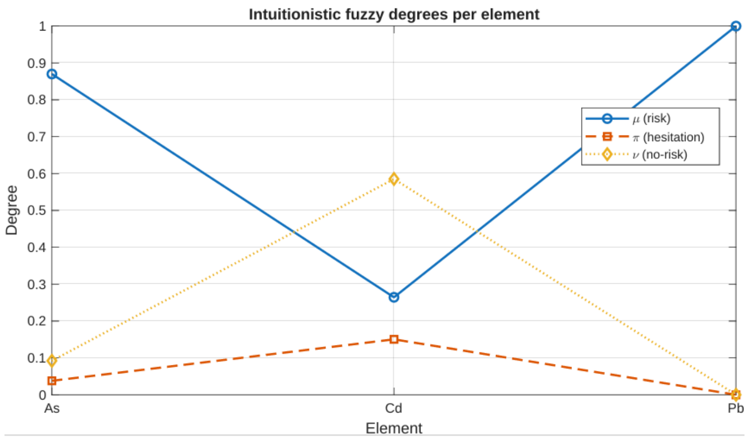

Figure 1. Intuitionistic fuzzy degrees per element

Monte Carlo simulation is used at the level of hazard quotients (HQ) to propagate uncertainty in exposure and toxicity parameters. For each element (As, Cd, Pb), the resulting HQ summaries are mapped to intuitionistic fuzzy membership (μ), non-membership (ν), and hesitation (π) degrees according to the proposed transformation. Figure 1 illustrates the corresponding element-level intuitionistic fuzzy degrees for the illustrative data used in this section.

Conceptual comparison with the classical hazard index

The methodological workflow follows a sequential structure:

1. Calculation of exposure and hazard quotients.
2. Uncertainty propagation using Monte Carlo simulation.
3. Mapping of risk indicators to intuitionistic fuzzy parameters.
4. Aggregation across multiple contaminants.
5. Decision support through risk ranking and prioritization.

Conceptual workflow:
HQ $\rightarrow$ Monte Carlo $\rightarrow$ IFS mapping $\rightarrow$ aggregation $\rightarrow$ decision support.

Figure 2. Conceptual flowchart of the proposed Monte Carlo–IFS framework.

The relationship between the classical hazard index (HI) and the intuitionistic fuzzy membership degree μ is monotonic by construction, as both are derived from the same underlying exposure indicators. However, unlike HI, the proposed framework explicitly incorporates uncertainty through the hesitation parameter π. Conceptually, a scatter representation of HI versus μ would reveal a positive association, while allowing differentiation between sites with similar HI values but different uncertainty levels. This additional dimension cannot be captured by deterministic risk indices.

The sensitivity scenarios summarized in Table 3 are intended to illustrate how changes in a, b, $\pi_{\max}$, and w_m may affect the resulting membership and score values. The corresponding numerical values and any claim of ranking robustness should be confirmed from the final implementation after all mapping corrections have been incorporated.

5 Conclusion

This study introduced a Monte Carlo–based intuitionistic fuzzy framework for human health risk assessment under uncertainty. By integrating probabilistic exposure modeling with intuitionistic fuzzy set theory, the proposed approach extends classical deterministic and probabilistic risk indices toward a more informative and decision-oriented representation of health risk.

The framework preserves the interpretability of traditional hazard quotients while explicitly incorporating uncertainty through the intuitionistic fuzzy parameters of membership, non-membership, and hesitation. Monte Carlo simulation enables the propagation of variability in exposure and toxicity parameters, whereas the intuitionistic fuzzy mapping translates probabilistic risk indicators into a structured representation suitable for aggregation and comparison. The component-wise weighted averaging scheme allows multiple contaminants to be combined into a single site-level assessment while retaining the aggregated hesitation component. In addition, the entropy-based measure provides a compact and normalized indicator of uncertainty in the aggregated risk evaluation. The numerical example demonstrates that the proposed framework can differentiate sites not only by risk magnitude but also by uncertainty, thereby offering additional insight beyond classical hazard index approaches. Sites with similar deterministic risk levels may exhibit substantially different hesitation degrees, which is critical for prioritization and risk management decisions. Conceptual comparison with the classical hazard index highlights that the intuitionistic fuzzy membership degree remains consistent with traditional risk signals, while the hesitation component captures information that is otherwise lost in deterministic frameworks.

Despite these advantages, several limitations should be acknowledged. The choice of mapping parameters and aggregation weights may influence the results and should be informed by expert judgment or regulatory guidance. Moreover, the present study focuses on non-carcinogenic risk pathways and illustrative numerical data; application to real-world datasets and carcinogenic risk metrics warrants further investigation.

Future research may extend the proposed framework by incorporating alternative intuitionistic fuzzy aggregation operators, dynamic or data-driven weight determination, and spatially explicit risk modeling. Integration with geographic information systems and stakeholder-oriented decision support tools also represents a promising direction. Overall, the Monte Carlo–intuitionistic fuzzy approach provides a robust and flexible methodological basis for health risk assessment in complex and uncertain environmental systems.

Acknowledgements

The work was partially supported by the Centre of Excellence in Informatics and ICT under the Grant No BG16RFPR002-1.014-0018-C01, financed by the Research, Innovation and Digitalization for Smart Transformation Programme 2021–2027 and co-financed by the European Union. The authors are grateful to the anonymous reviewers for their insightful comments.

References

[1] Atanassov, K. (1983). Intuitionistic fuzzy sets. *VII ITKR Session*, Sofia, 20–23 June 1983. Reprinted in *International Journal Bioautomation*, 20(S1) (2016), S1–S6.

[2] Atanassov, K. (1997). *Generalized Nets and Systems Theory*. Prof. M. Drinov Academic Publishing House, Sofia.

[3] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*. Springer, Berlin.

[4] Atanassov, K. T. (2014). *Index Matrices: Towards an Augmented Matrix Calculus*. Studies in Computational Intelligence 573, Springer, Cham.

[5] Atanassov, K., Szmidt, E., Kacprzyk, J. (2013). On intuitionistic fuzzy pairs. *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[6] Atanassov, K., Vassilev, P. (2020). Intuitionistic fuzzy sets and other fuzzy sets extensions representable by them. *Journal of Intelligent & Fuzzy Systems*, 38(1), 525–530.

[7] Atanassova, L., & Dworniczak, P. (2021). Operation as the tool for the de-i-fuzzification procedure and for the correction of the unconscientious experts' evaluations. *Notes on Intuitionistic Fuzzy Sets*, 27(4), 9–19.

[8] Atanassova, V., & Sotirov, S. (2012). A new formula for de-i-fuzzification of intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 18(3), 49–51.

[9] Ban, A., Kacprzyk, J., & Atanassov, K. (2008). On de-I-fuzzification of intuitionistic fuzzy sets. *Comptes Rendus de l'Academie bulgare des Sciences*, 61(12), 1535–1540.

[10] Briffa, J., Sinagra, E., & Blundell, R. (2020). Heavy metal pollution in the environment and their toxicological effects on humans. *Heliyon*, 6, Article ID e04691.

[11] Bulgarian State Standard (2005). *Water quality. Application of inductively coupled plasma mass spectrometry (ICP-MS).* BSS EN ISO 17294-2:2023.

[12] Ivanova, D., Nikolaeva, Z., & Dimitrov, A. (2022). Study of the concentration of heavy metals in silt loading from transport arteries in the city of Burgas, Bulgaria. *Oxidation Communications*, 45(2), 390–397.

[13] Ivanova, D., Tasheva, Y., Sotirova, E., Dimitrov, A., & Sotirov, S. (2023). Prediction of the granulometric composition of the silt loading on transport arteries in the city of Bourgas based on artificial neural networks. *Lecture Notes in Networks and Systems* (Vol. 658, pp. 21–31). Springer, Cham.

[14] Jayarathne, A., Egodawatta, P., Ayoko, G. A., & Goonetilleke, A. (2018). Assessment of ecological and human health risks of metals in urban road dust based on geochemical fractionation and potential bioavailability. *Science of the Total Environment*, 635, 1609–1619.

[15] Kroese, D. P., Brereton, T., Taimre, T., & Botev, Z. I. (2014). Why the Monte Carlo method is so important today. *WIREs Comput Stat.*, 6(6), 386–392.

[16] Li, J., Cui, D., Yang, Z., Ma, J., Liu, J., Yu, Y., Huang, X., & Xiang, P. (2024). Health risk assessment of heavy metal(loid)s in road dust via dermal exposure pathway from a low latitude plateau provincial capital city: The importance of toxicological verification. *Environmental Research*, 252, Article ID 118890.

[17] Mendel, J. M. (2001). *Uncertain Rule-based Fuzzy Logic Systems*. Prentice Hall, Upper Saddle River.

[18] Novák, V., Perfilieva, I., Močkoř, J. (1999). *Mathematical Principles of Fuzzy Logic*. Kluwer Academic Publishers, Dordrecht.

[19] Patrascu, V. (2024). Two de-I-fuzzification procedures for intuitionistic fuzzy information. *Notes on Intuitionistic Fuzzy Sets*, 30 (1), 18–25.

[20] Pedrycz, W., & Gomide, F. (2021). *Fuzzy Systems Engineering: Toward Human-Centric Computing*. Wiley-IEEE Press, Hoboken.

[21] Potocki, S., Rowinska-Zyrek, M., Witkowska, D., Pyrkosz, M., Szebesczyk, A., Krzywoszynska, K., & Kozlowski, H. (2012). Metal transport and homeostasis within the human body: Toxicity associated with transport abnormalities. *Current Medicinal Chemistry*, 19(17), 2738–2759.

[22] Ross, T. J. (2010). *Fuzzy Logic with Engineering Applications*. 3rd ed., Wiley, Chichester.

[23] Sotirov, S., Sotirova, E., Werner, M., Simeonov, S., Hardt, W., & Simeonova, N. (2016). Intuitionistic fuzzy estimation of the generalized nets model of spatial-temporal group scheduling problems. In: *Imprecision and Uncertainty in Information Representation and Processing*, Studies in Fuzziness and Soft Computing 332, Springer, Cham, 401–414.

[24] Todorov, V., Georgiev, S., Lazarova, M., Todorov, M., Sapundzhi, M., Traneva, V., & Petrov, M. (2026). An enhanced lattice rule with an optimized generating vector for high-dimensional sensitivity analysis. *Lecture Notes in Networks and Systems* (Vol. 1799, pp. 163–175), Springer, Cham.

[25] Todorov, V., & Zhivkov, P. (2025). Efficient Evaluation of Sobol' sensitivity indices via polynomial lattice rules and modified sobol' sequences. *Mathematics*, 13(21), Article ID 3402.

[26] Yager, R. R., & Zadeh, L. A. (Eds.) (1992). *An Introduction to Fuzzy Logic Applications in Intelligent Systems*. Springer, Boston.

[27] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8(3), 338–353.

[28] Zimmermann, H. J. (2001). *Fuzzy Set Theory and Its Applications*. 4th ed., Springer, Boston.