Notes on Intuitionistic Fuzzy Sets
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On the translational invariant intuitionistic fuzzy subset of a Γ-ring

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Abstract: In this paper, we introduce the notion of translational invariant intuitionistic fuzzy subset of a Γ-ring and generalize some notions of a ring to a Γ-ring. Also, we define ideals of a Γ-ring generated by an intuitionistic fuzzy subset with an element of Γ-ring and study their properties. The notion of units, associate, prime element, irreducible element are also generalized with respect to the intuitionistic fuzzy subset of a Γ-ring. Further, we study the properties of homomorphic image and pre-image of translational invariant intuitionistic fuzzy subset under the Γ-ring homomorphism and we prove that every homomorphic image of a prime ideal of a Γ-ring generated by an A_γ-prime element and translational invariant and f-invariant intuitionistic fuzzy subset is also a prime ideal.
Keywords: Γ-Ring, Translational invariant intuitionistic fuzzy subset (TIIFS), f-invariant intuitionistic fuzzy subset, A_γ-unit, A_γ-prime element.
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1 Introduction

The notion of a Γ-ring was introduced by N. Nobusawa [9] as more general than the notion of a ring. W. E. Barnes [2] weakened slightly the conditions in the definition of Γ-rings in the sense of N. Nobusawa. The structure of Γ-rings can be found in [11]. The notion of intuitionistic fuzzy set was introduced by K. T. Atanassov [1] to generalize the notion of fuzzy set given by L. A. Zadeh [14]. R. Biswas [4] was the first one to introduce the concept of intuitionistic fuzzy subgroup of a group and established many important properties. The notion of intuitionistic fuzzy subring and ideal in a ring was introduced by K. Hur et al. in [6, 7]. K. H. Kim et al. in [8] have studied intuitionistic fuzzy ideal of Γ-rings which was further studied by N. Palaniappan et al. in [10]. A. K. Ray [12] introduced the concept of translational invariant fuzzy subset in a ring. A. K. Ray and T. Ali in [13] also studied ideals and divisibility in a ring with respect to a fuzzy subset. Y. Bhargavi [3] studied the translational invariant vague set of a Γ-semiring. The purpose of this paper is to generalize some of the classical results of ring theory using the notion of a translational invariant intuitionistic fuzzy subset (TIIFS) of a Γ-ring.

2 Preliminaries

In this section, we list some basic concepts and definitions on Γ-rings theory and intuitionistic fuzzy sets theory, which are necessary for the better understanding of the paper.

Definition 2.1 ([2]). If $(M, +)$ and $(\Gamma, +)$ are additive Abelian groups, then M is called a Γ-ring if there exists a mapping $f : M \times \Gamma \times M \to M$, where $f(x, \alpha, y)$ is denoted by $x\alpha y$, $x, y \in M, \gamma \in \Gamma$ satisfying the following conditions:

(1) $x\alpha y \in M$.

(2) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$.

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$. for all $x, y, z \in M$, and $\gamma \in \Gamma$.

These conditions are further strengthened by defining another function $g : \Gamma \times M \times \Gamma \to \Gamma$, where $g(\alpha, x, \beta)$ is denoted by $\alpha x\beta$, $x \in M$, $\alpha, \beta \in \Gamma$, satisfying the following conditions for all $x, y, z \in M$ and for all $\alpha, \beta, \gamma \in \Gamma$,

$(1')$ $x\alpha y \in M$, $\alpha x\beta \in \Gamma$.

$(2')$ $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$.

$(3')$ $(x\alpha y)\beta z = x\alpha(y\beta z)$.

$(4')$ $x\alpha y = 0_M$ for all $x, y \in M$ implies $\alpha = 0_\Gamma$.

We then have a Γ-ring in the sense of Nobusawa [9].

Definition 2.2 ([2, 10]). A subset N of a Γ-ring M is a left (right) ideal of M if N is an additive subgroup of M and

$$M\Gamma N = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in N\}, (N\Gamma M)$$

is contained in N. If N is both a left and a right ideal, then N is a two-sided ideal, or simply an ideal of M.

Definition 2.3 ([13]). A Γ-ring M is said to be a commutative Γ-ring if $x\gamma y = y\gamma x, \forall x, y \in M$, $\gamma \in \Gamma$.

Definition 2.4 ([13]). Let M be a Γ-ring. An element $e \in M$ is said to be unity if for each $x \in M$ there exists $\gamma \in \Gamma$ such that $x\gamma e = e\gamma x = x$.

Definition 2.5 ([13]). An ideal P of a Γ-ring M is said to be prime ideal of M if for any $x, y \in M$, $\gamma \in \Gamma$, $x\gamma y \in P$ implies that $x \in P$ or $y \in P$.

Definition 2.6 ([2, 13]). Let M and $M^{'}$ be two Γ-rings. Then $f : M \to M^{'}$ is called a Γ-homomorphism if

- $f(x + y) = f(x) + f(y)$
- $f(x\gamma y) = f(x)\gamma f(y)$, for all $x, y \in M, \gamma \in \Gamma$.

Definition 2.7 ([1]). An intuitionistic fuzzy set A in X can be represented as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Remark 2.8 ([1]). (i) When $\mu_A(x) + \nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x) = \mu_{A^c}(x)$, then A is called a fuzzy set.
(ii) An intuitionistic fuzzy set (IFS) $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is shortly denoted by $A(x) = (\mu_A(x), \nu_A(x))$, for all $x \in X$.

Proposition 2.9 ([1]). *If A, B be two intuitionistic fuzzy sets of X, then*

(i) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$;

(ii) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$, i.e., $A(x) = B(x)$, for all $x \in X$.

Further if $f : X \to Y$ is a mapping and A, B be respectively IFS of X and Y, then the image $f(A)$ is an IFS of Y defined as $\mu_{f(A)}(y) = \sup\{\mu_A(x) : f(x) = y\}$, $\nu_{f(A)}(y) = \inf\{\nu_A(x) : f(x) = y\}$, for all $y \in Y$ and the inverse image $f^{-1}(B)$ is an IFS of X defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$, for all $x \in X$, i.e., $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$. Also the IFS A of X is said to be f-invariant if for any $x, y \in X$, whenever $f(x) = f(y)$ implies $A(x) = A(y)$.

3 Translational invariant intuitionistic fuzzy subset of a Γ-ring

Throughout this section, M is a Γ-ring with unities and the zero element θ.

Definition 3.1. Let A be an intuitionistic fuzzy subset of M. A is called a left translational invariant intuitionistic fuzzy subset with respect to the internal addition if $A(x) = A(y)$ implies that $A(x + m) = A(y + m)$, for all $x, y, m \in M$. Again A is called a left translational invariant intuitionistic fuzzy subset with respect to the external multiplication if $A(x) = A(y)$ implies that $A(m\gamma x) = A(m\gamma y)$, for all $x, y, m \in M$ and for all $\gamma \in \Gamma$. Similarly, we can define the notion of right translational invariant intuitionistic fuzzy subset with respect to the operation (addition, multiplication) in M.

Remark 3.2. An IFS A is said to be commutative under internal addition (or external multiplication) on M if $A(x + y) = A(y + x)$ (or $A(x\gamma y) = A(y\gamma x)$), for all $x, y \in M, \gamma \in \Gamma$. Therefore, when A is commutative, then the two notion coincides. In this case, we say that A is a translational invariant intuitionistic fuzzy subset (TIIFS) of M with respect to the operation $+$ (or $\times$).

Example 3.3. Consider the Γ-ring M, where $M = Z$ the ring of integers and $\Gamma = 2Z$, the ring of even integers and $x\gamma y$ denotes the usual product of integers x, γ, y. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of M defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \text{ is an even integer} \\ 0.5, & \text{if } x \text{ is an odd integer} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \text{ is an even integer} \\ 0.3, & \text{if } x \text{ is an odd integer.} \end{cases}$$

Then it is easy to verify that A is an TIIFS of M with respect to both the operation $+$ and $\times$.

Example 3.4. Consider the Γ-ring M, where $M = \{[a_{ij}] : a_{ij} \in Z_2, i = 1, j = 1, 2\}$, the set of (1×2) matrices whose entries are from Z_2 and $\Gamma = \{[a_{ij}] : a_{ij} \in Z_2, i = 1, 2, j = 1\}$, the set of (2×1) matrices whose entries are from Z_2. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of R defined by

$$\mu_A(a_{ij}) = \begin{cases} 0.7, & \text{if } a_{11} = a_{12} = 0 \\ 0.7, & \text{if } a_{11} = 1, a_{12} = 0 \\ 0.3, & \text{if } a_{11} = 0, a_{12} = 1 \\ 0.3, & \text{if } a_{11} = a_{12} = 1. \end{cases} ; \quad \nu_A(a_{ij}) = \begin{cases} 0.2, & \text{if } a_{11} = a_{12} = 0 \\ 0.2, & \text{if } a_{11} = 1, a_{12} = 0 \\ 0.5, & \text{if } a_{11} = 0, a_{12} = 1 \\ 0.5, & \text{if } a_{11} = a_{12} = 1. \end{cases}$$

Then it is easy to verify that A is an TIIFS of M with respect to both the operation addition of matrices and multiplication of matrices defined on M.

From this point onwards, every intuitionistic fuzzy subset A of a Γ-ring M satisfies the property $A(-x) = A(x)$, for all $x \in M$.

Proposition 3.5. *Let A be a TIIFS with respect to both internal addition and external multiplication operations defined on M. Then for any $m \in M$ the set*

$$L(m, \gamma, A) = \{x : x \in M \textit{ such that } A(x) = A(y\gamma m), \textit{ for some } y \in M\}$$

is a left ideal of M.

Proof. Clearly $L(m, \gamma, A) \neq \emptyset$, since $\theta \in L(m, \gamma, A)$ as $A(\theta) = A(\theta\gamma m)$. Let $x_1, x_2 \in L(m, \gamma, A)$. Then $A(x_1) = A(y_1\gamma m)$ and $A(x_2) = A(y_2\gamma m)$, for some $y_1, y_2 \in M$. Now

$$A(x_1) = A(y_1\gamma m) \Rightarrow A(x_1 - x_2) = A(y_1\gamma m - x_2) = A(x_2 - y_1\gamma m) \tag{i}$$

and

$$A(x_2) = A(y_2\gamma m) \Rightarrow A(x_2 - x_1) = A(y_2\gamma m - x_1) = A(x_1 - y_2\gamma m) \tag{ii}$$

From (i) and (ii) we get $A(x_1-y_2\gamma m) = A(x_2-y_1\gamma m) \Rightarrow A(x_1-x_2) = A(y_1\gamma m-y_2\gamma m) = A((y_1 - y_2)\gamma m)$. Thus, $x_1 - x_2 \in L(m, \gamma, A)$, since $(y_1 - y_2) \in M$.
Also, for any $y_3 \in M$ and $\gamma_1 \in \Gamma$, we have $A(y_3\gamma_1 x_1) = A(y_3\gamma_1(y_1\gamma m)) = A((y_3\gamma_1 y_1)\gamma m) \Rightarrow y_3\gamma_1 x_1 \in L(m, \gamma, A)$ for any $y_3 \in M$ and for any $\gamma_1 \in \Gamma$. Hence $L(m, \gamma, A)$ is a left ideal of M. □

Analogously we can prove:

Proposition 3.6. *Let A be a TIIFS with respect to both internal addition and external multiplication operations defined on M. Then for any $m \in M$ the set*

$$R(m, \gamma, A) = \{x : x \in M \textit{ such that } A(x) = A(m\gamma y), \textit{ for some } y \in M\}$$

is a right ideal of M.

Remark 3.7. If M is a commutative Γ-ring, then $L(m, \gamma, A) = R(m, \gamma, A), \forall m \in M$ and for all $\gamma \in \Gamma$.

Remark 3.8. We observe that for any $m \in M$ and $\gamma \in \Gamma$, the ideal $M\gamma m = \{x\gamma m : x \in M\}$ of M is contained in the left ideal $L(m, \gamma, A)$. Also for any $m \in M$ and $\gamma \in \Gamma$, the ideal $m\gamma M = \{m\gamma x : x \in M\}$ of M is contained in the right ideal $R(m, \gamma, A)$.

Definition 3.9. $L(m, \gamma, A)$ is called left A-principal ideal of M generated by m, γ and A, and $R(m, \gamma, A)$ is called right A-principal ideal of M generated by m, γ and A.

Definition 3.10. If $L(m, \gamma, A) = R(m, \gamma, A)$ for all $m \in M$ and $\gamma \in \Gamma$, then the ideal is denoted by $I(m, \gamma, A)$ and is called A-principal ideal of m generated by m, γ and A.

Definition 3.11. A Γ-ring M is called A-principal ideal Γ-ring if A is commutative and every ideal of M is an A-principal ideal generated by some $m \in M, \gamma \in \Gamma$ and A.

Definition 3.12. An element $a \in M$ with $A(a) \neq A(\theta)$ is called an A_γ-unit of M, where $\gamma \in \Gamma$ if there exists an element $a^{'} \in M$ such that $A(a^{'}) \neq A(\theta)$ and $A(a\gamma a^{'}\gamma m) = A(m) = A(a^{'}\gamma a\gamma m)$ for all $m \in M$.

From the definition it follows that $\gamma \neq 0_\Gamma$. In a Γ-field every element $a(\neq \theta)$ is an A_γ-unit for all $\gamma(\neq 0_\Gamma) \in \Gamma$.

Proposition 3.13. *If a is an A_γ-unit of M, then $L(m, \gamma, A) = R(m, \gamma, A) = M$, for all $\gamma \in \Gamma$.*

Proof. As a is an A_γ-unit of $M, \exists\ a^{'} \in M$ such that $A(a^{'}) \neq A(\theta)$ and $A(a\gamma a^{'}\gamma m) = A(m) = A(a^{'}\gamma a\gamma m)$, for all $m \in M$. Let $x \in M$. Then $A(x) = A(a\gamma a^{'}\gamma m) \Rightarrow x \in R(m, \gamma, A)$, since $a^{'}\gamma x \in M$. Therefore, $M \subseteq R(m, \gamma, A)$. Similarly, $M \subseteq L(m, \gamma, A)$. Hence $L(m, \gamma, A) = R(m, \gamma, A) = M$ for $\gamma \in \Gamma$, $\gamma \neq 0_\Gamma$. □

Proposition 3.14. *Let A be a TIIFS with respect to external multiplication defined on M and $a, b \in M$. Then, $a \in L(b, \gamma, A)$ for some $\gamma \in \Gamma \Rightarrow L(a, \gamma, A) \subseteq L(b, \gamma, A)$ and $a \in R(b, \gamma, A)$ for some $\gamma \in \Gamma \Rightarrow R(a, \gamma, A) \subseteq R(b, \gamma, A)$.*

Proof. Let $a \in L(b, \gamma, A)$, then $A(a) = A(x\gamma b)$, for some $x \in M$. Let $m \in L(a, \gamma, A)$. Then $A(m) = A(y\gamma a)$ for some $y \in M$.

Now $A(a) = A(x\gamma b) \Rightarrow A(y\gamma a) = A(y\gamma x\gamma b) \Rightarrow A(m) = A(y\gamma x\gamma b) \Rightarrow m \in L(b, \gamma, A)$.

Hence $L(a, \gamma, A) \subseteq L(b, \gamma, A)$. Similarly, we can prove $R(a, \gamma, A) \subseteq R(b, \gamma, A)$. □

Remark 3.15. We observe that $L(a, \gamma, A) = \{m \in M : A(m) = A(\theta)\} = M_A$, for any $\gamma \in \Gamma$.

Proposition 3.16. *Let A be a TIIFS with respect to the external multiplication defined on M and $a, b \in M$. Then $A(a) = A(b) \Rightarrow L(a, \gamma, A) = L(b, \gamma, A)$; $R(a, \gamma, A) = R(b, \gamma, A)$.*

Proof. Let $A(a) = A(b)$. Suppose $m \in L(a, \gamma, A)$. Then $A(m) = A(x\gamma a)$ for some $x \in M$. Now $A(a) = A(b)$ implies $A(x\gamma a) = A(x\gamma b)$. Hence, $A(m) = A(x\gamma b$, so $m \in L(a, \gamma, A)$. Thus, $L(a, \gamma, A) \subseteq L(b, \gamma, A)$.

Similarly, we can show that $L(b, \gamma, A) \subseteq L(a, \gamma, A)$. Consequently, $L(a, \gamma, A) = L(b, \gamma, A)$. In a similar way, we can prove $R(a, \gamma, A) = R(b, \gamma, A)$. □

In the next two sections, M is assumed to be a commutative Γ-ring with right and left unities and A is assumed to be a translational invariant intuitionistic fuzzy subset of M with respect to both internal addition and external multiplication defined on M satisfying $A(x) = A(-x)$, $\forall x \in M$. Henceforth, the ideal generated by an element $a \in M$, $\gamma \in \Gamma$ with respect to A will be denoted by $I(a, \gamma, A)$ and it will be assumed that $a \in I(a, \gamma, A)$ for all $\gamma \in \Gamma$.

4 A-divisors of zero, A-associates

Definition 4.1. An element $a \in M$ with $A(a) \neq A(\theta)$ is said to be an A_γ-divisor of zero for $\gamma \in \Gamma$, $\gamma \neq 0_\Gamma$ if there exists some $b \in M$ with $A(b) \neq A(\theta)$ such that $A(a\gamma b) = A(\theta)$.

Henceforth, we shall assume that M contains no A_γ-divisor of zero.

Definition 4.2. Let $a, b \in M$ and $A(a) \neq A(\theta)$. We say that a divides b with respect to A and $\gamma \in \Gamma$ or a is an A_γ- divisor of b, written as $(a/b)_{A_\gamma}$, if there exists $c \in M$ such that $A(b) = A(a\gamma c)$.

Theorem 4.3. *Let $a, b \in M$ be such that $A(a) \neq A(b)$ and $A(a) \neq A(\theta)$. Then $(a/b)_{A_\gamma}$ if and only if $I(b, \gamma, A) \subseteq I(a, \gamma, A)$, for $\gamma \in \Gamma$.*

Proof. Suppose that $(a/b)_{A_\gamma}$. Then $A(b) = A(c\gamma a)$ for some $c \in M$, which implies that $b \in I(a, \gamma, A)$ and, therefore, $I(b, \gamma, A) \subseteq I(a, \gamma, A)$.

Conversely, let $I(b, \gamma, A) \subseteq I(a, \gamma, A)$. As $b \in I(b, \gamma, A) \subseteq I(a, \gamma, A)$ hence, $A(b) = A(c\gamma a)$, for some $c \in M$. Also, $A(a) \neq A(\theta)$. Hence $(a/b)_{A_\gamma}$. □

Definition 4.4. Let $a, b \in M \backslash M_A$ be such that $A(a) \neq A(b)$. We say that a and b are A-associates with respect to $\gamma \in \Gamma$ if $(a/b)_{A_\gamma}$ and $(b/a)_{A_\gamma}$.

Proposition 4.5. *Let $a, b \in M \backslash M_A$. Then a, b are A-associates with respect to $\gamma \in \Gamma$ if and only if $A(a) = A(b\gamma u)$ for some A_γ-unit $u \in M$.*

Proof. Let a, b be A-associates with respect to γ. Then $(a/b)_{A_\gamma}$ and $(b/a)_{A_\gamma}$. So $A(b) = A(a\gamma d)$ and $A(a) = A(b\gamma c)$ for some $c, d \in M$. Hence
$A(a) = A(b\gamma c) = A(a\gamma d\gamma c)$
$\Rightarrow A(a\gamma x) = A(a\gamma d\gamma c\gamma x)$
$\Rightarrow A(a\gamma x - a\gamma d\gamma c\gamma x) = A(\theta)$
$\Rightarrow A(a\gamma(x - d\gamma c\gamma x)) = A(\theta)$
$\Rightarrow A(x - d\gamma c\gamma x) = A(\theta)$; since $A(a) \neq A(\theta)$ and R is without A_γ-divisor of zero.
$\Rightarrow A(x) = A(d\gamma c\gamma x)$, for all $x \in M$
$\Rightarrow c$ and d are A_γ-units in M. Hence $A(a) = A(b\gamma c)$, where c is an A_γ-unit in M.

Conversely, suppose that $A(a) = A(b\gamma u)$, for some A_γ-unit u in M.

Now, $A(a) = A(b\gamma u) \Rightarrow (b/a)_{A_\gamma}$. Since u is an A_γ-unit, there exists $v \in M \backslash M_A$ such that $A(u\gamma v\gamma x) = A(x)$, for all $x \in M$. Hence $A(a) = A(b\gamma u) \Rightarrow A(a\gamma v) = A(b\gamma u\gamma v) = A(b)$. This shows that $(a/b)_{A_\gamma}$. Thus we find $(a/b)_{A_\gamma}$ and $(b/a)_{A_\gamma}$. Hence a, b are A_γ-associates. □

Corollary 4.6. *Let $a, b \in M \backslash M_A$. If a, b are A_γ-associates, then $I(a, \gamma, A) = I(b, \gamma, A)$.*

Proof. Suppose that a and b are A_γ-associates. Then by Proposition 4.5, $A(a) = A(u\gamma b)$, for some A_γ-unit $u \in M$. Then, $a \in I(b, \gamma, A)$, and so $I(a, \gamma, A) \subseteq I(b, \gamma, A)$. Since u is an A_γ-unit of M, there exists $v \in M \backslash M_A$ such that $A(u\gamma v\gamma x) = A(x)$, for all $x \in M$. Hence $A(b\gamma u\gamma v) = A(b)$. Thus $A(b) = A(a\gamma v)$, and so $b \in I(a, \gamma, A)$. Therefore $I(b, \gamma, A) \subseteq I(a, \gamma, A)$. Consequently, $I(a, \gamma, A) = I(b, \gamma, A)$. □

Definition 4.7. Suppose $a \in M \backslash M_A$ and a is not an A_γ-unit for $\gamma \in \Gamma$. Then a is said to be A_γ-irreducible if $A(a) \neq A(b)$, $A(a) \neq A(c)$ and $A(a) = A(b\gamma c)$ implies either b or c is an A_γ-unit, where $b, c \in M$.

Definition 4.8. Suppose $a \in M \backslash M_A$ and a not an A_γ-unit for $\gamma \in \Gamma$. Then a is said to be A_γ-prime if $A(a) \neq A(b)$, $A(a) \neq A(c)$ and $(a/b\gamma_1 c)_{A_\gamma}$ implies $(a/b)_{A_\gamma}$ or $(a/c)_{A_\gamma}$, where $\gamma \in \Gamma$.

Proposition 4.9. *In the Γ-ring M with no A_γ-divisors of zero, any A_γ-prime is A_γ-irreducible.*

Proof. Let a be A_γ-prime. Suppose $A(a) \neq A(b)$, $A(a) \neq A(c)$ and $A(a) = A(b\gamma c)$ for $\gamma \in \Gamma$, $b, c \in M$. We can say that $(a/b\gamma c)_{A_\gamma}$. Since a is A_γ-prime, either $(a/b)_A$ or $(a/c)_A$.

Suppose $(a/b)_{A_\gamma}$. As $A(a) \neq A(b)$, $A(b) = A(a\gamma d)$ for some $d \in M$. Now
$A(a) = A(b\gamma c) = A(a\gamma d\gamma c)$
$\Rightarrow A(a\gamma x) = A(a\gamma d\gamma c\gamma x)$, for $x \in M$
$\Rightarrow A(a\gamma x - a\gamma d\gamma c\gamma x) = A(\theta) \Rightarrow A(a\gamma(x - d\gamma c\gamma x)) = A(\theta)$
$\Rightarrow A(x - d\gamma c\gamma x) = A(\theta)$, since $A(a) \neq A(\theta)$ and M is without A_γ-divisor of zero.
$\Rightarrow A(x) = A(d\gamma c\gamma x)$, for all x
$\Rightarrow c$ is a A_γ-unit.
Similarly, if $(a/c)_{A_\gamma}$ then we can show that b is an A_γ-unit. Hence a is A_γ-irreducible. □

Theorem 4.10. *Suppose that $a \in M \backslash M_A$ and a is not an A_γ-unit. Then a is A_γ-irreducible if and only if the ideal $I(a, \gamma, A)$ is maximal among all ideals $I(b, \gamma, A)$, where $b \in M$ and $A(a) \neq A(b)$.*

Proof. (i) Suppose that a is A_γ-irreducible. Let $I(a,\gamma,A) \subseteq I(b,\gamma,A) \neq M$ for some $b \in M$ with $A(b) \neq A(a)$. Now $a \in I(a,\gamma,A) \subseteq I(b,\gamma,A)$ and so $A(a) = A(c\gamma b)$ for some $c \in M \backslash M_A$. Now, if $A(a) = A(c)$, then $A(c) = A(c\gamma b)$, which implies $A(c\gamma x) = A(c\gamma b\gamma z)$, for all $x \in M$. Now M is without A_γ-divisor of zero and $A(c) \neq A(\theta)$, so $A(x) = A(b\gamma x)$ for all $x \in M$. Hence $I(b,\gamma,A) = M$, which is not the case. Hence $A(a) \neq A(c)$. As a is A_γ-irreducible, so either b is an A_γ-unit or c is an A_γ-unit. Since $I(b,\gamma,A) \neq M$ so by Proposition 4.9, we find that b is not an A_γ-unit. So there exists $u \in M \backslash M_A$ such that $A(c\gamma u\gamma x) = A(u\gamma c\gamma x) = A(x)$, for all $x \in M$. Thus $A(b) = A(c\gamma u\gamma b)$. Again, $A(a) = A(b\gamma c)$ implies $A(a\gamma u) = A(b)$. Hence $b \in I(a,\gamma,A)$ and so $I(b,\gamma,A) \subseteq I(a,\gamma,A)$. Consequently, $I(b,\gamma,A) = I(a,\gamma,A)$. Thus $I(a,\gamma,A)$ is maximal.

Conversely, assume that $I(a,\gamma,A)$ is maximal. Assume that $A(a) = A(c\gamma d)$ where $c, d \in M$ and $A(a) \neq A(c)$, $A(a) \neq A(d)$. Then $a \in I(d,\gamma,A)$ and so $I(a,\gamma,A) \subseteq I(d,\gamma,A)$. Hence by our hypothesis either $I(a,\gamma,A) = I(d,\gamma,A)$, or $I(d,\gamma,A) = M$. If $I(a,\gamma,A) = I(d,\gamma,A)$, then $d \in I(d,\gamma,A) = I(a,\gamma,A)$. Therefore, $A(d) = A(m\gamma a)$, for some $m \in M$. This gives $A(c\gamma d) = A(c\gamma m\gamma a)$. Thus we have $A(a) = A(c\gamma m\gamma a)$ and so $A(a\gamma(x - c\gamma m\gamma x)) = A(\theta)$, for all $x \in M$. Since M is without A_γ-divisors of zero and $A(a) \neq A(\theta)$, we have $A(x) = A(c\gamma m\gamma x)$, for all $x \in M$. This shows that c is an A_γ-unit. If $I(d,\gamma,A) = M$, then $A(d) = A(d\gamma m)$, for some $m \in M$. Again $A(m) = A(d\gamma y)$, for some $y \in M$. Therefore $A(d) = A(d\gamma m) = A(d\gamma d\gamma y)$. Hence, $A(d\gamma x) = A(d\gamma d\gamma y\gamma x)$. Thus $A(x) = A(d\gamma y\gamma x)$. This shows that d is an A_γ-unit. □

Theorem 4.11. *Suppose that $a \in M \backslash M_A$ and a is not an A_γ-unit. Then a is A_γ-prime if and only if for $x, y \in M, \gamma_1 \in \Gamma$, $x\gamma_1 y \in I(a,\gamma,A)$ implies either that $x \in I(a,\gamma,A)$ or $y \in I(a,\gamma,A)$, where $A(a) \neq A(x)$, $A(a) \neq A(y)$.*

Proof. Suppose that a is A_γ-prime and $x, y \in M$, $\gamma_1 \in \Gamma$, $x\gamma_1 y \in I(a,\gamma,A)$ implies either $x \in I(a,\gamma,A)$ or $y \in I(a,\gamma,A)$, where $A(a) \neq A(x)$, $A(a) \neq A(y)$. Then $A(x\gamma_1 y) = A(a\gamma m)$ for some $m \in M$, which shows that $(a/x\gamma_1 y)_{A_\gamma}$. As a is A_γ-prime, so either $(a/x)_{A_\gamma}$ or $(a/y)_{A_\gamma}$. If $(a/x)_{A_\gamma}$, then there exists $x_1 \in M$ such that $A(x) = A(a\gamma x_1)$ which implies $x \in I(a,\gamma,A)$. Similarly, if $(a/y)_{A_\gamma}$, then there exists $y_1 \in M$ such that $A(y) = A(a\gamma y_1)$, which implies $y \in I(a,\gamma,A)$.

Conversely, let for $x, y \in M, \gamma_1 \in \Gamma$, $x\gamma_1 y \in I(a,\gamma,A)$ implies either $x \in I(a,\gamma,A)$ or $y \in I(a,\gamma,A)$, where $A(a) \neq A(x)$, $A(a) \neq A(y)$. We have to prove a is A_γ-prime. Let $(a/x\gamma_1 y)_{A_\gamma}$, where $x, y \in M, \gamma_1 \in \Gamma$, $A(a) \neq A(x)$, $A(a) \neq A(y)$. $A(x\gamma_1 y) = A(a\gamma m)$, for some $m \in M$. Thus $x\gamma_1 y \in I(a,\gamma,A)$. Now from given condition either $x \in I(a,\gamma,A)$, or $y \in I(a,\gamma,A)$.

If $x \in I(a,\gamma,A)$, then $A(x) = A(a\gamma_1 m_1)$ for some $m_1 \in M$. Thus $(a/x)_{A_\gamma}$.

If $y \in I(a,\gamma,A)$, then $A(y) = A(a\gamma_1 m_2)$ for some $m_2 \in M$. Thus $(a/y)_{A_\gamma}$.

This proves that a is A_γ-prime. □

5 Images and inverse images under Γ-ring homomorphisms

In this section, we discuss the invariance of translational invariance property of an intuitionistic fuzzy subset under Γ-ring homomorphism. We also study the algebraic nature of ideals under Γ-ring homomorphism.

Proposition 5.1. *Let M and M' be Γ-rings and f be a Γ-homomorphism from M into M'. Let B be a TIIFS of M'. Then $f^{-1}(B)$ is a TIIFS of M.*

Proof. Let $a, b \in M$ and $f^{-1}(B)(a) = f^{-1}(B)(b)$. Then $B(f(a)) = B(f(b))$. Let $x \in M$ and $f(x) = y \in M'$. Since B is a TIIFS of M' and $B(f(a)) = B(f(b))$, we have $B(f(a) + y) = B(f(b) + y)$ and $B(f(a)\gamma y) = B(f(b)\gamma y), B(y\gamma f(a)) = B(y\gamma f(b))$. Now $B(f(a) + y) = B(f(b) + y)$ implies $B(f(a) + f(x)) = B(f(b) + f(x))$, and so $B(f(a + x)) = B(f(b + x))$. Hence $f^{-1}(B)(a + x) = f^{-1}(B)(b + x)$. On the other hand, from $B(f(a)\gamma y) = B(f(b)\gamma y)$ and $B(y\gamma f(a)) = B(y\gamma f(b))$, we get $B(f(a)\gamma f(x)) = B(f(b)\gamma f(x))$ and $B(f(x)\gamma f(a)) = B(f(x)\gamma f(b))$, and so $B(f(a\gamma x)) = B(f(b\gamma x))$ and $B(f(x\gamma a)) = B(f(x\gamma b))$. Thus, we have $f^{-1}(B)(a\gamma x) = f^{-1}(B)(b\gamma x)$ and $f^{-1}(B)(x\gamma a) = f^{-1}(B)(x\gamma b) \forall a, b, x \in M$ and $\gamma \in \Gamma$. Consequently, $f^{-1}(B)$ is TIIFS of M. □

Proposition 5.2. *Let M and M' be Γ-rings and f be a Γ-homomorphism from M onto M'. Let A be a TIIFS of M. If A is f-invariant, then $f(A)$ is a TIIFS of M'.*

Proof. Suppose that A is f-invariant. Then $\forall x, y \in M$, $f(x) = f(y)$ implies $A(x) = A(y)$.

As f is onto, for any $a \in M'$, $\mu_{f(A)}(a) = \sup\{\mu_A(x) : x \in M, f(x) = a\}$ and $\nu_{f(A)}(a) = \inf\{\nu_A(x) : x \in M, f(x) = a\}$. Let $x, y \in M$ and $f(x) = a = f(y)$. Then $f(x) = f(y)$, and so $A(x) = A(y)$. So $\mu_{f(A)}(a) = \mu_A(x)$ and $\nu_{f(A)}(a) = \nu_A(x)$. Hence $f(A)(a) = A(x)$, where $x \in M$ and $f(x) = a$. Thus $\forall a \in M'$, $f(A)(a) = A(x)$, where $x \in M$ and $f(x) = a$. Now, let $a, b \in M'$, and $f(A)(a) = f(A)(b)$. Then $A(x) = A(y)$, where $x, y \in M$, and $f(x) = a, f(y) = b$. Let $c \in M'$ be such that $f(z) = c$, where $z \in M$. Then, $a + c = f(x) + f(z) = f(x + z)$ and $b + c = f(y) + f(z) = f(y + z)$. Hence $f(A)(a + c) = A(x + z)$ and $f(A)(b + c) = A(y + z)$. Again, for $\gamma \in \Gamma$, $a\gamma c = f(x)\gamma f(z) = f(x\gamma z)$, $c\gamma a = f(z)\gamma f(x) = f(z\gamma x)$, $b\gamma c = f(y)\gamma f(z) = f(y\gamma z)$, and $c\gamma b = f(z)\gamma f(y) = f(z\gamma y)$. Since A is translational invariant, $A(x + z) = A(y + z)$, $A(x\gamma z) = A(y\gamma z)$, and $A(z\gamma x) = A(z\gamma y)$. Hence, $f(A)(a + c) = f(A)(b+c)$, $f(A)(a\gamma c) = f(A)(b\gamma c)$, and $f(A)(c\gamma a) = f(A)(c\gamma b)$ for all $c \in M'$. Hence, $f(A)$ is a TIIFS of M'. □

Theorem 5.3. *Let M and M' be Γ-rings and f be a Γ-homomorphism from M onto M' and A be a TIIFS of M. If A is f-invariant then,*

$$f(I(a, \gamma, A)) = I(f(a), \gamma, f(A)), \forall a \in M, \gamma \in \Gamma.$$

Proof. Suppose that A is f-invariant. Let $y \in I(f(a), \gamma, f(A))$. Then $f(A)(y) = f(A)(s\gamma f(a))$ for some $s \in M'$. Since $y, s \in M'$ and f is onto, there exist $x, r \in M$ such that $f(x) = y$ and $f(r) = s$. Thus $f(A)f(x) = f(A)(f(r)\gamma f(a)) = f(A)(f(r\gamma a))$. Since A is translational invariant, by what we have proved in Proposition 5.2, we get $f(A)(f(x)) = A(x)$ and

$f(A)(f(r\gamma a)) = A(r\gamma a)$. Thus $A(x) = A(r\gamma a)$, which implies $x \in I(a,\gamma,A)$, and so $f(x) \in f(I(a,\gamma,A))$, i.e., $y \in f(I(a,\gamma,A))$. Consequently, $I(f(a),\gamma,f(A)) \subseteq f(I(a,\gamma,A))$. Again, let $y \in f(I(a,\gamma,A))$. Then there exists $x \in I(a,\gamma,A)$ such that $f(x) = y$. Also, $x \in I(a,\gamma,A)$ implies $A(x) = A(a\gamma r)$ for some $r \in M$. Now,

$$\begin{aligned}\mu_{f(A)}(y) &= Sup\{\mu_A(x) : f(x) = y\} \\ &= \mu_A(x), \text{ since } A \text{ is } f\text{-invariant} \\ &= \mu_A(a\gamma r).\end{aligned}$$

Similarly, we can prove $\nu_{f(A)}(y) = \nu_A(a\gamma r)$. Hence $f(A)(y) = A(a\gamma r)$. Also, if $f(r) = s$, we have $f(A)(f(a)\gamma s) = f(A)(f(a)\gamma f(r)) = f(A)(f(a\gamma r)) = A(a\gamma r)$, since A is f-invariant so $f^{-1}f(a\gamma r) = a\gamma r$. Thus $f(A)(y) = f(A)(f(a)\gamma s)$ which implies $y \in I(f(a),\gamma,f(A))$. Hence $f(I(a,\gamma,A)) \subseteq I(f(a),\gamma,f(A))$, $a \in M$. Consequently, $f(I(a,\gamma,A)) = I(f(a),\gamma,f(A))$, for $a \in M, \gamma \in \Gamma$. □

Proposition 5.4. *Let M and M' be Γ-rings and f be a Γ-homomorphism from M onto M'. Let B be a TIIFS of M'. Let $a' \in M'$. Then $\forall a,b \in f^{-1}(a')$, $I(a,\gamma,f^{-1}(B)) = I(b,\gamma,f^{-1}(B))$; provided that $f^{-1}(a')$ contains more than one element.*

Proof. Let $x \in I(a,\gamma,f^{-1}(B))$. Then $f^{-1}(B)(x) = f^{-1}(B)(r\gamma a)$ for some $r \in M$ and so $f^{-1}(B)(x) = B(f(r\gamma a))$. Thus $f^{-1}(B)(x) = B(f(a)\gamma f(r))$. Since $a,b \in f^{-1}(a')$, $f(a) = f(b) = a'$ and hence we have $f^{-1}(B)(x) = B(f(b)\gamma f(r)) = B(f(b\gamma r)) = f^{-1}(B)(b\gamma r)$. This shows that $x \in I(b,\gamma,f^{-1}(B))$. Hence $I(a,\gamma,f^{-1}(B)) \subseteq I(b,\gamma,f^{-1}(B))$. Now let $y \in I(b,\gamma,f^{-1}(B))$. Then $f^{-1}(B)(y) = f^{-1}(B)(b\gamma r')$ for some $r' \in M$, and so $f^{-1}(B)(y) = B(f(b\gamma r')) = B(f(b)\gamma f(r'))$, Since $a,b \in f^{-1}(a')$, $f(a) = a' = f(b)$ and hence we have $f^{-1}(B)(y) = B(f(a)\gamma f(r')) = B(f(a\gamma r')) = f^{-1}(B)(a\gamma r')$. This shows that $y \in I(a,\gamma,f^{-1}(B))$. Hence, $I(b,\gamma,f^{-1}(B)) \subseteq I(a,\gamma,f^{-1}(B))$. Consequently, $I(a,\gamma,f^{-1}(B)) = I(b,\gamma,f^{-1}(B))\ \forall a,b \in f^{-1}(a')$. □

Theorem 5.5. *Let M and M' be Γ-rings and f be a Γ-isomorphism from M onto M'. Let B be a translational invariant intuitionistic fuzzy subset of M'. Then*

$$I(f^{-1}(y),\gamma.f^{-1}(B)) = f^{-1}(I(y,\gamma,B)),\ \forall y \in M', \gamma \in \Gamma.$$

Proof. Let $x \in I(f^{-1}(y),\gamma,f^{-1}(B))$. Then

$$\begin{aligned}f^{-1}(B)(x) &= f^{-1}(B)(f^{-1}(y)\gamma r) \text{ for some } r \in M. \\ &= f^{-1}(B)(f^{-1}(y)\gamma f^{-1}(s)), \text{ where } s \in M' \text{ such that } f(r) = s. \\ \Rightarrow B(f(x)) &= f^{-1}(B)(f^{-1}(y\gamma s)), \text{ since } f \text{ is bijective} \\ &= B(f(f^{-1}(y\gamma s))) \\ &= B(y\gamma s).\end{aligned}$$

So, we have $f(x) \in I(y,\gamma,B)$, i.e., $x \in f^{-1}(I(y,\gamma,B))$. Hence $I(f^{-1}(y),\gamma,f^{-1}(B)) \subseteq f^{-1}(I(y,\gamma,B)), \forall y \in M'$. Again, let $a \in f^{-1}(I(y,\gamma,B))$ then $f(a) \in I(y,\gamma,B)) \Rightarrow$

$B(f(a)) = B(y\gamma s)$, for some $s \in M'$. Also, $y, s \in M'$ and f is onto implies that there exist $x, r \in M$ such that $f(x) = y$ and $f(r) = s$. Now, $B(f(a)) = B(y\gamma s) \Rightarrow B(f(a)) = B(f(x)\gamma f(r)) = B(f(x\gamma r)) \Rightarrow f^{-1}(B)(a) = f^{-1}(B)(x\gamma r) = f^{-1}(B)(f^{-1}(y)\gamma r) \Rightarrow a \in I(f^{-1}(y), \gamma, f^{-1}(B))$. Thus, $f^{-1}(I(y,\gamma,B)) \subseteq I(f^{-1}(y), \gamma, f^{-1}(B)), \forall y \in M'$. Consequently, $I(f^{-1}(y), \gamma, f^{-1}(B)) = f^{-1}(I(y,\gamma,B)), \forall y \in M', \gamma \in \Gamma$. □

Theorem 5.6. *Let M and M' be Γ-rings and f be a Γ-homomorphism from M onto M'. If A is f-invariant and TIIFS of M. If p is an A_γ-prime element of M, then, $f(p)$ is a $f(A)_\gamma$-prime element of M'.*

Proof. Let f be a Γ-homomorphism from M onto M'. If A is f-invariant and TIIFS of M. Then by Proposition 5.2 $f(A)$ is a TIIFS of M'. Suppose that p is an A_γ-prime element of M. Let $(f(p)/x\gamma y)_{f(A)_\gamma}$, where $x, y \in M'$. Since f is onto, there exists $a, b \in M$ such that $f(a) = x$, $f(b) = y$.

Now $(f(p)/x\gamma y)_{f(A)_\gamma} \Rightarrow \exists c \in M$ such that

$$\begin{aligned} & f(A)(x\gamma y) = f(A)(f(p)\gamma f(c)) \\ & \Rightarrow f(A)(f(a)\gamma f(b)) = f(A)(f(p\gamma c)) \\ & \Rightarrow f(A)(f(a\gamma b)) = f(A)(f(p\gamma c)) \\ & \Rightarrow A(a\gamma b) = A(p\gamma c) \text{ and so } (p/a\gamma b)_{A_\gamma}. \end{aligned}$$

Since p is an A_γ-prime element of M, we have $(p/a)_{A_\gamma}$ or $(p/b)_{A_\gamma}$
$\Rightarrow A(a) = A(p\gamma m)$ or $A(b) = A(p\gamma n)$, for some $m, n \in M, \forall \gamma \in \Gamma$
$\Rightarrow f(A)(f(a)) = f(A)(f(p\gamma m))$ or $f(A)(f(b)) = f(A)(f(p\gamma n))$
$\Rightarrow f(A)(f(a)) = f(A)(f(p)\gamma f(m))$ or $f(A)(f(b)) = f(A)(f(p)\gamma f(n))$
$\Rightarrow (f(p)/f(a))_{f(A)}$ or $(f(p)/f(b))_{f(A)}$. Thus, $f(p)$ is a $f(A)$-prime element of M'. □

Theorem 5.7. *Let f be a homomorphism of a Γ-ring M onto a Γ-ring M'. Let A be an f-invariant and TIIFS of M. If p is an A_γ-prime element of M, then the homomorphic image of $I(p, \gamma, A)$ is a prime ideal of M'.*

Proof. Let f be a Γ-homomorphism from M onto M'. If A is f-invariant and TIIFS of M, then by Proposition 5.2 $f(A)$ is TIIFS of M'.

By Theorem 4.11 $I(p, \gamma, A)$ is a prime ideal of M.
By Theorem 5.3 $f(I(p, \gamma, A)) = I(f(p), \gamma, f(A))$.
By Theorem 5.6 $f(p)$ is a $f(A)_\gamma$-prime element of M'
By Theorem 4.11 $f(I(p, \gamma, A))$ is a prime ideal of M'. □

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