

On intuitionistic fuzzy implications, negations and the law $(\neg A \supset \neg B) \supset ((\neg A \supset \neg \neg B) \supset \neg \neg A)$

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Abstract: In a series of papers, the intuitionistic fuzzy logic is extended with different negations and implications. Here, all implications and generated by them negations, that satisfy the laws

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg \neg B) \supset \neg \neg A)$$

and

$$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$$

are described.

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1 Introduction

In a series of papers of the author, starting with [4, 5], 138 intuitionistic fuzzy implications were defined (see Table 1) and some of their basic properties were studied. Meantime, L. Atanassova and P. Dwornizak in [7, 8, 9, 10, 11, 12] defined other intuitionistic fuzzy implications, that will be studied for the discussed here property, in a next research.

Here, we check the validity of the axiom from title

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg \neg B) \supset \neg \neg A).$$

It plays special role in intuitionistic fuzzy sets and logics theories. In 1987, the author defined two implications and he saw that implication \rightarrow_1 , defined below satisfies the axiom, while implication

\rightarrow_3 – does not satisfy it. This was one of the reasons that for a long time the implication \rightarrow_3 , having intuitionistic behaviour, was outside author's interests. Now, having in mind the results from [3], we show that it can have another form.

In intuitionistic fuzzy logic, if x is a variable then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b \rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x . Any other formula is estimated by analogy.

Everywhere below we shall assume that for the two variables A and B there hold the equalities: $V(A) = \langle a, b \rangle$, $V(B) = \langle c, d \rangle$, $(a, b, c, d, a + b, c + d \in [0, 1])$.

For the needs of the discussion below, we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1, 2]) by:

$$x \text{ is an IFT if and only if } a \geq b,$$

while x will be a tautology iff $a = 1$ and $b = 0$.

In some definitions, we shall use the functions sg and $\overline{\text{sg}}$:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad \overline{\text{sg}}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

2 Intuitionistic fuzzy implications and negations

Following [6], we give the list of 138 different intuitionistic fuzzy implications (see Table 1) and 34 different intuitionistic fuzzy negations (see Table 2), generated by the intuitionistic fuzzy implications. The relations between the negations and implications are shown on Table 3.

Table 1

\rightarrow_1	$\langle \max(b, \min(a, c)), \min(a, d) \rangle$
\rightarrow_2	$\langle \overline{\text{sg}}(a - c), d \cdot \text{sg}(a - c) \rangle$
\rightarrow_3	$\langle 1 - (1 - c) \cdot \text{sg}(a - c), d \cdot \text{sg}(a - c) \rangle$
\rightarrow_4	$\langle \max(b, c), \min(a, d) \rangle$
\rightarrow_5	$\langle \min(1, b + c), \max(0, a + d - 1) \rangle$
\rightarrow_6	$\langle b + ac, ad \rangle$
\rightarrow_7	$\langle \min(\max(b, c), \max(a, b), \max(c, d)), \max(\min(a, d), \min(a, b), \min(c, d)) \rangle$
\rightarrow_8	$\langle 1 - (1 - \min(b, c)) \cdot \text{sg}(a - c), \max(a, d) \cdot \text{sg}(a - c), \text{sg}(d - b) \rangle$
\rightarrow_9	$\langle b + a^2c, ab + a^2d \rangle$
\rightarrow_{10}	$\langle c \cdot \overline{\text{sg}}(1 - a) + \text{sg}(1 - a) \cdot (\overline{\text{sg}}(1 - c) + b \cdot \text{sg}(1 - c)), d \cdot \overline{\text{sg}}(1 - a) + a \cdot \text{sg}(1 - a) \cdot \text{sg}(1 - c) \rangle$
\rightarrow_{11}	$\langle 1 - (1 - c) \cdot \text{sg}(a - c), d \cdot \text{sg}(a - c) \cdot \text{sg}(d - b) \rangle$
\rightarrow_{12}	$\langle \max(b, c), 1 - \max(b, c) \rangle$
\rightarrow_{13}	$\langle b + c - b.c, a.d \rangle$

\rightarrow_{14}	$\langle 1 - (1 - c).\text{sg}(a - c) - d.\overline{\text{sg}}(a - c).\text{sg}(d - b), d.\text{sg}(d - b) \rangle$
\rightarrow_{15}	$\langle 1 - (1 - \min(b, c)).\text{sg}(a - c).\text{sg}(d - b) - \min(b, c).\text{sg}(a - c).\text{sg}(d - b),$ $1 - (1 - \max(a, d)).\text{sg}(\overline{\text{sg}}(a - c) + \overline{\text{sg}}(d - b)) - \max(a, d).\overline{\text{sg}}(a - c).\overline{\text{sg}}(d - b) \rangle$
\rightarrow_{16}	$\langle \max(\overline{\text{sg}}(a), c), \min(\text{sg}(a), d) \rangle$
\rightarrow_{17}	$\langle \max(b, c), \min(a.b + a^2, d) \rangle$
\rightarrow_{18}	$\langle \max(b, c), \min(1 - b, d) \rangle$
\rightarrow_{19}	$\langle \max(1 - \text{sg}(\text{sg}(a) + \text{sg}(1 - b)), c), \min(\text{sg}(1 - b), d) \rangle$
\rightarrow_{20}	$\langle \max(\overline{\text{sg}}(a), \text{sg}(c)), \min(\text{sg}(a), \overline{\text{sg}}(c)) \rangle$
\rightarrow_{21}	$\langle \max(b, c.(c + d)), \min(a.(a + b), d.(c^2 + d + c.d)) \rangle$
\rightarrow_{22}	$\langle \max(b, 1 - d), 1 - \max(b, 1 - d) \rangle$
\rightarrow_{23}	$\langle 1 - \min(\text{sg}(1 - b), \overline{\text{sg}}(1 - d)), \min(\text{sg}(1 - b), \overline{\text{sg}}(1 - d)) \rangle$
\rightarrow_{24}	$\langle \overline{\text{sg}}(a - c).\overline{\text{sg}}(d - b), \text{sg}(a - c).\text{sg}(d - b) \rangle$
\rightarrow_{25}	$\langle \max(b, \overline{\text{sg}}(a).\overline{\text{sg}}(1 - b), c.\overline{\text{sg}}(d).\overline{\text{sg}}(1 - c)), \min(a, d) \rangle$
\rightarrow_{26}	$\langle \max(\overline{\text{sg}}(1 - b), c), \min(\text{sg}(a), d) \rangle$
\rightarrow_{27}	$\langle \max(\overline{\text{sg}}(1 - b), \text{sg}(c)), \min(\text{sg}(a), \overline{\text{sg}}(1 - d)) \rangle$
\rightarrow_{28}	$\langle \max(\overline{\text{sg}}(1 - b), c), \min(a, d) \rangle$
\rightarrow_{29}	$\langle \max(\overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - c)), \min(a, \overline{\text{sg}}(1 - d)) \rangle$
\rightarrow_{30}	$\langle \max(1 - a, \min(a, 1 - d)), \min(a, d) \rangle$
\rightarrow_{31}	$\langle \overline{\text{sg}}(a + d - 1), d.\text{sg}(a + d - 1) \rangle$
\rightarrow_{32}	$\langle 1 - d.\text{sg}(a + d - 1), d.\text{sg}(a + d - 1) \rangle$
\rightarrow_{33}	$\langle 1 - \min(a, d), \min(a, d) \rangle$
\rightarrow_{34}	$\langle \min(1, 2 - a - d), \max(0, a + d - 1) \rangle$
\rightarrow_{35}	$\langle 1 - a.d, a.d \rangle$
\rightarrow_{36}	$\langle \min(1 - \min(a, d), \max(a, 1 - a), \max(1 - d, d)),$ $\max(\min(a, d), \min(a, 1 - a), \min(1 - d, d)) \rangle$
\rightarrow_{37}	$\langle 1 - \max(a, d).\text{sg}(a + d - 1), \max(a, d).\text{sg}(a + d - 1) \rangle$
\rightarrow_{38}	$\langle 1 - a + (a^2.(1 - d)), a.(1 - a) + a^2.d \rangle$
\rightarrow_{39}	$\langle (1 - d).\overline{\text{sg}}(1 - a) + \text{sg}(1 - a).(\overline{\text{sg}}(d) + (1 - a).\text{sg}(d)),$ $d.\overline{\text{sg}}(1 - a) + a.\text{sg}(1 - a).\text{sg}(d) \rangle$
\rightarrow_{40}	$\langle 1 - \text{sg}(a + d - 1), 1 - \overline{\text{sg}}(a + d - 1) \rangle$
\rightarrow_{41}	$\langle \max(\overline{\text{sg}}(a), 1 - d), \min(\text{sg}(a), d) \rangle$
\rightarrow_{42}	$\langle \max(\overline{\text{sg}}(a), \text{sg}(1 - d)), \min(\text{sg}(a), \overline{\text{sg}}(1 - d)) \rangle$
\rightarrow_{43}	$\langle \max(\overline{\text{sg}}(a), 1 - d), \min(\text{sg}(a), d) \rangle$
\rightarrow_{44}	$\langle \max(\overline{\text{sg}}(a), 1 - d), \min(a, d) \rangle$
\rightarrow_{45}	$\langle \max(\overline{\text{sg}}(a), \overline{\text{sg}}(d)), \min(a, \overline{\text{sg}}(1 - d)) \rangle$
\rightarrow_{46}	$\langle \max(b, \min(1 - b, c)), 1 - \max(b, c) \rangle$
\rightarrow_{47}	$\langle \overline{\text{sg}}(1 - b - c), (1 - c).\text{sg}(1 - b - c) \rangle$
\rightarrow_{48}	$\langle 1 - (1 - c).\text{sg}(1 - b - c), (1 - c).\text{sg}(1 - b - c) \rangle$
\rightarrow_{49}	$\langle \min(1, b + c), \max(0, 1 - b - c) \rangle$
\rightarrow_{50}	$\langle b + c - b.c, 1 - b - c + b.c \rangle$

\rightarrow_{51}	$\langle \min(\max(b, c), \max(1 - b, b), \max(c, 1 - c)),$ $\max(1 - \max(b, c), \min(1 - b, b), \min(c, 1 - c)) \rangle$
\rightarrow_{52}	$\langle 1 - (1 - \min(b, c)).\text{sg}(1 - b - c), 1 - \min(b, c).\text{sg}(1 - b - c) \rangle$
\rightarrow_{53}	$\langle b + (1 - b)^2.c, (1 - b).b + (1 - b)^2.(1 - c) \rangle$
\rightarrow_{54}	$\langle c.\overline{\text{sg}}(b)) + \text{sg}(b).(\overline{\text{sg}}(1 - c) + b.\text{sg}(1 - c)), (1 - c).\overline{\text{sg}}(b) + (1 - b).\text{sg}(b).\text{sg}(1 - c) \rangle$
\rightarrow_{55}	$\langle 1 - \text{sg}(1 - b - c), 1 - \overline{\text{sg}}(1 - b - c) \rangle$
\rightarrow_{56}	$\langle \max(\overline{\text{sg}}(1 - b), c), \min(\text{sg}(1 - b), (1 - c)) \rangle$
\rightarrow_{57}	$\langle \max(\overline{\text{sg}}(1 - b), \text{sg}(c)), \min(\text{sg}(1 - b), \overline{\text{sg}}(c)) \rangle$
\rightarrow_{58}	$\langle \max(\overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - c)), 1 - \max(b, c) \rangle$
\rightarrow_{59}	$\langle \max(\overline{\text{sg}}(1 - b), c), (1 - \max(b, c)) \rangle$
\rightarrow_{60}	$\langle \max(\overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - c)), \min((1 - b), \overline{\text{sg}}(c)) \rangle$
\rightarrow_{61}	$\langle \max(c, \min(b, d)), \min(a, d) \rangle$
\rightarrow_{62}	$\langle \overline{\text{sg}}(d - b), a.\text{sg}(d - b) \rangle$
\rightarrow_{63}	$\langle 1 - (1 - b).\text{sg}(d - b), a.\text{sg}(d - b) \rangle$
\rightarrow_{64}	$\langle c + b.d, a.d \rangle$
\rightarrow_{65}	$\langle 1 - (1 - \min(b, c)).\text{sg}(d - b), \max(a, d).\text{sg}(d - b).\text{sg}(a - c) \rangle$
\rightarrow_{66}	$\langle c + d^2.b, b.d + d^2.a \rangle$
\rightarrow_{67}	$\langle b.\overline{\text{sg}}(1 - d) + \text{sg}(1 - d).(\overline{\text{sg}}(1 - b) + c.\text{sg}(1 - b)),$ $a.\overline{\text{sg}}(1 - d) + d.\text{sg}(1 - d).\text{sg}(1 - b) \rangle$
\rightarrow_{68}	$\langle 1 - (1 - b).\text{sg}(d - b), a.\text{sg}(d - b).\text{sg}(a - c) \rangle$
\rightarrow_{69}	$\langle 1 - (1 - b).\text{sg}(d - b) - a.\overline{\text{sg}}(d - b).\text{sg}(a - c), a.\text{sg}(a - c) \rangle$
\rightarrow_{70}	$\langle \max(\overline{\text{sg}}(d), b), \min(\text{sg}(d), a) \rangle$
\rightarrow_{71}	$\langle \max(b, c), \min(c.d + d^2, a) \rangle$
\rightarrow_{72}	$\langle \max(b, c), \min(1 - c, a) \rangle$
\rightarrow_{73}	$\langle \max(1 - \max(\text{sg}(d), \text{sg}(1 - c)), b), \min(\text{sg}(1 - c), a) \rangle$
\rightarrow_{74}	$\langle \max(\text{sg}(b), \overline{\text{sg}}(d)), \min(\overline{\text{sg}}(b), \text{sg}(d)) \rangle$
\rightarrow_{75}	$\langle \max(c, b.(a + b)), \min(d.(c + d), a.(b^2 + a) + a.b) \rangle$
\rightarrow_{76}	$\langle \max(c, 1 - a), \min(1 - c, a) \rangle$
\rightarrow_{77}	$\langle (1 - \min(\overline{\text{sg}}(1 - a), \text{sg}(1 - c))), \min(\overline{\text{sg}}(1 - a), \text{sg}(1 - c)) \rangle$
\rightarrow_{78}	$\langle \max(\overline{\text{sg}}(1 - c), b), \min(\text{sg}(d), a) \rangle$
\rightarrow_{79}	$\langle \max(\overline{\text{sg}}(1 - c), \text{sg}(b)), \min(\text{sg}(d), \overline{\text{sg}}(1 - a)) \rangle$
\rightarrow_{80}	$\langle \max(\overline{\text{sg}}(1 - c), b), \min(d, a) \rangle$
\rightarrow_{81}	$\langle \max(\overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - c)), \min(d, \overline{\text{sg}}(1 - a)) \rangle$
\rightarrow_{82}	$\langle \max(1 - d, \min(d, 1 - a)), \min(d, a) \rangle$
\rightarrow_{83}	$\langle \overline{\text{sg}}(a + d - 1), a.\text{sg}(a + d - 1) \rangle$
\rightarrow_{84}	$\langle 1 - a.\text{sg}(a + d + 1), a.\text{sg}(a + d + 1) \rangle$
\rightarrow_{85}	$\langle 1 - d + d^2.(1 - a), d.(1 - d) + d^2. \rangle$
\rightarrow_{86}	$\langle (1 - a).\overline{\text{sg}}(1 - d) + \text{sg}(1 - d).\overline{\text{sg}}(a + \min(1 - d, \text{sg}(a))),$ $a.\overline{\text{sg}}(1 - d) + d.\text{sg}(1 - d).\text{sg}(a) \rangle$
\rightarrow_{87}	$\langle \max(\overline{\text{sg}}(d), 1 - a), \min(\text{sg}(d), a) \rangle$
\rightarrow_{88}	$\langle \max(\overline{\text{sg}}(d), \text{sg}(1 - a)), \min(\text{sg}(d), \overline{\text{sg}}(1 - a)) \rangle$

\rightarrow_{89}	$\langle \max(\overline{\text{sg}}(d), 1 - a), \min(d, a) \rangle$
\rightarrow_{90}	$\langle \max(\overline{\text{sg}}(a), \overline{\text{sg}}(d)), \min(d, \overline{\text{sg}}(1 - a)) \rangle$
\rightarrow_{91}	$\langle \max(c, \min(1 - c, b)), 1 - \max(b, c) \rangle$
\rightarrow_{92}	$\langle \overline{\text{sg}}(1 - b - c), \min(1 - b, \text{sg}(1 - b - c)) \rangle$
\rightarrow_{93}	$\langle (1 - \min(1 - b, \text{sg}(1 - b - c)), \min(1 - b, \text{sg}(1 - b - c)) \rangle$
\rightarrow_{94}	$\langle c + (1 - c)^2.b, (1 - c).c + (1 - c)^2.(1 - b) \rangle$
\rightarrow_{95}	$\langle \min(b, \overline{\text{sg}}(c)) + \text{sg}(c).(\overline{\text{sg}}(1 - b) + \min(c, \text{sg}(1 - b))), (\min(1 - b, \overline{\text{sg}}(c)) + \min(1 - c, \text{sg}(c), \text{sg}(1 - b))) \rangle$
\rightarrow_{96}	$\langle \max(\overline{\text{sg}}(1 - c), b), \min(\text{sg}(1 - b), 1 - c) \rangle$
\rightarrow_{97}	$\langle \max(\overline{\text{sg}}(1 - c), \text{sg}(b)), \min(\text{sg}(1 - c), \overline{\text{sg}}(b)) \rangle$
\rightarrow_{98}	$\langle \max(\overline{\text{sg}}(1 - c), b), 1 - \max(b, c) \rangle$
\rightarrow_{99}	$\langle \max(\overline{\text{sg}}(1 - c), \overline{\text{sg}}(1 - b)), \min(1 - c, \overline{\text{sg}}(b)) \rangle$
\rightarrow_{100}	$\langle \max(\min(b, \text{sg}(a)), c), \min(a, \text{sg}(b), d) \rangle$
\rightarrow_{101}	$\langle \max(\min(b, \text{sg}(a)), \min(c, \text{sg}(d))), \min(a, \text{sg}(b), \text{sg}(c), d) \rangle$
\rightarrow_{102}	$\langle \max(b, \min(c, \text{sg}(d))), \min(a, \text{sg}(c), d) \rangle$
\rightarrow_{103}	$\langle \max(\min(1 - a, \text{sg}(a)), 1 - d), \min(a, \text{sg}(1 - a), d) \rangle$
\rightarrow_{104}	$\langle \max(\min(1 - a, \text{sg}(a)), \min(1 - d, \text{sg}(d))), \min(a, \text{sg}(1 - a), d, \text{sg}(1 - d)) \rangle$
\rightarrow_{105}	$\langle \max(1 - a, \min(1 - d, \text{sg}(d))), \min(a, d, \text{sg}(1 - d)) \rangle$
\rightarrow_{106}	$\langle \max(\min(b, \text{sg}(1 - b)), c), \min(1 - b, \text{sg}(b), 1 - c) \rangle$
\rightarrow_{107}	$\langle \max(\min(b, \text{sg}(1 - b)), \min(c, \text{sg}(1 - c))), \min(1 - b, \text{sg}(b), 1 - c, \text{sg}(c)) \rangle$
\rightarrow_{108}	$\langle \max(b, \min(c, \text{sg}(1 - c))), \min(1 - b, 1 - c, \text{sg}(c)) \rangle$
\rightarrow_{109}	$\langle b + \min(\overline{\text{sg}}(1 - a), c), a.b + \min(\overline{\text{sg}}(1 - a), d) \rangle$
\rightarrow_{110}	$\langle \max(b, c), \min(a.b + \overline{\text{sg}}(1 - a), d) \rangle$
\rightarrow_{111}	$\langle \max(b, c.d + \overline{\text{sg}}(1 - c)), \min(a.b + \overline{\text{sg}}(1 - a), d.(c.d + \overline{\text{sg}}(1 - c)) + \overline{\text{sg}}(1 - d)) \rangle$
\rightarrow_{112}	$\langle b + c - b.c, a.b + \overline{\text{sg}}(1 - a).d \rangle$
\rightarrow_{113}	$\langle b + c.d - b.(c.d + \overline{\text{sg}}(1 - c)), (a.b + \overline{\text{sg}}(1 - a)).(d.(c.d + \overline{\text{sg}}(1 - c)) + \overline{\text{sg}}(1 - d)) \rangle$
\rightarrow_{114}	$\langle 1 - a + \min(\overline{\text{sg}}(1 - a), 1 - d), a.(1 - a) + \min(\overline{\text{sg}}(1 - a), d) \rangle$
\rightarrow_{115}	$\langle 1 - \min(a, d), \min(a.(1 - a) + \overline{\text{sg}}(1 - a), d) \rangle$
\rightarrow_{116}	$\langle \max(1 - a, (1 - d).d + \overline{\text{sg}}(d)), \min(a.(1 - a) + \overline{\text{sg}}(1 - a), d.((1 - d).d + \overline{\text{sg}}(d)) + \overline{\text{sg}}(1 - d)) \rangle$
\rightarrow_{117}	$\langle 1 - a - d + a.d, (a.(1 - a) + \overline{\text{sg}}(1 - a)).d \rangle$
\rightarrow_{118}	$\langle 1 - a + (1 - d).d - (1 - a).((1 - d).d + \overline{\text{sg}}(d)), (a.(1 - a) + \overline{\text{sg}}(1 - a)).d.((1 - d).d + \overline{\text{sg}}(d)) + \overline{\text{sg}}(1 - d) \rangle$
\rightarrow_{119}	$\langle b + \min(\overline{\text{sg}}(b), c), (1 - b).b + \min(\overline{\text{sg}}(b), 1 - c) \rangle$
\rightarrow_{120}	$\langle \max(b, c), \min((1 - b).b + \overline{\text{sg}}(b), 1 - c) \rangle$
\rightarrow_{121}	$\langle \max(b, c.(1 - c) + \overline{\text{sg}}(1 - c)), \min((1 - b).b + \overline{\text{sg}}(b), (1 - c).(c.(1 - c) + \overline{\text{sg}}(1 - c))) + \overline{\text{sg}}(c) \rangle$
\rightarrow_{122}	$\langle b + c - b.c, ((1 - c).b + \overline{\text{sg}}(b)).(1 - c) \rangle$

\rightarrow_{123}	$\langle b + c.(1 - c) - (b.(c.(1 - c) + \overline{\text{sg}}(1 - c))),$ $((1 - b).b + \overline{\text{sg}}(b)).(((1 - c).(c.(1 - c) + \overline{\text{sg}}(1 - c))) + \overline{\text{sg}}(c)) \rangle$
\rightarrow_{124}	$\langle c + \min(\overline{\text{sg}}(1 - d), b), c.d + \min(\overline{\text{sg}}(1 - d), a) \rangle$
\rightarrow_{125}	$\langle \max(b, c), \min(c.d + \overline{\text{sg}}(1 - d), a) \rangle$
\rightarrow_{126}	$\langle \max(c, a.b + \overline{\text{sg}}(1 - b)), \min(c.d + \overline{\text{sg}}(1 - d), a.(a.b + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(1 - a)) \rangle$
\rightarrow_{127}	$\langle b + c - b.c, (c.d + \overline{\text{sg}}(1 - d)).a \rangle$
\rightarrow_{128}	$\langle c + a.b - c.(a.b + \overline{\text{sg}}(1 - b)), (c.d + \overline{\text{sg}}(1 - d)).(a.(a.b + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(1 - a)) \rangle$
\rightarrow_{129}	$\langle 1 - d + \min(\overline{\text{sg}}(1 - d), 1 - a), d.(1 - d) + \min(\overline{\text{sg}}(1 - d), a) \rangle$
\rightarrow_{130}	$\langle 1 - \min(d, a), \min(d.(1 - d) + \overline{\text{sg}}(1 - d), a) \rangle$
\rightarrow_{131}	$\langle \max(1 - d, (1 - a).a + \overline{\text{sg}}(a)),$ $\min(d.(1 - d) + \overline{\text{sg}}(1 - d), a.((1 - a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1 - a)) \rangle$
\rightarrow_{132}	$\langle 1 - a.d, (d.(1 - d) + \overline{\text{sg}}(1 - d)).a \rangle$
\rightarrow_{133}	$\langle 1 - d + (1 - a).a - (1 - d).((1 - a).a + \overline{\text{sg}}(a)),$ $(d.(1 - d) + \overline{\text{sg}}(1 - d)).(a.((1 - a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1 - a)) \rangle$
\rightarrow_{134}	$\langle c + \min(\overline{\text{sg}}(c), b), (1 - c).c + \min(\overline{\text{sg}}(c), (1 - b)) \rangle$
\rightarrow_{135}	$\langle \max(b, c), \min((1 - c).c + \overline{\text{sg}}(c), 1 - b) \rangle$
\rightarrow_{136}	$\langle \max(c, (b.(1 - b) + \overline{\text{sg}}(1 - b))),$ $\min((1 - c).c + \overline{\text{sg}}(c), (1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b)) \rangle$
\rightarrow_{137}	$\langle b + c - b.c, ((1 - c).c + \overline{\text{sg}}(c)).(1 - b) \rangle$
\rightarrow_{138}	$\langle c + b.(1 - b) - c.(b.(1 - b) + \overline{\text{sg}}(1 - b)),$ $((1 - c).c + \overline{\text{sg}}(c)).((1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b)) \rangle$

Table 2

\neg_1	$\langle x, b, a \rangle$
\neg_2	$\langle x, \overline{\text{sg}}(a), \text{sg}(a) \rangle$
\neg_3	$\langle x, b, a.b + a^2 \rangle$
\neg_4	$\langle x, b, 1 - b \rangle$
\neg_5	$\langle x, \overline{\text{sg}}(1 - b), \text{sg}(1 - b) \rangle$
\neg_6	$\langle x, \overline{\text{sg}}(1 - b), \text{sg}(a) \rangle$
\neg_7	$\langle x, \overline{\text{sg}}(1 - b), a \rangle$
\neg_8	$\langle x, 1 - a, a \rangle$
\neg_9	$\langle x, \overline{\text{sg}}(a), a \rangle$
\neg_{10}	$\langle x, \overline{\text{sg}}(1 - b), 1 - b \rangle$
\neg_{11}	$\langle x, \text{sg}(b), \overline{\text{sg}}(b) \rangle$
\neg_{12}	$\langle x, b.(b + a), \min(1, a.(b^2 + a + b.a)) \rangle$
\neg_{13}	$\langle x, \text{sg}(1 - a), \overline{\text{sg}}(1 - a) \rangle$
\neg_{14}	$\langle x, \text{sg}(b), \overline{\text{sg}}(1 - a) \rangle$
\neg_{15}	$\langle x, \overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - a) \rangle$
\neg_{16}	$\langle x, \overline{\text{sg}}(a), \overline{\text{sg}}(1 - a) \rangle$
\neg_{17}	$\langle x, \overline{\text{sg}}(1 - b), \overline{\text{sg}}(b) \rangle$
\neg_{18}	$\langle x, b.\text{sg}(a), a.\text{sg}(b) \rangle$

\neg_{19}	$\langle x, b.\text{sg}(a), 0 \rangle$
\neg_{20}	$\langle x, b, 0 \rangle$
\neg_{21}	$\langle x, \min(1 - a, \text{sg}(a)), \min(a, \text{sg}(1 - a)) \rangle$
\neg_{22}	$\langle x, \min(1 - a, \text{sg}(a)), 0 \rangle$
\neg_{23}	$\langle x, 1 - a, 0 \rangle$
\neg_{24}	$\langle x, \min(b, \text{sg}(1 - b)), \min(1 - b, \text{sg}(b)) \rangle$
\neg_{25}	$\langle x, \min(b, \text{sg}(1 - b)), 0 \rangle$
\neg_{26}	$\langle x, b, a.b + \overline{\text{sg}}(1 - a) \rangle$
\neg_{27}	$\langle x, 1 - a, a.(1 - a) + \overline{\text{sg}}(1 - a) \rangle$
\neg_{28}	$\langle x, b, (1 - b).b + \overline{\text{sg}}(b) \rangle$
\neg_{29}	$\langle x, \max(0, b.a + \overline{\text{sg}}(1 - b)), \min(1, a.(b.a + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(1 - a)) \rangle$
\neg_{30}	$\langle x, a.b, a.(a.b + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(1 - a) \rangle$
\neg_{31}	$\langle x, \max(0, (1 - a).a + \overline{\text{sg}}(a)), \min(1, a.((1 - a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1 - a)) \rangle$
\neg_{32}	$\langle x, (1 - a).a, a.((1 - a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1 - a) \rangle$
\neg_{33}	$\langle x, b.(1 - b) + \overline{\text{sg}}(1 - b), (1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b) \rangle$
\neg_{34}	$\langle x, b.(1 - b), (1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b) \rangle$

Table 3

\neg_1	$\rightarrow_1, \rightarrow_4, \rightarrow_5, \rightarrow_6, \rightarrow_7, \rightarrow_{10}, \rightarrow_{13}, \rightarrow_{61}, \rightarrow_{63}, \rightarrow_{64}, \rightarrow_{66}, \rightarrow_{67}, \rightarrow_{68}, \rightarrow_{69}, \rightarrow_{70}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{73}, \rightarrow_{78}, \rightarrow_{80}, \rightarrow_{124}, \rightarrow_{125}, \rightarrow_{127}$
\neg_2	$\rightarrow_2, \rightarrow_3, \rightarrow_8, \rightarrow_{11}, \rightarrow_{16}, \rightarrow_{20}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{37}, \rightarrow_{40}, \rightarrow_{41}, \rightarrow_{42}$
\neg_3	$\rightarrow_9, \rightarrow_{17}, \rightarrow_{21}$
\neg_4	$\rightarrow_{12}, \rightarrow_{18}, \rightarrow_{22}, \rightarrow_{46}, \rightarrow_{49}, \rightarrow_{50}, \rightarrow_{51}, \rightarrow_{53}, \rightarrow_{54}, \rightarrow_{91}, \rightarrow_{93}, \rightarrow_{94}, \rightarrow_{95}, \rightarrow_{96}, \rightarrow_{98}, \rightarrow_{134}, \rightarrow_{135}, \rightarrow_{137}$
\neg_5	$\rightarrow_{14}, \rightarrow_{15}, \rightarrow_{19}, \rightarrow_{23}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{52}, \rightarrow_{55}, \rightarrow_{56}, \rightarrow_{57}$
\neg_6	$\rightarrow_{24}, \rightarrow_{26}, \rightarrow_{27}, \rightarrow_{65}$
\neg_7	$\rightarrow_{25}, \rightarrow_{28}, \rightarrow_{29}, \rightarrow_{62}$
\neg_8	$\rightarrow_{30}, \rightarrow_{33}, \rightarrow_{34}, \rightarrow_{35}, \rightarrow_{36}, \rightarrow_{38}, \rightarrow_{39}, \rightarrow_{76}, \rightarrow_{82}, \rightarrow_{84}, \rightarrow_{85}, \rightarrow_{86}, \rightarrow_{87}, \rightarrow_{89}, \rightarrow_{129}, \rightarrow_{130}, \rightarrow_{132}$
\neg_9	$\rightarrow_{43}, \rightarrow_{44}, \rightarrow_{45}, \rightarrow_{83}$
\neg_{10}	$\rightarrow_{58}, \rightarrow_{59}, \rightarrow_{60}, \rightarrow_{92}$
\neg_{11}	$\rightarrow_{74}, \rightarrow_{97}$
\neg_{12}	\rightarrow_{75}
\neg_{13}	$\rightarrow_{77}, \rightarrow_{88}$
\neg_{14}	\rightarrow_{79}
\neg_{15}	\rightarrow_{81}
\neg_{16}	\rightarrow_{90}
\neg_{17}	\rightarrow_{99}
\neg_{18}	\rightarrow_{100}
\neg_{19}	\rightarrow_{101}
\neg_{20}	$\rightarrow_{102}, \rightarrow_{108}$

\neg_{21}	\rightarrow_{103}
\neg_{22}	\rightarrow_{104}
\neg_{23}	\rightarrow_{105}
\neg_{24}	\rightarrow_{106}
\neg_{25}	\rightarrow_{107}
\neg_{26}	$\rightarrow_{109}, \rightarrow_{110}, \rightarrow_{111}, \rightarrow_{112}, \rightarrow_{113}$
\neg_{27}	$\rightarrow_{114}, \rightarrow_{115}, \rightarrow_{116}, \rightarrow_{117}, \rightarrow_{118}$
\neg_{28}	$\rightarrow_{119}, \rightarrow_{120}, \rightarrow_{121}, \rightarrow_{122}, \rightarrow_{123}$
\neg_{29}	\rightarrow_{126}
\neg_{30}	\rightarrow_{128}
\neg_{31}	\rightarrow_{131}
\neg_{32}	\rightarrow_{133}
\neg_{33}	\rightarrow_{136}
\neg_{34}	\rightarrow_{138}

3 Main results

In [3], the following forms of De Morgan's Laws

$$\neg(\neg x \vee \neg y) = x \wedge y$$

$$\neg(\neg x \wedge \neg y) = x \vee y$$

and

$$\neg(\neg x \vee \neg y) = \neg\neg x \wedge \neg\neg y$$

$$\neg(\neg x \wedge \neg y) = \neg\neg x \vee \neg\neg y$$

and the following forms of the Law for Excluded Third are discussed: $x \vee \neg x$, $\neg\neg x \vee \neg x$.

In [1], it is proved for the first time that the axiom

$$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$$

is an IFT, when \supset is \rightarrow_1 , while it is not valid for implication \rightarrow_3 .

Having in mind the new forms of De Morgan's Laws and the Law of Excluded Third, the above axiom can be modified to the form

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset \neg\neg A).$$

Here, we formulate four theorems and prove one case of the fourth theorem. Now, it is clear that this proof had to be given 25 years ago.

Theorem 1. For every two variables A and B , the expression

$$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$$

is an IFT for implications $\rightarrow_1, \rightarrow_4, \rightarrow_5, \rightarrow_7, \rightarrow_9, \rightarrow_{13}, \rightarrow_{18}, \rightarrow_{20}, \dots, \rightarrow_{23}, \rightarrow_{25}, \rightarrow_{27}, \dots, \rightarrow_{29}, \rightarrow_{61}, \rightarrow_{66}, \rightarrow_{71}, \rightarrow_{74}, \rightarrow_{76}, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \rightarrow_{100}, \dots, \rightarrow_{102}, \rightarrow_{104}, \rightarrow_{105}, \rightarrow_{107}, \rightarrow_{109}, \dots, \rightarrow_{113}, \rightarrow_{118}, \rightarrow_{124}, \dots, \rightarrow_{128}, \rightarrow_{133}$.

Theorem 2. For every two variables A and B , the expression

$$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$$

is a tautology for implications $\rightarrow_{20}, \rightarrow_{23}, \rightarrow_{74}, \rightarrow_{77}$.

Theorem 3. For every two variables A and B , the expression

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg \neg B) \supset \neg \neg A)$$

is an IFT for implications $\rightarrow_1, \dots, \rightarrow_5, \rightarrow_7, \dots, \rightarrow_9, \rightarrow_{11}, \dots, \rightarrow_{38}, \rightarrow_{40}, \dots, \rightarrow_{43}, \rightarrow_{45}, \dots, \rightarrow_{53}, \rightarrow_{55}, \dots, \rightarrow_{57}, \rightarrow_{61}, \rightarrow_{66}, \rightarrow_{71}, \rightarrow_{74}, \dots, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \dots, \rightarrow_{83}, \rightarrow_{85}, \rightarrow_{88}, \rightarrow_{91}, \rightarrow_{94}, \rightarrow_{97}, \rightarrow_{99}, \dots, \rightarrow_{119}, \rightarrow_{121}, \rightarrow_{124}, \dots, \rightarrow_{137}$.

Theorem 4. For every two variables A and B , the expression

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg \neg B) \supset \neg \neg A)$$

is a tautology for implications $\rightarrow_2, \rightarrow_3, \rightarrow_8, \rightarrow_{11}, \rightarrow_{14}, \dots, \rightarrow_{16}, \rightarrow_{19}, \rightarrow_{20}, \rightarrow_{23}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{37}, \rightarrow_{40}, \dots, \rightarrow_{43}, \rightarrow_{45}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{52}, \rightarrow_{55}, \dots, \rightarrow_{57}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{83}, \rightarrow_{88}, \rightarrow_{97}, \rightarrow_{99}$.

Proof of Theorem 4: Here, we prove Theorem 4 for the case of \rightarrow_3 . Let A and B be two variables.

$$\begin{aligned} & (\neg A \supset \neg B) \supset ((\neg A \supset \neg \neg B) \supset \neg \neg A) \\ &= (\langle 1 - \text{sg}(a), \text{sg}(a) \rangle \supset \langle 1 - \text{sg}(c), \text{sg}(c) \rangle) \\ &\supset ((\langle 1 - \text{sg}(a), \text{sg}(a) \rangle \supset \langle \text{sg}(c), 1 - \text{sg}(c) \rangle) \supset \langle \text{sg}(a), 1 - \text{sg}(a) \rangle) \\ &= (\langle 1 - \text{sg}(a), \text{sg}(a) \rangle \supset \langle 1 - \text{sg}(c), \text{sg}(c) \rangle) \\ &\supset (\langle 1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)), (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)).\text{sg}(1 - \text{sg}(c) - \text{sg}(a)) \rangle \\ &\quad \supset \langle \text{sg}(a), 1 - \text{sg}(a) \rangle) \\ &= (\langle 1 - (1 - 1 + \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - 1 + \text{sg}(c)), \text{sg}(c).\text{sg}(1 - \text{sg}(a) - 1 + \text{sg}(c)).\text{sg}(\text{sg}(c) - \text{sg}(a)) \rangle) \\ &\supset (\langle 1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)), (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) \rangle \\ &\quad \supset \langle \text{sg}(a), 1 - \text{sg}(a) \rangle) \\ &= \langle 1 - \text{sg}(c).\text{sg}(\text{sg}(c) - \text{sg}(a)), \text{sg}(c).\text{sg}(\text{sg}(c) - \text{sg}(a)) \rangle \\ &\supset \langle 1 - (1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a)), \\ &\quad (1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a)) \\ &\quad .\text{sg}(1 - \text{sg}(a) - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c))) \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle 1 - (1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a)) \\
&\quad .\text{sg}(1 - \text{sg}(c).\text{sg}(\text{sg}(c) - \text{sg}(a))) - 1 + (1 - \text{sg}(a)) \\
&\quad .\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a))), \\
&\quad (1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a)) \\
&\quad .\text{sg}(1 - \text{sg}(a) - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c))) \\
&.\text{sg}(1 - \text{sg}(c).\text{sg}(\text{sg}(c) - \text{sg}(a))) - 1 + (1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a))) \\
&\quad .\text{sg}((1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a))) \\
&\quad .\text{sg}(1 - \text{sg}(a) - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c))) - \text{sg}(c).\text{sg}(\text{sg}(c) - \text{sg}(a))) \rangle.
\end{aligned}$$

Let

$$\begin{aligned}
X \equiv & 1 - (1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a)) \\
& .\text{sg}(1 - \text{sg}(c).\text{sg}(\text{sg}(c) - \text{sg}(a)) - 1 + (1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a))) \\
& -(1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a)) \\
& .\text{sg}(1 - \text{sg}(a) - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c))) \\
& .\text{sg}(1 - \text{sg}(c).\text{sg}(\text{sg}(c) - \text{sg}(a)) - 1 + (1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a))) \\
& .\text{sg}((1 - \text{sg}(a)).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c)) - \text{sg}(a)) \\
& .\text{sg}(1 - \text{sg}(a) - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(a) - \text{sg}(c))) - \text{sg}(c).\text{sg}(\text{sg}(c) - \text{sg}(a)))
\end{aligned}$$

Let $a = 0$. Then

$$\begin{aligned}
X = & 1 - \text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))).\text{sg}(1 - \text{sg}(c).\text{sg}(\text{sg}(c)) - 1 + \text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c)))) \\
& - \text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))) \\
& .\text{sg}(1 - \text{sg}(c).\text{sg}(\text{sg}(c)) - 1 + \text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c)))) \\
& .\text{sg}(\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))) - \text{sg}(c).\text{sg}(\text{sg}(c))) \\
= & 1 - \text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))).\text{sg}(\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))) - \text{sg}(c)) \\
& - \text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))) \\
& .\text{sg}(\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))) - \text{sg}(c)) \\
& .\text{sg}(\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))).\text{sg}(1 - (1 - \text{sg}(c)).\text{sg}(1 - \text{sg}(c))) - \text{sg}(c).\text{sg}(\text{sg}(c)))
\end{aligned}$$

(because for every real number $x \geq 0$, $x.\text{sg}(x) = x$)

$$\begin{aligned}
&= 1 - \text{sg}(1 - (1 - \text{sg}(c))).\text{sg}(\text{sg}(1 - (1 - \text{sg}(c))) - \text{sg}(c)) - \text{sg}(1 - (1 - \text{sg}(c))).\text{sg}(1 - (1 - \text{sg}(c))) \\
&\quad .\text{sg}(\text{sg}(1 - (1 - \text{sg}(c))) - \text{sg}(c)).\text{sg}(1 - (1 - \text{sg}(c))).\text{sg}(1 - (1 - \text{sg}(c))) - sg(c)) \\
&= 1 - \text{sg}(\text{sg}(c)).\text{sg}(\text{sg}(\text{sg}(c)) - \text{sg}(c)) - \text{sg}(\text{sg}(c)).\text{sg}(\text{sg}(c))
\end{aligned}$$

$$\begin{aligned}
& \cdot \text{sg}(\text{sg}(\text{sg}(c)) - \text{sg}(c)) \cdot \text{sg}(\text{sg}(c)) \cdot \text{sg}(\text{sg}(c)) - \text{sg}(c)) \\
& = 1 - \text{sg}(c) \cdot \text{sg}(\text{sg}(c) - \text{sg}(c)) - \text{sg}(c) \cdot \text{sg}(c) \cdot \text{sg}(\text{sg}(c) - \text{sg}(c)) \cdot \text{sg}(\text{sg}(c) - \text{sg}(c)) \\
& = 1 - \text{sg}(c) \cdot \text{sg}(0) \cdot \text{sg}(0) = 1.
\end{aligned}$$

Let $a > 0$. Then from $1 - \text{sg}(a) = 0$ it follows directly that

$$X = 1.$$

Therefore, the axiom $(\neg A \supset \neg B) \supset ((\neg A \supset \neg \neg B) \supset \neg \neg A)$ is a tautology and hence it is an IFT, too. \square

All other assertions are proved analogously.

References

- [1] Atanassov, K. Two variants of intuitionistic fuzzy propositional calculus. *Preprint IM-MFAIS-5-88*, Sofia, 1988.
- [2] Atanassov, K. *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer, Heidelberg, 1999.
- [3] Atanassov, K. On intuitionistic fuzzy negations and De Morgan laws. *Proc. of Eleventh International Conf. IPMU 2006*, Paris, July 2–7, 2006, 2399–2404.
- [4] Atanassov, K. A new intuitionistic fuzzy implication from a modal type. *Advanced Studies in Contemporary Mathematics*, Vol. 12, 2006, No. 1, 117–122.
- [5] Atanassov, K. On some intuitionistic fuzzy implications. *Comptes Rendus de l'Academie bulgare des Sciences*, Vol. 59, 2006, No. 1, 19–24.
- [6] Atanassov, K. *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin, 2012.
- [7] Atanassova, L. A new intuitionistic fuzzy implication. *Cybernetics and Information Technologies*, Vol. 9, 2009, No. 2, 21–25.
- [8] Atanassova, L. On two modifications of the intuitionistic fuzzy implication $\rightarrow_{@}$. *Notes on Intuitionistic Fuzzy Sets*, Vol. 18, 2012, No. 2, 26–30.
- [9] Atanassova, L. On the modal form of the intuitionistic fuzzy implications $\rightarrow'_{@}$ and $\rightarrow''_{@}$. *Issues in Intuitionistic Fuzzy Sets and Generalized Nets*, Vol. 10, 5–11.
- [10] Dworniczak, P. Some remarks about the L. Atanassova's paper “A new intuitionistic fuzzy implication”. *Cybernetics and Information Technologies*, Vol. 10, 2010, No. 3, 3–9.
- [11] Dworniczak, P. On one class of intuitionistic fuzzy implications. *Cybernetics and Information Technologies*, Vol. 10, 2010, No. 4, 13–21.
- [12] Dworniczak, P. On some two-parametric intuitionistic fuzzy implication. *Notes on Intuitionistic Fuzzy Sets*, Vol. 17, 2011, No. 2, 8–16.