Notes on Intuitionistic Fuzzy Sets
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Numerical solution of an intuitionistic fuzzy parabolic partial differential equation using an explicit cubic spline method

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Abstract: This paper presents a numerical approach for solving intuitionistic fuzzy parabolic partial differential equations (IFPPDEs) using the explicit cubic spline method. Intuitionistic fuzzy systems, which extend classical fuzzy sets by incorporating a degree of non-membership, provide a more flexible framework for modelling uncertainty in real-world phenomena. The initial and boundary conditions of the intuitionistic fuzzy parabolic partial differential equation are considered intuitionistic triangular fuzzy numbers. The proposed cubic spline-based scheme ensures smooth and accurate approximations of the solution while maintaining stability and convergence properties. A discretization strategy is developed to transform the IFPPDE into a solvable system, and an iterative algorithm is introduced to handle the intuitionistic fuzzy parameters effectively. The efficiency and accuracy of the method are demonstrated through numerical experiments, comparing the results with the exact solution. The findings suggest that the cubic spline method provides good accuracy and computational efficiency, making it a promising tool for solving intuitionistic fuzzy parabolic partial differential problems in various scientific and engineering applications.

Keywords: Intuitionistic triangular fuzzy number, Fuzzy finite difference method, Stability, Convergence, Explicit cubic spline method.
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1 Introduction

Partial differential equations (PDEs) play a fundamental role in modelling various real-world phenomena in physics, engineering, finance, and biological systems. However, in many practical scenarios, uncertainties arise due to imprecise measurements, ambiguous initial or boundary conditions, and inherent vagueness in system parameters. Traditional PDE models fail to capture such uncertainties effectively. To address this, fuzzy set theory [12] has been widely applied, leading to the development of fuzzy partial differential equations (FPDEs). Among different fuzzy models, intuitionistic fuzzy sets (IFSs), introduced by Atanassov [5], provide an advanced mathematical framework by incorporating membership, non-membership, and hesitation degrees, making them more suitable for handling real-world imprecision. This has led to growing interest in intuitionistic fuzzy parabolic partial differential equations (IFPPDEs) as a powerful tool for modelling dynamic systems under uncertainty.

The study of intuitionistic fuzzy differential equations (IFDEs) has experienced substantial growth, driven by significant theoretical advancements and sophisticated computational methodologies. This paper provides a critical review of the key developments that have shaped the evolution of IFDEs, with an emphasis on both mathematical formulations and numerical techniques. The formal introduction of IFDEs in 2000 by Melliani and Chadli [8] marked a fundamental step in the field, incorporating intuitionistic fuzzy numbers into differential equations and establishing the groundwork for subsequent research. In 2015, Mondal and Roy [9] extended this framework by solving first-order homogeneous ordinary differential equations using triangular intuitionistic fuzzy numbers, thereby expanding the analytical scope of IFDEs. Further refinements emerged in 2015 when Nirmala and Parimala [10] enhanced the mathematical framework of IFDEs, strengthening their theoretical foundation. In 2018, Akın and Bayeğ [1] applied the intuitionistic Zadeh extension principle to address second-order fuzzy initial value problems, contributing to the methodological advancements in the field. A significant breakthrough occurred in 2020 with the work of Bouchra et al. [4] who rigorously established existence and uniqueness results for boundary value problems using Banach's fixed point theorem. This development reinforced the mathematical robustness of IFDEs and provided a solid theoretical basis for further exploration. Computational approaches saw substantial progress in 2021 when Sushanta et al. [7] introduced an explicit finite difference method to solve the intuitionistic fuzzy heat equation, accompanied by a stability analysis using the Von Neumann method. Later that year, the same authors, along with Anupama, extended their work to the intuitionistic fuzzy Poisson's partial differential equation, further validating the applicability of numerical techniques in solving complex IFDEs. The most recent advancement, reported in 2023 by Sushanta Man et al. [6] employed a finite difference approach to solve the intuitionistic fuzzy Poisson equation. This contribution underscored the growing importance of numerical methods in tackling IFDEs, highlighting their effectiveness in addressing complex differential systems. The evolution of IFDEs has been characterized by continuous theoretical refinements and computational innovations. The integration of rigorous mathematical principles with advanced numerical techniques ensures that IFDEs remain at the forefront of applied mathematics and engineering research. Future investigations will likely focus on enhancing stability, accuracy, and computational efficiency, further solidifying the relevance and applicability of IFDEs across diverse scientific disciplines. The main objective of this paper is to propose and analyse an explicit cubic spline method for solving IFPPDEs. The study aims to:

- Develop a numerical scheme that handles the uncertainty inherent in IFPPDEs.
- Solve parabolic PDEs under intuitionistic fuzzy conditions with high accuracy and efficiency.
- Develop a numerical scheme using explicit cubic splines to effectively approximate the solution of IFPPDEs.

This paper begins by introducing fundamental terminology essential to the study. Subsequently, a finite difference scheme based on the cubic spline method is formulated to solve parabolic partial differential equations in a crisp environment. The approach is then extended to develop an intuitionistic fuzzy finite difference method for solving IFPPDEs. The stability and convergence of the proposed method are analysed, followed by its validation through a numerical example.

2 Preliminaries

Definition 2.1. [7] *An intuitionistic fuzzy set is defined as a set $\tilde{A} \in X$ and and represented as $\tilde{A} = (x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X)$ and characterized by a membership function $\mu_{\tilde{A}} : X \to [0,1]$ and non-membership function $\nu_{\tilde{A}}(x) : X \to [0,1]$ for each element $x \in X$ such that $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$.*

Definition 2.2. [7] *The (α, β)-cut (level) of an intuitionistic fuzzy set $\tilde{A}$ is represented as $\tilde{A}_{\alpha,\beta}$ and defined as $\tilde{A}_{\alpha,\beta}$=$\{x \in X : \mu_{\tilde{A}}(x) \geq \alpha$,$\nu_{\tilde{A}}(x) \leq \beta\}$ as $\alpha \in [0,1]$, $\beta \in [0,1]$ such that $\alpha + \beta \leq 1$ where $\tilde{A}_{\alpha,\beta}$ is a crisp set that represent the $\tilde{A}$ at least to the degree of α and which does not belong to $\tilde{A}$ at most to the degree of β.*

Definition 2.3. [7] *An intuitionistic fuzzy set $\tilde{A}$ is said to be an intuitionistic fuzzy number if it satisfies the following properties:*
(1) there exist $x_0 \in \boldsymbol{R}$ such that membership value $\mu_{\tilde{A}}(x_0) = 1$ and non-membership value $\nu_{\tilde{A}}(x_0) = 0$.
(2) The membership function $\mu_{\tilde{A}}$ is convex fuzzy set if

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}; \forall x_1, x_2 \in \boldsymbol{R}, \lambda \in [0,1].$$

(3) The non-membership function $\nu_{\tilde{A}}$ is concave fuzzy set if

$$\nu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \leq max\{\nu_{\tilde{A}}(x_1), \nu_{\tilde{A}}(x_2)\}; \forall x_1, x_2 \in \boldsymbol{R}, \lambda \in [0,1].$$

Definition 2.4. [7] *A fuzzy number* $\tilde{A} =< (\sigma_1, \sigma_2, \sigma_3), (\sigma'_1, \sigma'_2, \sigma'_3) >$ *is known as an intuitionistic triangular fuzzy number if both its membership and non-membership functions satisfy the conditions mentioned below*

$$\mu_{\tilde{A}}(x; \sigma_1, \sigma_2, \sigma_3) = \begin{cases} 0, & for\, x < \sigma_1; \\ \frac{x-\sigma_1}{\sigma_2-\sigma_1}, & for\, \sigma_1 \leq x \leq \sigma_2; \\ \frac{\sigma_3-x}{\sigma_3-\sigma_2}, & for\, \sigma_2 \leq x \leq \sigma_3; \\ 0, & for\, x > \sigma_3; \end{cases}$$

$$\nu_{\tilde{A}}(x; \sigma'_1, \sigma'_2, \sigma'_3) = \begin{cases} 1, & for\, x < \sigma'_1; \\ \frac{\sigma_2-x}{\sigma_2-\sigma'_1}, & for\, \sigma'_1 \leq x \leq \sigma_2; \\ \frac{x-\sigma_2}{\sigma'_3-\sigma_2}, & for\, \sigma_2 \leq x \leq \sigma'_3; \\ 1, & for\, x > \sigma'_3; \end{cases}$$

where $\sigma'_1 \leq \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \sigma'_3$. *The* (α, β)*-cut of* $\tilde{A} =< (\sigma_1, \sigma_2, \sigma_3), (\sigma'_1, \sigma'_2, \sigma'_3) >$ *may be represented as,* $\tilde{A}_{\alpha,\beta} =< [\underline{A}(\alpha), \overline{A}(\alpha)], [\underline{A}'(\beta), \overline{A}'(\beta)] >$*, where* $\underline{A}(\alpha) = \sigma_1 + \alpha(\sigma_2 - \sigma_1)$, $\overline{A}(\alpha) = \sigma_3 - \alpha(\sigma_3 - \sigma 2)$, $\underline{A}'(\beta) = \sigma_2 - \beta(\sigma_2 - \sigma'_1)$, $\overline{A}'(\beta) = \sigma_2 + \beta(\sigma'_3 - \sigma_2)$.

Definition 2.5. [7] *Since we know that fuzzy numbers can be converted into intervals using the parametric form . So for any fuzzy number* $\tilde{\rho} = [\underline{\rho}(\alpha), \overline{\rho}(\alpha)]$, *and* $\tilde{\sigma} = [\underline{\sigma}(\alpha), \overline{\sigma}(\alpha)]$, *and scalar k, we have the interval-based fuzzy arithmetic as*

(i) $\tilde{\rho} = \tilde{\sigma}$ *if and only* $\underline{\rho}(\alpha) = \underline{\sigma}(\alpha)$ *and* $\overline{\rho}(\alpha) = \overline{\sigma}(\alpha)$,

(ii) $\tilde{\rho} + \tilde{\sigma} = [\underline{\rho}(\alpha) + \underline{\sigma}(\alpha), \overline{\rho}(\alpha) + \overline{\sigma}(\alpha)]$,

(iii) $\tilde{\rho} - \tilde{\sigma} = [\underline{\rho}(\alpha), \overline{\rho}(\alpha)] - [\underline{\sigma}(\alpha), \overline{\sigma}(\alpha)] = [\underline{\rho}(\alpha) - \overline{\sigma}(\alpha), \overline{\rho}(\alpha) - \underline{\sigma}(\alpha)]$,

(iv) $\tilde{\rho} \times \tilde{\sigma} = [\min(\underline{\rho}(\alpha) \times \underline{\sigma}(\alpha), \underline{\rho}(\alpha) \times \overline{y\sigma}(\alpha), \overline{\rho}(\alpha) \times \underline{\sigma}(\alpha), \overline{\rho}(\alpha) \times \overline{\sigma}(\alpha)), \max(\underline{\rho}(\alpha) \times \underline{\sigma}(\alpha), \underline{\rho}(\alpha) \times \overline{\sigma}(\alpha), \overline{\rho}(\alpha) \times \underline{\sigma}(\alpha), \overline{\rho}(\alpha) \times \overline{\sigma}(\alpha))]$,

(v) $k\tilde{\rho} = \begin{cases} [k\underline{\rho}(\alpha), k\overline{\rho}(\alpha)], & for\, k \geq 0 \\ [k\overline{\rho}(\alpha), k\underline{\rho}(\alpha)], & for\, k < 0. \end{cases}$

Definition 2.6. [5] *The degree of non-determinacy in an intuitionistic fuzzy set can be calculated as* $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$ *where* $x \in X$ *and* $\tilde{A}$ *is the intuitionistic fuzzy set.*

3 Finite difference formulation using the explicit cubic spline method for parabolic partial differential equation in the crisp environment

To derive the finite difference method using the cubic spline method to find the solution of the heat conduction equation, the time derivative is explicitly replaced using the forward difference operator while the space derivatives are approximated with a natural cubic spline. A cubic spline approximation to the heat conduction equation produces a finite difference formula. Let $U_{i,j}$ be the discrete approximation of $U(x,t)$ at the point (x_i, t_j). $S_j(x_i)$ represent the cubic spline interpolating the value of $U_{i,j}$ at the j^{th} time level. Replacing the heat conduction equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

by

$$M_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{k}, \tag{1}$$

$$M_{i,j} = S_j''(x_i), \tag{2}$$

and

$$S_j(x_i) = M_{i-1,j}\frac{(x_i - x)^3}{6h} + (U_{i-1,j} - \frac{h^2}{6}M_{i-1,j})\frac{(x_i - x)}{h} + (U_{i,j} - \frac{h^2}{6}M_{i,j})\frac{(x - x_{i-1})}{h}, \tag{3}$$

where $i = 1, 2, 3, ...N$.

$$S_j'(x_i^+) = \frac{-h}{3}M_{i,j} - \frac{-h}{6}M_{i+1,j} + \frac{U_{i+1,j} - U_{i,j}}{h}, \tag{4}$$

$$S_j'(x_i^-) = \frac{h}{3}M_{i,j} + \frac{h}{6}M_{i-1,j} + \frac{U_{i,j} - U_{i-1,j}}{h}, \tag{5}$$

so, the continuity of the first derivative implies

$$\frac{1}{6}M_{i-1,j} + \frac{2}{3}M_{i,j} + \frac{1}{6}M_{i+1,j} = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2}. \tag{6}$$

Now replace the time derivative with the forward difference operator by

$$M_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{k}$$

in the equation (6). So, we have the finite difference equation as:

$$U_{i-1,j+1} + 4U_{i,j+1} + U_{i+1,j+1} = (1 + 6\Gamma)U_{i-1,j} + 4(1 - 3\Gamma)U_{i,j} + (1 + 6\Gamma)U_{i+1,j} \tag{7}$$

such that $\Gamma = \frac{k}{h^2}$. The finite difference method presented is derived from the explicit cubic spline approach within a crisp environment, enabling the determination of numerical solutions for crisp parabolic PDEs.

4 Formulation of the intuitionistic fuzzy finite difference scheme

Let $\tilde{U}$ be an intuitionistic fuzzy function of the independent crisp variable x and t, which satisfies the IFPPDEs as

$$(D_t - b^2 D_x^2)\tilde{U} = \tilde{0}, \tag{8}$$

the boundary condition is given as $\tilde{U}(0,t) = \tilde{U}(l,t) = \tilde{0}$, also, the initial condition is given as $\tilde{U}(x,0) = \tilde{f}(x)$. We define the domain as $I = \{(x,t) : 0 \leq x \leq l, 0 \leq t \leq T\}$. An (α, β) cut of $\tilde{U}(x,t)$ and its parametric form will be

$$\tilde{U}(x,t)[\alpha,\beta] = \langle[\underline{U}(x,t;\alpha), \overline{U}(x,t;\alpha)], [\underline{U'}(x,t;\beta), \overline{U}'(x,t;\beta)]\rangle. \tag{9}$$

We assume that the continuous partial derivative of $\underline{U}(x,t;\alpha),\overline{U}(x,t;\alpha),\underline{U'}(x,t;\beta),\overline{U}'(x,t;\beta)$ exists with respect to x and t. Thus, for any $(x,t) \in I$, $(D_t - b^2D_x^2)\underline{U}(x,t;\alpha)$, $(D_t - b^2D_x^2)\overline{U}(x,t;\alpha),(D_t - b^2D_x^2)\underline{U'}(x,t;\beta),(D_t - b^2D_x^2)\overline{U}'(x,t;\beta)$ are continuous for any $\alpha \in [0,1]$, $\beta \in [0,1]$.

As we know, to find the finite difference method, the time derivative is explicitly replaced using the forward difference operator, while the space derivatives are approximated with a natural cubic spline. So, using the standard finite difference method and definition of Taylor's theorem [2].

$$\begin{cases} D_x^2\underline{U}(x,t;\alpha)|_{i,j} \approx \dfrac{\underline{U}_{i+1,j} - 2\underline{U}_{i,j} + \underline{U}_{i-1,j}}{h^2}, \\ D_x^2\overline{U}(x,t;\alpha)|_{i,j} \approx \dfrac{\overline{U}_{i+1,j} - 2\overline{U}_{i,j} + \overline{U}_{i-1,j}}{h^2}, \\ D_x^2\underline{U'}(x,t;\beta)|_{i,j} \approx \dfrac{\underline{U'}_{i+1,j} - 2\underline{U'}_{i,j} + \underline{U'}_{i-1,j}}{h^2}, \\ D_x^2\overline{U}'(x,t;\beta)|_{i,j} \approx \dfrac{\overline{U}'_{i+1,j} - 2\overline{U}'_{i,j} + \overline{U}'_{i-1,j}}{h^2}. \end{cases} \tag{10}$$

The error of the above finite difference scheme $O(h^2)$.The forward difference scheme for $(D_t)\tilde{U}$, we have

$$\begin{cases} D_t\underline{U}(x,t;\alpha)|_{i,j} \approx \dfrac{\underline{U}_{i,j+1} - \underline{U}_{i,j}}{k}, \\ D_t\overline{U}(x,t;\alpha)|_{i,j} \approx \dfrac{\overline{U}_{i,j+1} - \overline{U}_{i,j}}{k}, \\ D_t\underline{U'}(x,t;\beta)|_{i,j} \approx \dfrac{\underline{U'}_{i,j+1} - \underline{U'}_{i,j}}{k}, \\ D_t\overline{U}'(x,t;\beta)|_{i,j} \approx \dfrac{\overline{U}'_{i,j+1} - \overline{U}'_{i,j}}{k}. \end{cases} \tag{11}$$

The error of the above finite difference schemes $O(k)$. Now, the time derivative is explicitly replaced using the forward difference operator:

$$\tilde{M}_{i,j} = \frac{\tilde{U}_{i,j+1} - \tilde{U}_{i,j}}{k}, \tag{12}$$

$$\tilde{M}_{i+1,j} = \frac{\tilde{U}_{i+1,j+1} - \tilde{U}_{i+1,j}}{k}, \tag{13}$$

$$\tilde{M}_{i-1,j} = \frac{\tilde{U}_{i-1,j+1} - \tilde{U}_{i-1,j}}{k}. \tag{14}$$

The fuzzy linear parabolic partial differential equation is solved using the cubic spline method:

$$\frac{1}{6}\tilde{M}_{i-1,j} + \frac{2}{3}\tilde{M}_{i,j} + \frac{1}{6}\tilde{M}_{i+1,j} = b^2(\frac{\tilde{U}_{i-1,j} - 2\tilde{U}_{i,j} + \tilde{U}_{i+1,j}}{h^2}). \tag{15}$$

Now we implement equations (12), (13), (14) in the above equation (15), and we get

$$(\frac{\tilde{U}_{i-1,j+1} - \tilde{U}_{i-1,j}}{k}) + \frac{4}{k}(\frac{\tilde{U}_{i,j+1} - \tilde{U}_{i,j}}{k}) + \frac{1}{k}(\frac{\tilde{U}_{i+1,j+1} - \tilde{U}_{i+1,j}}{k}) = 6b^2(\frac{\tilde{U}_{i-1,j} - 2\tilde{U}_{i,j} + \tilde{U}_{i+1,j}}{h^2}). \tag{16}$$

Now, taking the (α, β) level of the above finite difference form, we get

$$\begin{cases} (\frac{\underline{U}_{i-1,j+1} - \underline{U}_{i-1,j}}{k}) + \frac{4}{k}(\frac{\underline{U}_{i,j+1} - \underline{U}_{i,j}}{k}) + \frac{1}{k}(\frac{\underline{U}_{i+1,j+1} - \underline{U}_{i+1,j}}{k}) = 6b^2(\frac{\underline{U}_{i-1,j} - 2\underline{U}_{i,j} + \underline{U}_{i+1,j}}{h^2}), \\ (\frac{\overline{U}_{i-1,j+1} - \overline{U}_{i-1,j}}{k}) + \frac{4}{k}(\frac{\overline{U}_{i,j+1} - \overline{U}_{i,j}}{k}) + \frac{1}{k}(\frac{\overline{U}_{i+1,j+1} - \overline{U}_{i+1,j}}{k}) = 6b^2(\frac{\overline{U}_{i-1,j} - 2\overline{U}_{i,j} + \overline{U}_{i+1,j}}{h^2}), \\ (\frac{\underline{U}'_{i-1,j+1} - \underline{U}'_{i-1,j}}{k}) + \frac{4}{k}(\frac{\underline{U}'_{i,j+1} - \underline{U}'_{i,j}}{k}) + \frac{1}{k}(\frac{\underline{U}'_{i+1,j+1} - \underline{U}'_{i+1,j}}{k}) = 6b^2(\frac{\underline{U}'_{i-1,j} - 2\underline{U}'_{i,j} + \underline{U}'_{i+1,j}}{h^2}), \\ (\frac{\overline{U}'_{i-1,j+1} - \overline{U}'_{i-1,j}}{k}) + \frac{4}{k}(\frac{\overline{U}'_{i,j+1} - \overline{U}'_{i,j}}{k}) + \frac{1}{k}(\frac{\overline{U}'_{i+1,j+1} - \overline{U}'_{i+1,j}}{k}) = 6b^2(\frac{\overline{U}'_{i-1,j} - 2\overline{U}'_{i,j} + \overline{U}'_{i+1,j}}{h^2}). \end{cases} \tag{17}$$

Now, taking the(α, β) level of the finite difference form :

$$\begin{cases} \underline{U}_{i-1,j+1} + 4\underline{U}_{i,j+1} + \underline{U}_{i+1,j+1} = (1+6\Gamma)\underline{U}_{i-1,j} + 4(1-3\Gamma)\underline{U}_{i,j} + (1+6\Gamma)\underline{U}_{i+1,j}, \\ \overline{U}_{i-1,j+1} + 4\overline{U}_{i,j+1} + \overline{U}_{i+1,j+1} = (1+6\Gamma)\overline{U}_{i-1,j} + 4(1-3\Gamma)\overline{U}_{i,j} + (1+6\Gamma)\overline{U}_{i+1,j}, \\ \underline{U}'_{i-1,j+1} + 4\underline{U}'_{i,j+1} + \underline{U}'_{i+1,j+1} = (1+6\Gamma)\underline{U}'_{i-1,j} + 4(1-3\Gamma)\underline{U}'_{i,j} + (1+6\Gamma)\underline{U}'_{i+1,j}, \\ \overline{U}'_{i-1,j+1} + 4\overline{U}'_{i,j+1} + \overline{U}'_{i+1,j+1} = (1+6\Gamma)\overline{U}'_{i-1,j} + 4(1-3\Gamma)\overline{U}'_{i,j} + (1+6\Gamma)\overline{U}'_{i+1,j}, \end{cases} \tag{18}$$

where $\Gamma = \frac{kb^2}{h^2}$.

5 Convergence analysis

Assume $\tilde{U}(x_i, t_j)$ and $\tilde{U}_{i,j}$ is the exact solution and approximate solution at the same point (x_i, t_j) is obtained by the proposed finite difference method respectively. Error at the point (x_i, t_j) is given as

$$\tilde{\epsilon}_{i,j} = \tilde{U}_{i,j} - \tilde{U}(x_i, t_j). \tag{19}$$

The difference between $\tilde{U}_{i,j}$ and $\tilde{U}(x_i, t_j)$ is a local truncation error, which is given by

$$\tilde{\epsilon}_{i-1,j+1} + 4\tilde{\epsilon}_{i,j+1} + \tilde{\epsilon}_{i+1,j+1} = (1+6\Gamma)\tilde{\epsilon}_{i-1,j} + 4(1-3\Gamma)\tilde{\epsilon}_{i,j} + (1+6\Gamma)\tilde{\epsilon}_{i+1,j} + O(k+h^2)H(x,t), \tag{20}$$

where $\tilde{\epsilon}_{i,j} = \tilde{U}_{i,j} - \tilde{U}(x_i, t_j)$ and $H(x,t)$ is the remainder term, i.e., derivative of $\tilde{U}(x_i, t_j)$. When both step sizes h and k, approach to zero, the error function also approaches to zero. To verify this, we show that the norm of error function at the discretised points intends zero. The properties of $\tilde{\epsilon}(x_i, t_j)$ and the solution function $\tilde{U}(x_i, t_j)$ are identical and have the same properties. Hence,

$$\partial_t \tilde{\epsilon}(x,t)|_{i,j} - \partial_{xx}\tilde{\epsilon}(x,t)|_{i,j} = (k+h^2)H_{i,j}, \tag{21}$$

where

$$H_{i,j} \propto \frac{\partial^3 \tilde{U}}{\partial t^3}(x_i, t_j).$$

Let $h_{i,j} = (k+h^2)H_{i,j}$ and consider

$$M = \max\{\frac{\partial^3 \tilde{U}}{\partial t^3}(x_i, t_j)\}, \tag{22}$$

hence $h_{i,j} \propto M(k+h^2)$. A system of linear equations is obtained using the error equation, which can be formulated in matrix form as AX=B, Where $A = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}$ and $X = \begin{pmatrix} E \\ \overline{E} \end{pmatrix}$ and $B = (K + h^2) \begin{pmatrix} h \\ \overline{h} \end{pmatrix}$, where M_1 is the block diagonal sparse matrix. If the matrix $A = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}$ is non-singular, then the vector $X = \begin{pmatrix} E \\ \overline{E} \end{pmatrix}$ can be founded by the determinant of $A = \det(M_1 + M_2).\det(M_1 - M_2)$ and we have $\det \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} \neq 0$ and hence

$$\begin{pmatrix} E \\ \overline{E} \end{pmatrix} = (K + h^2) \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}^{-1} \begin{pmatrix} h \\ \overline{h} \end{pmatrix}. \tag{23}$$

It is sufficient to show that the system is convergence under the vector and matrix infinity norm.

$$\left\| \begin{matrix} E \\ \overline{E} \end{matrix} \right\|_\infty \leq (K + h^2) \left\| \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} \right\|_\infty^{-1} \left\| \begin{matrix} h \\ \overline{h} \end{matrix} \right\|_\infty \tag{24}$$

If $M = \left\| \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}^{-1} \right\|$, $m = \left\| \begin{matrix} h \\ \overline{h} \end{matrix} \right\|_\infty$, then $0 < \left\| \begin{matrix} E \\ \overline{E} \end{matrix} \right\|_\infty \leq Mm(K + h^2)$, which means that as $h \to 0, K \to 0, \implies \left\| \begin{matrix} E \\ \overline{E} \end{matrix} \right\|_\infty \to 0$. Hence the considered system is convergence as $h \to 0$, $K \to 0$.

6 Stability

Before analyzing the stability of the proposed method, let us first review some theorems.

Theorem 6.1. [3] *Let the matrix M have the special structure as* $\begin{pmatrix} K & L \\ L & K \end{pmatrix}$ *then the eigenvalues of the matrix M are the union of eigenvalues of* $K + L$ *and* $K - L$ *matrices.*

Theorem 6.2. [11] *The difference equation is stable if and only if it meets the necessary and sufficient condition* $\eta(M) \leq 1$.

Remark 6.1. *Here largest eigenvalue of the matrix M is represented by* $\eta(M)$.

Now, let us discuss the stability of the proposed method. For stability, it is sufficient to show that $\eta(M) \leq 1$. Since matrix representation of the system of equations for is as follows:

$$A \begin{pmatrix} U_{j+1} \\ \overline{U}_{j+1} \end{pmatrix} = B \begin{pmatrix} U_j \\ \overline{U}_j \end{pmatrix},$$

where

$$A = \begin{pmatrix} K & L \\ L & K \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} P & N \\ N & P \end{pmatrix}.$$

Here A and B are the block matrices. To show the stability, we need to find the eigenvalue of $K+L, K-L, P+N, P-N$ matrices. Since,

$$K+L = K-L = \begin{pmatrix} 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 4 \end{pmatrix},$$

$$P+N = P-N = \begin{pmatrix} 4(1-3\Gamma) & (1+6\Gamma) & 0 & \cdots & 0 & 0 \\ (1+6\Gamma) & 4(1-3\Gamma) & (1+6\Gamma) & \cdots & 0 & 0 \\ 0 & (1+6\Gamma) & 4(1-3\Gamma) & (1+6\Gamma) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & (1+6\Gamma) & 4(1-3\Gamma) & (1+6\Gamma) \\ 0 & 0 & \cdots & 0 & (1+6\Gamma) & 4(1-3\Gamma) \end{pmatrix}.$$

Here, $K+L, K-L, P+N, P-N$ are the square tridiagonal matrices of size $(n-1)$. For the matrices $K+L$ or $K-L$, let $a=4, b=1, c=1$. Since, by the matrix method of stability, the eigenvalues are given by the formula

$$\lambda_{K+L} = a + 2\sqrt{bc}\cos(\frac{\pi r}{n+1}), r = 1, 2, 3, ..., N.$$

hence,

$$\lambda_{K+L} = 4 + 2\cos(\frac{\pi r}{N+1}) \tag{25}$$

or

$$\lambda_{K+L} = 6 - 4\sin^2(\frac{\pi r}{2(N+1)}) \tag{26}$$

similarly,

$$\lambda_{K-L} = 6 - 4\sin^2(\frac{\pi r}{2(N+1)}) \tag{27}$$

similarly for the matrices $P+N, P-N$, $a = 4(1-3\Gamma), b = 1+6\Gamma, c = 1+6\Gamma$

$$\lambda_{P+N} = \lambda_{P-N} = 6 - 4(1+6\Gamma)\sin^2(\frac{\pi r}{2(N+1)}), \tag{28}$$

where $\Gamma = \frac{kb^2}{h^2}$. As we know, when the $\eta(M) = \eta(A^{-1}B)$, then the finite difference method exhibits stability. Since for $\Gamma \leq \frac{1}{6}$,

$$\eta(M) = \max_r \left| \frac{6 - 4(1+6\Gamma)\sin^2(\frac{\pi r}{2(N+1)})}{6 - 4\sin^2(\frac{\pi r}{2(N+1)})} \right| < 1.$$

Hence, the finite difference method is conditionally stable for $\Gamma \leq \dfrac{1}{6}$.

7 Numerical illustrations

To test the efficiency of the proposed numerical scheme, we applied the scheme on an example of an IFPPDE (example: intuitionistic fuzzy heat conduction equation)

$$(D_t - D_x^2)\tilde{U} = \tilde{0}, x \in [0,1], t > 0, \tag{29}$$

subject to the boundary conditions are $\tilde{U}(0,t) = \tilde{U}(1,t) = 0, t > 0$ and initial condition is

$$\tilde{U}(x,0) = \tilde{g}(x) = \tilde{\kappa}\cos(\pi x - \frac{\pi}{2}), x \in [0,1].$$

Here, $\tilde{\kappa}[\alpha,\beta] = \langle[\underline{\kappa}(\alpha), \overline{\kappa}(\alpha)], [\underline{\kappa}(\beta), \overline{\kappa}(\beta)]\rangle = \langle[\alpha - 1, 1 - \alpha], [-1.25\beta, 1.25\beta]\rangle$.
The exact solution for

$$\begin{cases} \dfrac{\partial \underline{U}}{\partial t}(x,t;\alpha) = \dfrac{\partial^2 \underline{U}}{\partial x^2}(x,t;\alpha), \\ \dfrac{\partial \overline{U}}{\partial t}(x,t;\alpha) = \dfrac{\partial^2 \overline{U}}{\partial x^2}(x,t;\alpha), \\ \dfrac{\partial \underline{U}'}{\partial t}(x,t;\beta) = \dfrac{\partial^2 \underline{U}'}{\partial x^2}(x,t;\beta), \\ \dfrac{\partial \overline{U}'}{\partial t}(x,t;\beta) = \dfrac{\partial^2 \overline{U}'}{\partial x^2}(x,t;\beta), \end{cases} \tag{30}$$

where $x \in [0,1], t > 0$, are respectively

$$\begin{cases} \underline{U}(x,t;\alpha) = \underline{\kappa}(\alpha)e^{-\pi^2 t}\cos(\pi x - \frac{\pi}{2}), \\ \overline{U}(x,t;\alpha) = \overline{\kappa}(\alpha)e^{-\pi^2 t}\cos(\pi x - \frac{\pi}{2}), \\ \underline{U}'(x,t;\beta) = \underline{\kappa}(\beta)e^{-\pi^2 t}\cos(\pi x - \frac{\pi}{2}), \\ \overline{U}'(x,t;\beta) = \overline{\kappa}(\beta)e^{-\pi^2 t}\cos(\pi x - \frac{\pi}{2}). \end{cases} \tag{31}$$

The above system of equations can also be written like

$$\begin{cases} \underline{U}(x,t;\alpha) = (\alpha - 1)e^{-\pi^2 t}cos(\pi x - \frac{\pi}{2}), \\ \overline{U}(x,t;\alpha) = (1 - \alpha)e^{-\pi^2 t}cos(\pi x - \frac{\pi}{2}), \\ \underline{U}'(x,t;\beta) = (-1.25\beta)e^{-\pi^2 t}cos(\pi x - \frac{\pi}{2}), \\ \overline{U}'(x,t;\beta) = (1.25\beta)e^{-\pi^2 t}cos(\pi x - \frac{\pi}{2}). \end{cases} \tag{32}$$

Now, we will implement the finite difference, which we derived using the cubic spline method to approximate the IFPPDE. Let us consider the $h = 0.1$ and $k = 0.000001$ for the following system of finite difference schemes.

$$\begin{cases} \underline{U}_{i-1,j+1} + 4\underline{U}_{i,j+1} + \underline{U}_{i+1,j+1} = (1+6\Gamma)\underline{U}_{i-1,j} + 4(1-3\Gamma)\underline{U}_{i,j} + (1+6\Gamma)\underline{U}_{i+1,j}, \\ \overline{U}_{i-1,j+1} + 4\overline{U}_{i,j+1} + \overline{U}_{i+1,j+1} = (1+6\Gamma)\overline{U}_{i-1,j} + 4(1-3\Gamma)\overline{U}_{i,j} + (1+6\Gamma)\overline{U}_{i+1,j}, \\ \underline{U}'_{i-1,j+1} + 4\underline{U}'_{i,j+1} + \underline{U}'_{i+1,j+1} = (1+6\Gamma)\underline{U}'_{i-1,j} + 4(1-3\Gamma)\underline{U}'_{i,j} + (1+6\Gamma)\underline{U}'_{i+1,j}, \\ \overline{U}'_{i-1,j+1} + 4\overline{U}'_{i,j+1} + \overline{U}'_{i+1,j+1} = (1+6\Gamma)\overline{U}'_{i-1,j} + 4(1-3\Gamma)\overline{U}'_{i,j} + (1+6\Gamma)\overline{U}'_{i+1,j}, \end{cases} \tag{33}$$

where $\Gamma = \frac{k}{h^2}$. We have the following results after solving the above system of finite difference scheme at the $h = 0.1$ and $k = 0.000001$.

Tables 1–4 give a detailed explanation of the solution of the IFPPDE for the approximate solution $U_{1,1}$ and exact solution $U(1,1)$ at each upper and lower (α, β) levels at $h = 0.1$ and $k = 0.000001$. These tables also give the absolute error of the approximate solution at each (α, β) levels. It is also clear from Tables 1–4 that the solution at each α levels give seven decimal places of accuracy where the solution at each β levels give five decimal places of accuracy. Hence, the explicit cubic spline for the finite difference method is an effective method that can be used to approximate the solution of the IFPPDEs.

Table 1. Comparison of the intuitionistic fuzzy heat equation's approximate solution $U_{1,1}$ and the exact solution $U(1,1)$ at different lower α levels at $h = 0.1$ and $k = 0.000001$

Sr.no	**α levels**	**$U_{1,1}$ lower**	**Exact $U(1,1)$ lower**	**Absolute error**
1	0	-0.309013919333556	-0.309013944514510	0.0000000251809540
2	0.1	-0.27811252740020	-0.278112550063059	0.0000000226628590
3	0.2	-0.247211135466844	-0.247211155611608	0.0000000201447640
4	0.3	-0.216309743533489	-0.216309761160157	0.0000000176266680
5	0.4	-0.185408351600133	-0.185408366708706	0.0000000151085730
6	0.5	-0.154506959666778	-0.154506972257255	0.0000000125904770
7	0.6	-0.123605567733422	-0.123605577805804	0.0000000100723820
8	0.7	-0.092704175800067	-0.092704183354353	0.0000000075542860
9	0.8	-0.061802783866711	-0.061802788902902	0.0000000050361910
10	0.9	-0.030901391933356	-0.030901394451451	0.0000000025180950
11	1	0	0	0

Table 2. Comparison of the intuitionistic fuzzy heat equation's approximate solution $U_{1,1}$ and the exact solution $U(1,1)$ at different upper α levels at $h = 0.1$ and $k = 0.000001$

Sr.no	**α levels**	**$U_{1,1}$ lower**	**Exact $U(1,1)$ lower**	**Absolute error**
1	0	0.309013919333556	0.309013944514510	0.0000000251809540
2	0.1	0.27811252740020	0.278112550063059	0.0000000226628590
3	0.2	0.247211135466844	0.247211155611608	0.0000000201447640
4	0.3	0.216309743533489	0.216309761160157	0.0000000176266680
5	0.4	0.185408351600133	0.185408366708706	0.0000000151085730
6	0.5	0.154506959666778	0.154506972257255	0.0000000125904770
7	0.6	0.123605567733422	0.123605577805804	0.0000000100723820
8	0.7	0.092704175800067	0.092704183354353	0.0000000075542860
9	0.8	0.061802783866711	0.061802788902902	0.0000000050361910
10	0.9	0.030901391933356	0.030901394451451	0.0000000025180950
11	1	0	0	0

Table 3. Comparison of the intuitionistic fuzzy heat equation's approximate solution $U_{1,1}$ and the exact solution $U(1,1)$ at different lower β levels at $h = 0.1$ and $k = 0.000001$

Sr.no	**β levels**	**$U_{1,1}$ lower**	**Exact $U(1,1)$ lower**	**Absolute error**
1	0	0	0	0
2	0.1	−0.038626739916694	−0.038626361835522	0.000000378081172
3	0.2	−0.077253479833389	−0.077252723671043	0.000000756162346
4	0.3	−0.115880219750083	−0.115879085506565	0.000001134243518
5	0.4	−0.154506959666778	−0.154505447342087	0.000001512324691
6	0.5	−0.193133699583472	−0.193131809177609	0.000001890405863
7	0.6	−0.231760439500167	−0.231758171013130	0.000002268487037
8	0.7	−0.270387179416861	−0.270384532848652	0.000002646568209
9	0.8	−0.309013919333556	−0.309010894684174	0.000003024649382
10	0.9	−0.347640659250250	−0.347637256519695	0.000003402730555
11	1	−0.386267399166945	−0.386263618355217	0.000003780811728

Table 4. Comparison of the intuitionistic fuzzy heat equation's approximate solution $U_{1,1}$ and the exact solution $U(1,1)$ at different upper β levels at $h = 0.1$ and $k = 0.000001$

Sr.no	**β levels**	**$U_{1,1}$ lower**	**Exact $U(1,1)$ lower**	**Absolute error**
1	0	0	0	0
2	0.1	0.038626739916694	0.038626361835522	0.000000378081172
3	0.2	0.077253479833389	0.077252723671043	0.000000756162346
4	0.3	0.115880219750083	0.115879085506565	0.000001134243518
5	0.4	0.154506959666778	0.154505447342087	0.000001512324691
6	0.5	0.193133699583472	0.193131809177609	0.000001890405863
7	0.6	0.231760439500167	0.231758171013130	0.000002268487037
8	0.7	0.270387179416861	0.270384532848652	0.000002646568209
9	0.8	0.309013919333556	0.309010894684174	0.000003024649382
10	0.9	0.347640659250250	0.347637256519695	0.000003402730555
11	1	0.386267399166945	0.386263618355217	0.000003780811728

Similarly, Tables 5–8 give good approximate solutions for $U(2,1)$ and $U_{2,1}$ as the same accuracy as we got for $U(1,1)$ and $U_{1,1}$.

Since our problem contains the initial and boundary values in terms of the intuitionistic triangular fuzzy number, and Figure 1 and Figure 2 represent the nature and properties of the fuzzy solutions. Both figures represent the solution to the problem considered in terms of the intuitionistic triangular fuzzy number. we can also observe that solution width or fuzziness of the solution increases as we move from $U_{1,1}$ to $U_{2,1}$.

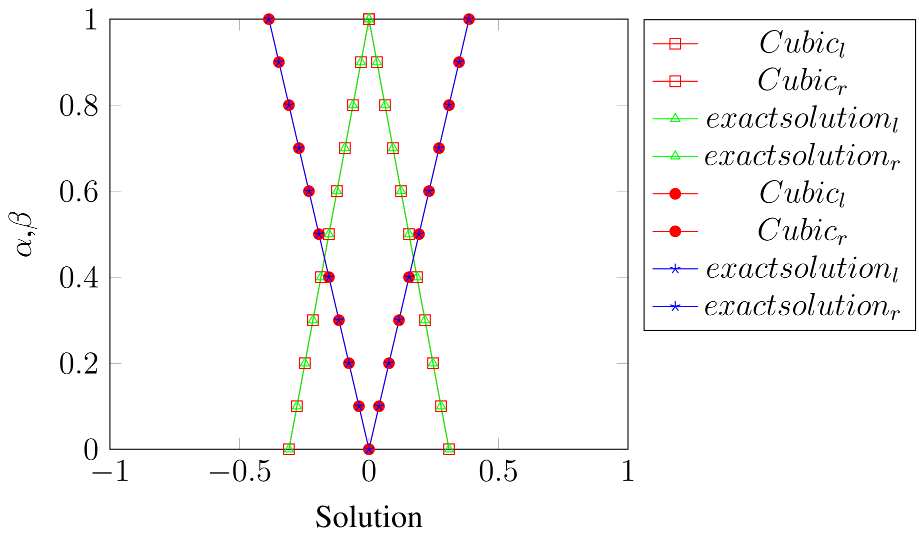

Figure 1. Comparison of intuitionistic explicit cubic spline's solution $U_{1,1}$ and exact solutions $U(1,1)$ at different α and β levels

Table 5. Comparison of the intuitionistic fuzzy heat equation's approximate solution $U_{2,1}$ and the exact solution $U(2,1)$ at different lower α levels at $h = 0.1$ and $k = 0.000001$

Sr.no	**α levels**	**$U_{2,1}$ lower**	**Exact $U(2,1)$ lower**	**Absolute error**
1	0	−0.587779403216166	−0.587779451113188	0.00000004789702202
2	0.1	−0.529001462894550	−0.529001506001869	0.00000004310731905
3	0.2	−0.470223522572933	−0.470223560890550	0.00000003831761702
4	0.3	−0.411445582251316	−0.411445615779232	0.00000003352791600
5	0.4	−0.352667641929700	−0.352667670667913	0.00000002873821298
6	0.5	−0.293889701608083	−0.293889725556594	0.00000002394851101
7	0.6	−0.235111761286466	−0.235111780445275	0.00000001915880901
8	0.7	−0.176333820964850	−0.176333835333956	0.00000001436910599
9	0.8	−0.117555880643233	−0.117555890222638	0.00000000957940501
10	0.9	−0.058777940321617	−0.058777945111319	0.00000000478970200
11	1	0	0	0

Table 6. Comparison of the intuitionistic fuzzy heat equation's approximate solution $U_{2,1}$ and the exact solution $U(2,1)$ at different upper α levels at $h = 0.1$ and $k = 0.000001$

Sr.no	α levels	$U_{2,1}$ lower	Exact $U(2,1)$ lower	Absolute error
1	0	0.587779403216166	0.587779451113188	0.00000004789702202
2	0.1	0.529001462894550	0.529001506001869	0.00000004310731905
3	0.2	0.470223522572933	0.470223560890550	0.00000003831761702
4	0.3	0.411445582251316	0.411445615779232	0.00000003352791600
5	0.4	0.352667641929700	0.352667670667913	0.00000002873821298
6	0.5	0.293889701608083	0.293889725556594	0.00000002394851101
7	0.6	0.235111761286466	0.235111780445275	0.00000001915880901
8	0.7	0.176333820964850	0.176333835333956	0.00000001436910599
9	0.8	0.117555880643233	0.117555890222638	0.00000000957940501
10	0.9	0.058777940321617	0.058777945111319	0.00000000478970200
11	1	0	0	0

Table 7. Comparison of the intuitionistic fuzzy heat equation's approximate solution $U_{2,1}$ and the exact solution $U(2,1)$ at different lower β levels at $h = 0.1$ and $k = 0.000001$

Sr.no	β levels	$U_{2,1}$ lower	Exact $U(2,1)$ lower	Absolute error
1	0	0	0	0
2	0.1	-0.073472425402021	-0.073471706248895	0.00000071915312599
3	0.2	-0.146944850804041	-0.146943412497789	0.00000143830625199
4	0.3	-0.220417276206062	-0.220415118746684	0.00000215745937798
5	0.4	-0.293889701608083	-0.293886824995579	0.00000287661250398
6	0.5	-0.367362127010104	-0.367358531244474	0.00000359576563003
7	0.6	-0.440834552412125	-0.440830237493368	0.00000431491875696
8	0.7	-0.514306977814145	-0.514301943742263	0.00000503407188202
9	0.8	-0.587779403216166	-0.587773649991158	0.00000575322500795
10	0.9	-0.661251828618187	-0.661245356240052	0.00000647237813500
11	1	-0.734724254020208	-0.734717062488947	0.00000719153126105

Table 8. Comparison of the intuitionistic fuzzy heat equation's approximate solution $U_{2,1}$ and the exact solution $U(2,1)$ at different upper β levels at $h = 0.1$ and $k = 0.000001$

Sr.no	**β levels**	**$U_{2,1}$ lower**	**Exact $U(2,1)$ lower**	**Absolute error**
1	0	0	0	0
2	0.1	0.073472425402021	0.073471706248895	0.00000071915312599
3	0.2	0.146944850804041	0.146943412497789	0.00000143830625199
4	0.3	0.220417276206062	0.220415118746684	0.00000215745937798
5	0.4	0.293889701608083	0.293886824995579	0.00000287661250398
6	0.5	0.367362127010104	0.367358531244474	0.00000359576563003
7	0.6	0.440834552412125	0.440830237493368	0.00000431491875696
8	0.7	0.514306977814145	0.514301943742263	0.00000503407188202
9	0.8	0.587779403216166	0.587773649991158	0.00000575322500795
10	0.9	0.661251828618187	0.661245356240052	0.00000647237813500
11	1	0.734724254020208	0.734717062488947	0.00000719153126105

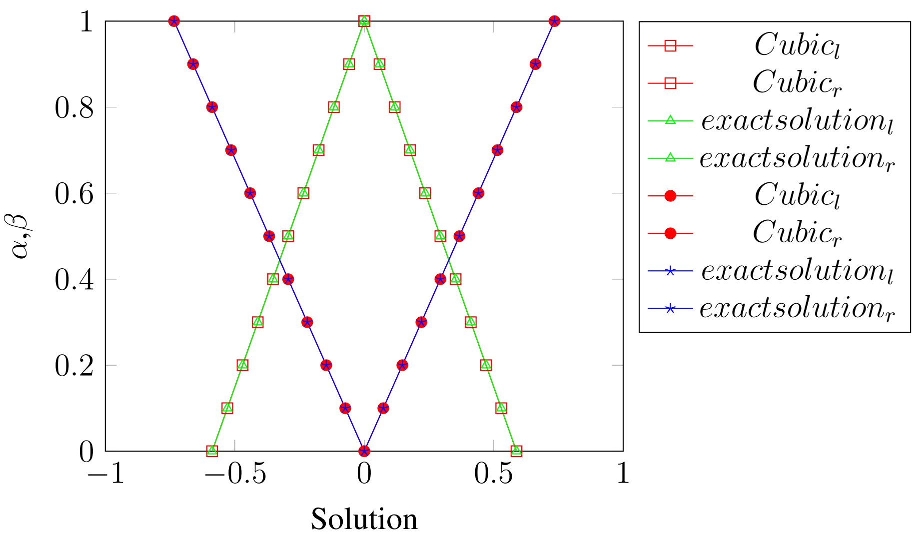

Figure 2. Comparison of intuitionistic explicit cubic spline's solution $U_{2,1}$ and exact solutions $U(2,1)$ at different α and β levels

8 Conclusion

This study offers a comprehensive numerical analysis of the one-dimensional IFPPDE using an explicit cubic spline method. A detailed convergence analysis confirms the reliability of the proposed method. Moreover, the explicit cubic spline method, which generates fuzzy finite differences, is proven to be conditionally stable through the matrix stability method. The effectiveness of the approach is further demonstrated through a numerical example, where solutions at various levels are compared using graphical and tabular representations. These findings underscore the robustness and accuracy of the method in handling intuitionistic fuzzy uncertainty, making it a valuable tool for solving IFPPDEs.

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