# Intuitionistic Fuzzy Sets or Orthopair Fuzzy Sets?

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## Abstract

Intuitionistic Fuzzy Sets (IFS) are defined as pairs of mutually orthogonal fuzzy sets. We discuss this approach from an algebraic point of view. As a result we characterize two implication operators on the collection of IFS, which on a particular subset of IFS behave as a Łukasiewicz and a Gödel implication.

**Keywords:** fuzzy sets, intuitionistic fuzzy sets, orthogonality, rough approximations.

## 1 Introduction

In the other work presented to this conference [6] we introduced some generalized notions of orthocomplementation and some algebraic structures to model them. Here, we will make reference to those notions. In particular, we consider Heyting Wajsberg algebras and we discuss their relation with Intuitionistic Fuzzy Sets.

## 2 Rough Approximations in BK Lattices

In any wBD lattice $\langle \Sigma, \wedge, \vee, ', ^\sim, 0, 1\rangle$ (and hence in any HW algebra $\langle \Sigma, \to_L, \to_G, 0\rangle$ which has a natural structure of BK lattice with respect to the lattice operations $a \wedge b = ((a' \to_L b') \to_L b')'$, $a \vee b = (a \to_L b) \to_L b$, the complementations $a' := a \to_L 0$, $a^\sim := a \to_G 0$, and the constant element $1 := 0'$ [6]) it is possible to introduce, by suitable compositions of the two complementations $'$ (the de Morgan or fuzzy one) and $^\sim$ (the weak Brouwer or intuitionistic one), the modal operators of *necessity* $\nu$ and *possibility* $\mu$ defined for any element $a \in \Sigma$ respectively as $\nu(a) := a'^\sim$ and $\mu(a) := a^{\sim\prime}$. It can be easily seen that $a^\sim = (\mu(a))'$ and $a^\flat = (\nu(a))'$. That is, similarly to the modal interpretation of intuitionistic logic, the Brouwer complement $^\sim$ can be interpreted as the negation of possibility or *impossibility*, and the anti–Brouwer complement as the negation of necessity or *contingency*.

Our modal operations $\nu$ and $\mu$ turn out to have an $S_5$-like behavior (see [8]), based on a de Morgan algebra instead of on a Boolean one .

**Proposition 2.1.** *In any wBD lattice the following conditions hold:*

$$\nu(x) \leq x \leq \mu(x) \qquad \textit{[T–principle]}$$
$$\nu(\nu(x)) = \nu(x) \qquad \textit{[}S_4\textit{–principle]}$$
$$\mu(\mu(x)) = \mu(x)$$
$$x \leq \nu(\mu(x)) \qquad \textit{[B–principle]}$$
$$\mu(x) = \nu(\mu(x)) \qquad \textit{[}S_5\textit{–principle]}$$
$$\nu(x) = \mu(\nu(x))$$

These modal operators can be used to give an approximation of any element of the lattice through exact elements. An element $e$ of a wBD lattice is *exact* (also *sharp*, *crisp*) iff $e = e^{\sim\sim}$ or equivalently, iff $e = e^{\flat\flat}$ iff $\nu(e) = e$ iff $e = \mu(e)$. Clearly, this is a classical situation where necessity, actuality and possibility coincide. We remark that the so defined exact elements have some other interesting classical properties. First of all on the set of all exact elements the two complementations coincide: for every exact element $e$: $e' = e^{\sim}$. In particular, one has that it satisfies the double negation law for the Brouwer complementation, $e = e^{\sim\sim}$. Further, in the case of a BK lattice it is also satisfied the excluded middle law (equivalently, non contradiction law) for the Kleene negation, $e \vee e' = 1$ (equivalently, $e \wedge e' = 0$). In the sequel we use the symbol $'$ to denote this standard complementation on exact elements and the following result gives the expected characterization of the collection of all exact set.

**Proposition 2.2.** *Let* $\langle \Sigma, \wedge, \vee, ', {}^{\sim}, 0, 1\rangle$ *be a wBD (resp., BK) lattice. The set* $\Sigma_e := \{e \in \Sigma : e = e^{\sim\sim}\}$ *of all its exact (crisp) elements has the structure* $\langle \Sigma_e, \wedge, \vee, ', 0, 1\rangle$ *of de Morgan (resp., Boolean) lattice.*

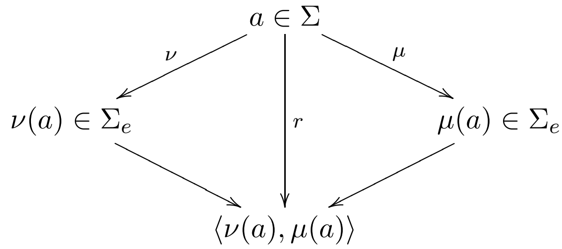
**Definition 2.3.** Let $\langle \Sigma, \wedge, \vee, ', {}^{\sim}, 0, 1\rangle$ be a wBD (resp., BK) lattice. The induced rough approximation space ([4]) is the structure $\mathbb{R} = \langle \Sigma, \Sigma_e, \nu, \mu\rangle$, where

- $\Sigma$ is the set of approximable elements;
- $\Sigma_e \subseteq \Sigma$ is the de Morgan (resp., Boolean) lattice of exact elements;
- $\nu : \Sigma \to \Sigma_e$ is the *inner approximation map*;
- $\mu : \Sigma \to \Sigma_e$ is the *outer approximation map.*

For any element $a \in \Sigma$, its *rough approximation* is defined as the pair:

$$r(a) := \langle \nu(a), \mu(a)\rangle \quad [\text{with} \quad \nu(a) \le a \le \mu(a)]$$

drawn in the following diagram:




The concepts of rough approximation and rough approximation space were developed in order to give an abstract approach to the concrete rough sets on information systems introduced by Pawlak ([9, 10]). Indeed, it turns out that the power set of the objects of an information system is a BK lattice, once opportunely defined the primitive operators (see [4]) and the abstract rough approximation of necessity–possibility corresponds to the usual lower–upper approximation of rough sets theory. Thus, the underlying idea of a rough approximation is, as in rough sets theory, to approximate a fuzzy (imprecise, vague) element with a pair of sharp (exact, crisp) ones.
Note that an element $e \in \Sigma$ is said to be exact (crisp) with respect to $\mathbb{R}$ iff $r(e) = \langle e, e\rangle$, i.e., iff it is exact (crisp) with respect to $\Sigma$.
The approximation $r(a)$ is the best approximation through exact elements of the element a. In fact, for any element $a \in \Sigma$, $\nu(a)$ and $\mu(a)$ are exact elements, that is $\nu(a), \mu(a) \in \Sigma_e$; $\nu(a)$ (resp., $\mu(a)$) is an inner (resp., outer) approximation of $a$, i.e., $\nu(a) \le a$ (resp., $a \le \mu(a)$); moreover, $\nu(a)$ (resp., $\mu(a)$) is the best inner (resp., outer) approximation of $a$ by sharp elements: if a sharp $e \in \Sigma_e$ is such that $e \le a$ (resp., $a \le e$) than $e \le \nu(a)$ (resp., $\mu(a) \le e$).

An equivalent way to define a rough approximation is to consider the pair necessity – impossibility:

$$r_\perp(a) := \langle \nu(a), a^{\sim}\rangle \quad [\text{with } \nu(a) \le (a^{\sim})' = \mu(a)]$$

Needless to stress, from the approximation $r_\perp(a)$ one can uniquely obtain the approximation $r(a)$ and vice versa.

Let $a, b$ be two elements of a wBD lattice. They are said to be *orthogonal*, written as $a \perp b$, iff $a \leq b'$. Now, let us consider the collection $\mathbb{O}(\Sigma)$ of all pairs $\langle a_i, a_e \rangle$ where $a_i \perp a_e$ and $a_i, a_e \in \Sigma_e$, i.e., of all orthogonal pairs (also *orthopairs*) of exact elements from a wBD lattice. Note that, in particular, all rough approximations $r_\perp(a)$, for $a \in \Sigma$, are elements of $\mathbb{O}(\Sigma)$ with $a_i := \nu(a)$ the *interior* and $a_e := a^\sim$ the *exterior* of $a$.

**Proposition 2.4.** *Let $\langle \Sigma, \wedge, \vee, ', ^\sim, 0, 1 \rangle$ be a wBD lattice and $\mathbb{O}(\Sigma)$ the collection of all orthopairs of exact elements on $\Sigma$. Once defined the operators*

$$\begin{aligned}\langle a_i, a_e \rangle \Rightarrow_L \langle b_i, b_e \rangle \\ &:= \langle (a_i' \wedge b_e') \vee (a_e \vee b_i), a_i \wedge b_e \rangle \\ \langle a_i, a_e \rangle \Rightarrow_G \langle b_i, b_e \rangle \\ &:= \langle (a_i' \wedge b_e') \vee (a_e \vee b_i), a_e' \wedge b_e \rangle\end{aligned}$$

*and the element $\mathbf{0} = \langle 0, 1 \rangle$, the structure $\langle \mathbb{O}(\Sigma), \Rightarrow_L, \Rightarrow_G, \mathbf{0} \rangle$ is a HW algebra.*

The collection of rough approximations is now endowed with a very rich algebraic structure. The induced BK lattice structure is $\langle \mathbb{O}(\Sigma), \sqcap, \sqcup, ^-, ^\approx, \mathbf{0}, \mathbf{1} \rangle$, with:

- the lattice operations

  $$\begin{aligned}\langle a_i, a_e \rangle \sqcap \langle b_i, b_e \rangle &= \langle a_i \wedge b_i, a_e \vee b_e \rangle \\ \langle a_i, a_e \rangle \sqcup \langle b_i, b_e \rangle &= \langle a_i \vee b_i, a_e \wedge b_e \rangle\end{aligned}$$

  whose induced partial ordering is

  $$\langle a_i, a_e \rangle \sqsubseteq \langle b_i, b_e \rangle \quad \text{iff} \quad a_i \leq b_i \text{ and } b_e \leq a_e.$$

  We remark that this is the same partial order obtained by the implications:

  $$\begin{aligned}\langle a_i, a_e \rangle \sqsubseteq \langle b_i, b_e \rangle \text{ iff } \langle a_i, a_e \rangle \Rightarrow_L \langle b_i, b_e \rangle = 1 \\ \text{iff } \langle a_i, a_e \rangle \Rightarrow_G \langle b_i, b_e \rangle = 1.\end{aligned}$$

  With respect to this partial order relation the least element is $\mathbf{0} = \langle 0, 1 \rangle$ and the greatest element is $\mathbf{1} = \langle 1, 0 \rangle$;

- the two complementations

  $$\begin{aligned}\langle a_i, a_e \rangle^- &= \langle a_e, a_i \rangle && \text{(Kleene)} \\ \langle a_i, a_e \rangle^\approx &= \langle a_e, a_e' \rangle && \text{(Brouwer)}\end{aligned}$$

Thus, the necessity and possibility operators (in this context denoted by $\Box$ and $\Diamond$ respectively) are

$$\begin{aligned}\Box(\langle a_i, a_e \rangle) &= \langle a_i, a_i' \rangle = r_\perp(a_i) \\ \Diamond(\langle a_i, a_e \rangle) &= \langle a_e', a_e \rangle = r_\perp(a_e').\end{aligned}$$

Hence, an orthopair $\langle a_i, a_e \rangle$ is *exact* iff $a_e = a_i'$ and thus the generic sharp orthopair is of the form $\langle a, a' \rangle$ with $a$ exact element from $\Sigma$ ($a \in \Sigma_e$). The mapping $r_\perp : \Sigma_e \to \mathbb{O}(\Sigma)_e$ associating to any exact element $a \in \Sigma_e$ its rough approximation $r_\perp(a) = \langle a, a' \rangle$ is a Boolean algebra isomorphism.

## 3 Rough Approximation of Fuzzy Sets

Let us consider the collection of all fuzzy sets on a domain $X$: $\mathcal{F}(X) = [0, 1]^X$. For any such fuzzy set $f \in \mathcal{F}(X)$ we consider the two subsets of $X$: the *certainly–yes (interior) domain* $A_i(f) := \{x \in X : f(x) = 1\}$ and the *certainly–no (exterior) domain* $A_e(f) := \{x \in X : f(x) = 0\}$ of fuzzy set $f$.

In [6] we showed how to give the structure of a HW algebra to $\mathcal{F}(X)$. It turns out that HW exact fuzzy sets are just *crisp* sets, that is $\{0, 1\}$–valued functionals, i.e., $\mathcal{F}(X)_e = \{0, 1\}^X$.

Notice that for any subset $A \subseteq X$ one can introduce the associated *characteristic functional* $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise, which is a crisp set. On the other hand, given any crisp set $h \in \{0, 1\}^X$, one has $h = \chi_{A_i(h)}$, i.e., any crisp set is the characteristic functional of its certainly–yes domain. The mapping $\chi : \mathcal{P}(X) \to \{0, 1\}^X$ associating to any subset of $X$, $A \in \mathcal{P}(X)$, the corresponding characteristic functional $\chi_A \in \{0, 1\}^X$ is a Boolean algebra isomorphism.

The modal operators of necessity and possibility of a fuzzy set $f$ induced by the HW structure are respectively

$$\nu(f) = \chi_{A_i(f)} \quad \text{and} \quad \mu(f) = \chi_{A_e(f)^c}$$

Now we are able to define a rough approximation of a given fuzzy set through exact elements. Precisely, let $f$ be a fuzzy set, then its rough approximation is the *orthopair of crisp* sets $r_{\perp}(f) = \langle \nu(f), f^{\sim} \rangle = \langle \chi_{A_i(f)}, \chi_{A_e(f)} \rangle$, where $\nu(f) \leq \mu(f) = (f^{\sim})'$ (i.e., $A_i(f) \cap A_e(f) = \emptyset$).
Further, if we consider the collection $\mathbb{O}([0,1]^X)$ of all *orthopairs of crisp* fuzzy sets, i.e., all pairs $\langle \chi_{A_i}, \chi_{A_e} \rangle$ of crisp sets ($\chi_{A_i}, \chi_{A_e} \in \{0,1\}^X$) such that $\chi_{A_i} \leq (\chi_{A_e})' = 1 - \chi_{A_e}$ (i.e., $A_i \cap A_e = \emptyset$), we can apply Proposition 2.4 to this collection and give to it the structure of a HW algebra, which contains the set $r_{\perp}(\mathcal{F}(X))$. In particular, according to section 2, we can define the Kleene and the Brouwer negations and the discussed BK structure can be derived.

Let us note that any orthopair of crisp fuzzy sets $\langle \chi_{A_i}, \chi_{A_e} \rangle$ with $\chi_{A_i} \leq (\chi_{A_e})'$ can be identified with the pair $\langle A_i, A_e \rangle$ of mutually disjoint $A_i \cap A_e = \emptyset$ subsets of the universe $X$. The collection of all such disjoint pairs of subsets of $X$ can be canonically equipped with the HW structure inherited from $\mathbb{O}([0,1]^X)$.

## 4 Orthopair Fuzzy Sets and IFS

In this section, we turn our attention to Intuitionistic Fuzzy Sets (IFS) and their relation with the previously discussed structures.
Let us recall that an IFS is defined as an *orthopair* of fuzzy (non necessarily crisp) sets, i.e., as a pair $\langle f_A, g_A \rangle$ of fuzzy sets ($f_A, g_A \in [0,1]^X$) such that for all $x \in X$, $f_A(x) \leq g'_A(x) = 1 - g_A(x)$.
Let us denote the collection of all IFSs on X as $\mathcal{IF}(X)$. In $\mathcal{IF}(X)$, similarly to section 2, we can introduce the lattice operators:

$$\langle f_A, g_A \rangle \cap \langle f_B, g_B \rangle = \langle f_A \wedge f_B, g_A \vee g_B \rangle$$
$$\langle f_A, g_A \rangle \cup \langle f_B, g_B \rangle = \langle f_A \vee f_B, g_A \wedge g_B \rangle$$

whose induced partial order is

$\langle f_A, g_A \rangle \subseteq \langle f_B, g_B \rangle$ iff $f_A \leq f_B$ and $g_B \leq g_A$.

Given an element $x \in X$ by $\langle f_A, g_A \rangle \subseteq_x \langle f_B, g_B \rangle$ we mean $f_A(x) \leq f_B(x)$ and $g_B(x) \leq g_A(x)$, thus trivially $\langle f_A, g_A \rangle \subseteq \langle f_B, g_B \rangle$ iff for all $x \in X$ $\langle f_A, g_A \rangle \subseteq_x \langle f_B, g_B \rangle$.
Moreover, we can define the two unusual negations respectively as: $\langle f_A, g_A \rangle^- = \langle g_A, f_A \rangle$ and $\langle f_A, g_A \rangle^{\approx} = \langle g_A, g'_A \rangle$. We remark that the operation $^-$ is the usual negation operator defined on IFS ([1]). The negation $^-$ *is a de Morgan negation and not a Kleene one* since property (K3) $A \cap A^- \subseteq B \cup B^-$ does not hold (see [6]).
Further, the operation $^{\approx}$ *is no more a Brouwer negation* since only some weaker results can be proved about it. In fact, in general it does not satisfy the non contradiction property $\forall A,\ A \wedge A^{\approx} = \mathbf{0}$. For instance, $(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \cap (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})^{\approx} = (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \neq (\mathbf{0}, \mathbf{1})$.

However, it still satisfies

- the weak double negation law $A \subseteq A^{\approx\approx}$,
- the de Morgan law $(A \cup B)^{\approx} = A^{\approx} \cap B^{\approx}$,
- the interconnection rule: $A^{\approx\approx} = A^{\approx -}$.

So, the structure $\langle \mathcal{IF}(X), \cap, \cup, ^-, ^{\approx}, \mathbf{0}, \mathbf{1} \rangle$ is a de Morgan lattice with respect to the complementation $^-$, equipped with another operation $^{\approx}$ which gives rise to a weaker form of Brouwer negation, where in particular the non contradiction principle is not required to hold (this structure has been called weak Brouwer de Morgan lattice).
We remark that both the operators $^-$ and $^{\approx}$ satisfy the general definition of intuitionistic fuzzy complementation in the sense of Definition 8 given in [3]:

**Definition 4.1.** An *intuitionistic fuzzy complementation* is a function $N : \mathcal{IF}(X) \mapsto \mathcal{IF}(X)$ satisfying for any pair of IFSs $\langle f_A, g_A \rangle, \langle f_B, g_B \rangle \in \mathcal{IF}(X)$ the following conditions

(N1) If $\langle f_A, g_A \rangle (x) = \langle 0, 1 \rangle$ then $N(\langle f_A, g_A \rangle)(x) = \langle 1, 0 \rangle$ and if $\langle f_A, g_A \rangle (x) = \langle 1, 0 \rangle$ then $N(\langle f_A, g_A \rangle)(x) = \langle 0, 1 \rangle$, (N2) If $\langle f_A, g_A\rangle \subseteq_x \langle f_B, g_B\rangle$ then $N(\langle f_B, g_B\rangle) \subseteq_x N(\langle f_A, g_A\rangle)$.

Also the modal operators, defined according to our approach by compositions of the two negations, correspond to the usual necessity and possibility on IFS introduced in [1]: $\Box\langle f_A, g_A\rangle = \langle f_A, f'_A\rangle$ and $\Diamond\langle f_A, g_A\rangle = \langle g'_A, g_A\rangle$ and they have an $S_5$ modal behavior on a wBD lattice.
Moreover, with regards to the operators $\Rightarrow_L$ and $\Rightarrow_G$, once considered over the whole space of IFS, they are still well defined in the sense that given two IFSs $A, B \in \mathcal{IF}(X)$ then $A \Rightarrow_L B \in \mathcal{IF}(X)$ and $A \Rightarrow_G B \in \mathcal{IF}(X)$. However, they are no more a Łukasiewicz and Gödel implication, in fact they satisfy only some of the axioms of HW algebras. In particular they do not satisfy the axioms (HW1), (HW4) and (HW9) as shown in the following example. Let us set $A = \langle \mathbf{0.1}, \mathbf{0.6}\rangle$ and $B = \langle \mathbf{0.2}, \mathbf{0.7}\rangle$ then

(HW1) $A \Rightarrow_G A = \langle \mathbf{0.6}, \mathbf{0.4}\rangle \neq \langle \mathbf{1}, \mathbf{0}\rangle$;
(HW4) $A \cap (A \Rightarrow_G B) = \langle \mathbf{0.1}, \mathbf{0.6}\rangle \neq \langle \mathbf{0.1}, \mathbf{0.7}\rangle = A \cap B$;
(HW9) $(A \Rightarrow_G B) \Rightarrow_L (A \Rightarrow_L B) = \langle \mathbf{0.6}, \mathbf{0.4}\rangle \Rightarrow_L \langle \mathbf{0.6}, \mathbf{0.1}\rangle = \langle \mathbf{0.6}, \mathbf{0.1}\rangle \neq \langle \mathbf{1}, \mathbf{0}\rangle$.

As a consequence the lattice operators $\cap$ and $\cup$ cannot be derived by the Łukasiewicz implication as in HW algebras (see Definition 7.1 in [6]). Further, it is not possible to induce a partial order relation by the implications in the standard way $a \leq b$ iff $a \to b = 1$, since in general it does not hold neither $A \Rightarrow_G A = \langle \mathbf{1}, \mathbf{0}\rangle$ nor $A \Rightarrow_L A = \langle \mathbf{1}, \mathbf{0}\rangle$.

However, it can be shown that both operators $\Rightarrow_L$ and $\Rightarrow_G$ satisfy the following definition of intuitionistic fuzzy implication which extends to $[0,1]^X$ the one given in [2] on the unit interval $[0,1]$:

**Definition 4.2.** An *intuitionistic fuzzy implication* is a function $I : \mathcal{IF}(X) \times \mathcal{IF}(X) \mapsto \mathcal{IF}(X)$ which satisfies for any two IFS $A = \langle f_A, g_A\rangle$ and $B = \langle f_B, g_B\rangle$ the following properties

(I1) If $f_A(x) + g_A(x) = 1$ and $f_B(x) + g_B(x) = 1$ then $\pi_{I(A,B)(x)} = 0$ (where $\pi_{C(x)}$ is the intuitionistic fuzzy index of the element $x$ relative to the IFS C defined as $\pi_{C(x)} = 1 - f_C(x) - g_C(x)$);

(I2) If $\langle f_A, g_A\rangle \subseteq_x \langle f_B, g_B\rangle$ then for all $C = \langle f_C, g_C\rangle \in \mathcal{IF}(X)$ $I(\langle f_B, g_B\rangle, \langle f_C, g_C\rangle) \subseteq_x I(\langle f_A, g_A\rangle, \langle f_C, g_C\rangle)(x)$;

(I3) If $\langle f_A, g_A\rangle \subseteq_x \langle f_B, g_B\rangle$ then for all $C = \langle f_C, g_C\rangle \in \mathcal{IF}(X)$ $I(\langle f_C, g_C\rangle, \langle f_A, g_A\rangle) \subseteq_x I(\langle f_C, g_C\rangle, \langle f_B, g_B\rangle)$;

(I4) If $\langle f_A, g_A\rangle(x) = \langle 0, 1\rangle$ then $I(\langle f_A, g_A\rangle(x), \langle f_B, g_B\rangle)(x) = \langle 1, 0\rangle$;

(I5) If $\langle f_B, g_B\rangle(x) = \langle 1, 0\rangle$ then $I(\langle f_A, g_A\rangle(x), \langle f_B, g_B\rangle)(x) = \langle 1, 0\rangle$;

(I6) If $\langle f_A, g_A\rangle(x) = \langle 1, 0\rangle$ and $\langle f_B, g_B\rangle(x) = \langle 0, 1\rangle$ then $I(\langle f_A, g_A\rangle(x), \langle f_B, g_B\rangle)(x) = \langle 0, 1\rangle$.

As proposed by Deschrijver (private communication relative to the unit interval $[0,1]$, here extended to $\mathcal{F}(X)$) it is possible to define a Gödel implication also on $\mathcal{IF}(X)$ as follows

$$(\langle f_A, g_A\rangle \Rightarrow_G \langle f_B, g_B\rangle)(x) := \begin{cases} \langle 1, 0\rangle & \text{if } f_A(x) \leq f_B(x) \\ & \text{and } g_A(x) \geq g_B(x) \\ \langle 1 - g_B(x), g_B(x)\rangle & \text{if } f_A(x) \leq f_B(x) \\ & \text{and } g_A(x) < g_B(x) \\ \langle f_B(x), 0\rangle & \text{if } f_A(x) > f_B(x) \\ & \text{and } g_A(x) \geq g_B(x) \\ \langle f_B(x), g_B(x)\rangle & \text{if } f_A(x) > f_B(x) \\ & \text{and } g_A(x) < g_B(x) \end{cases}$$

We remark that on the collection of "exact" IFSs, i.e., in the case that $f_A, g_A, f_B, g_B \in \{0,1\}^X$ this implication coincides with the one introduced in Proposition 2.4 on the collection of all orthopairs of crisp fuzzy sets.

The structure $\langle \mathcal{IF}(X), \cap, \cup, \Rightarrow_G, \langle \mathbf{0}, \mathbf{1} \rangle \rangle$ is a Heyting algebra. So, the negation induced by the implication $\Rightarrow_G$ in the usual manner $A^{\approx} = A \Rightarrow_G \langle \mathbf{0}, \mathbf{1} \rangle$

$$A^{\approx}(x) = \begin{cases} \langle 1, 0 \rangle & \text{if} \quad \langle f_A, g_A \rangle (x) = \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \text{otherwise} \end{cases}$$

is a Brouwer negation. Further, considering also the de Morgan negation on IFS $^-$, it is possible to show that the structure $\langle \mathcal{IF}(X), \cap, \cup, \Rightarrow_G, ^- \rangle$ is a symmetric Heyting algebra.

## 5 Conclusion

A rough approximation space has been introduced in the wBD lattice structure. Then, we showed that the collection of all rough approximations and, more generally, of all orthopairs of exact elements, gives rise to a HW algebra. This result is linked to IFS theory since an "exact" IFS is an orthopair of fuzzy exact sets on a BK lattice, thus the collection of all such IFSs is endowed with a HW structure. On the whole set of IFS, i.e., on the collection of all orthopairs of fuzzy sets, only some weaker algebras can be defined.

As a future work, it will be interesting to give an HW algebraic structure to the collection of all IFS, with the result to have a common algebraic framework for fuzzy sets, rough sets and intuitionistic fuzzy sets. A first step toward this goal has already been done with the definition of a Gödel implication on $\mathcal{IF}(X)$, as shown above, but as far as we know, a Łukasiewicz implication on $\mathcal{IF}(X)$ has still to be defined.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