Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2022, Volume 28, Number 2, Pages 113–119
DOI: 10.7546/nifs.2022.28.2.113-119

On modal forms of the two-parametric weak intuitionistic fuzzy implication

Piotr Dworniczak

Department of Economics, The Great Poland University of Social and Economics
ul. Surzyńskich 2, 63-000 Środa Wlkp., Poland
e-mail: p.dworniczak@wwsse.pl

Received: 21 October 2021 **Accepted:** 22 May 2022

Abstract: In this paper new modal forms of the two-parametric weak intuitionistic fuzzy implication are introduced. Fulfillment of some properties, together with *Modus Ponens* and *Modus Tollens* inference rules, are investigated. Negations generated by implications are presented.

Keywords: Intuitionistic fuzzy logic, Weak intuitionistic fuzzy implication, Intuitionistic fuzzy negation.
2020 Mathematics Subject Classification: 03E72.

1 Introduction

The Intuitionistic Fuzzy Logic (shortly: IFL) is the extension of the classical, two valued, logic. The main results in the area of the IFL are collected in the monograph [2]. In the IFL the truth-value of variable x is given by ordered pair $\langle a, b\rangle$, where $a, b, a + b \in [0, 1]$.

Such a pair is called an Intuitionistic Fuzzy Pair (shortly: IFP), [4]. The numbers a and b are interpreted as the degrees of validity and non-validity of x. We denote the truth-value of x by $V(x)$. Especially, the variable with truth-value *true* in the classical logic we denote by $\underline{1}$ and the variable *false* by $\underline{0}$. For this variables holds $V(\underline{1}) = \langle 1, 0\rangle$ and $V(\underline{0}) = \langle 0, 1\rangle$.

We call the variable x an Intuitionistic Fuzzy Tautology (shortly: IFT), if and only if (shortly: iff) when for $V(x) = \langle a, b\rangle$ holds: $a \geq b$ and, similar, an Intuitionistic Fuzzy co-Tautology (shortly: IFcT), iff holds: $a \leq b$.

For every x we can define the value of negation of x in the typical form $V(\neg x) = \langle b, a\rangle$.

An important operator of IFL is Intuitionistic Fuzzy Implication. In [3] are noticed nearly 200 different intuitionistic fuzzy implications. One of these is given in [7]. Such a type of implication is called later [6] a Weak Intuitionistic Fuzzy Implication (shortly: WIFI).

Before the introduction of the definition of a WIFI we must remind the order relation in the set of IFPs. For $V(x) = \langle a, b\rangle$ and $V(y) = \langle c, d\rangle$ we define the order relation as follows:

$$V(x) \preceq V(y) \text{ iff } a \leq c \text{ and } b \geq d.$$

Definition 1. The logical connective $\Rightarrow$ is the WIFI if it fulfills the conditions (i1)–(i5) in the form:

(i1) if $V(x_1) \preceq V(x_2)$, then $V(x_1 \Rightarrow y) \succeq V(x_2 \Rightarrow y)$,

(i2) if $V(y_1) \preceq V(y_2)$, then $V(x \Rightarrow y_1) \preceq V(x \Rightarrow y_2)$,

(i3) $\underline{0} \Rightarrow y$ is an IFT,

(i4) $x \Rightarrow \underline{1}$ is an IFT,

(i5) $\underline{1} \Rightarrow \underline{0}$ is an IFcT.

2 Main results

In [7] were introduced a class of two-parametric intuitionistic fuzzy implications. For $V(x) = \langle a, b\rangle$ and $V(y) = \langle c, d\rangle$ such implication is the intuitionistic logical connective with the truth-value:

$$V(x \rightarrow_{\alpha,\beta} y) = \left\langle \frac{b+c+\alpha-1}{\alpha+\beta}, \frac{a+d+\beta-1}{\alpha+\beta} \right\rangle,$$

where $\alpha, \beta \in \Re$, $\alpha, \beta \geq 1$, and $\beta \in [\alpha - 2, \alpha]$.

The implication fulfills the Definition 1. Therefore, it is a WIFI. We denote it as (α, β)-WIFI.

One of the fundamental tautologies of classical logic is the relationship between the implication and negation. This relationship says that the truth-value of negation of the variable x is equal to the value of the logical implications of the antecedent x and the consequent *false*. Symbolically, this tautology is written in the form of $N(x) \Leftrightarrow (x \Rightarrow 0)$. Using this relationship, we can, for every intuitionistic fuzzy implication, designate a corresponding negation, called a generated (induced) negation. The negation $N_{\alpha,\beta}$ generated by an (α, β)-WIFI is given by the formula:

$$V(N_{\alpha,\beta}(x)) = \left\langle \frac{b+\alpha-1}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta} \right\rangle.$$

There exist in the literature two basic modal operators over intuitionistic fuzzy sets or intuitionistic fuzzy pairs. We remind them both in the form for IFPs.

The operators $\Box$ and $\Diamond$, called necessity and possibility operators, are defined (see, e.g., [1]) as follows:

$$\Box\langle a,b\rangle = \langle a, 1-a\rangle,$$

$$\Diamond\langle a,b\rangle = \langle 1-b, b\rangle.$$

Now, using the (α, β)-WIFI and the above operators, we will define four new WIFIs. For simplicity we write $\Box x$ and $\Diamond x$ instead of $\Box V(x) = \Box\langle a, b\rangle$ and $\Diamond V(x) = \Diamond\langle a, b\rangle$.

Definition 2. For any variables x and y

$$V(x \to_{\Box\Box} y) = V(\Box x \to_{\alpha,\beta} \Box y) = \left\langle \frac{c-a+\alpha}{\alpha+\beta}, \frac{a-c+\beta}{\alpha+\beta} \right\rangle,$$

$$V(x \to_{\Box\Diamond} y) = V(\Box x \to_{\alpha,\beta} \Diamond y) = \left\langle \frac{1-a-d+\alpha}{\alpha+\beta}, \frac{a+d+\beta-1}{\alpha+\beta} \right\rangle,$$

$$V(x \to_{\Diamond\Box} y) = V(\Diamond x \to_{\alpha,\beta} \Box y) = \left\langle \frac{b+c+\alpha-1}{\alpha+\beta}, \frac{1-b-c+\beta}{\alpha+\beta} \right\rangle,$$

$$V(x \to_{\Diamond\Diamond} y) = V(\Diamond x \to_{\alpha,\beta} \Diamond y) = \left\langle \frac{b-d+\alpha}{\alpha+\beta}, \frac{d-b+\beta}{\alpha+\beta} \right\rangle.$$

Theorem 1. *The logical connectives $x \to_{\Box\Box} y$, $x \to_{\Box\Diamond} y$, $x \to_{\Diamond\Box} y$ and $x \to_{\Diamond\Diamond} y$ with truth-values given in Definition 2 are the WIFIs.*

Proof. The reasoning will be presented only for the first connective. For the other three connectives, the reasoning is analogous.

Let us note firstly that the value $\left\langle \frac{c-a+\alpha}{\alpha+\beta}, \frac{a-c+\beta}{\alpha+\beta} \right\rangle$ is an IFP.

We check now the properties (i1)–(i5) from Definition 1.

(i1) Let $V(x_1) \preceq V(x_2)$, therefore $\langle a_1, b_1 \rangle \preceq \langle a_2, b_2 \rangle$, which means that $a_1 \le a_2$ and $b_1 \ge b_2$. Now,

$$V(x_1 \to_{\Box\Box} y) = V(\Box x_1 \to_{\alpha,\beta} \Box y) = \left\langle \frac{c-a_1+\alpha}{\alpha+\beta}, \frac{a_1-c+\beta}{\alpha+\beta} \right\rangle,$$

$$V(x_2 \to_{\Box\Box} y) = V(\Box x_2 \to_{\alpha,\beta} \Box y) = \left\langle \frac{c-a_2+\alpha}{\alpha+\beta}, \frac{a_2-c+\beta}{\alpha+\beta} \right\rangle.$$

Since inequalities

$$\frac{c-a_1+\alpha}{\alpha+\beta} \ge \frac{c-a_2+\alpha}{\alpha+\beta} \quad \text{and} \quad \frac{a_1-c+\beta}{\alpha+\beta} \le \frac{a_2-c+\beta}{\alpha+\beta}$$

are equivalent to

$$a_1 \le a_2,$$

and this, by assumption, holds, then the property (i1) is fulfilled.

(i2) Let $V(y_1) \preceq V(y_2)$, therefore $\langle c_1, d_1 \rangle \preceq \langle c_2, d_2 \rangle$ what means $c_1 \le c_2$ and $d_1 \ge d_2$. Now,

$$V(x \to_{\Box\Box} y_1) = V(\Box x \to_{\alpha,\beta} \Box y_1) = \left\langle \frac{c_1-a+\alpha}{\alpha+\beta}, \frac{a-c_1+\beta}{\alpha+\beta} \right\rangle,$$

$$V(x \to_{\Box\Box} y_2) = V(\Box x \to_{\alpha,\beta} \Box y_2) = \left\langle \frac{c_2-a+\alpha}{\alpha+\beta}, \frac{a-c_2+\beta}{\alpha+\beta} \right\rangle.$$

Since inequalities

$$\frac{c_1-a+\alpha}{\alpha+\beta} \le \frac{c_2-a+\alpha}{\alpha+\beta} \quad \text{and} \quad \frac{a-c_1+\beta}{\alpha+\beta} \ge \frac{a-c_2+\beta}{\alpha+\beta}$$

are equivalent to

$$c_1 \le c_2,$$

and this, by assumption, holds, then the property (i2) is fulfilled.

(i3) For any y it holds that

$$V(\underline{0} \rightarrow_{\square\square} y) = \left\langle \frac{c-0+\alpha}{\alpha+\beta}, \frac{0-c+\beta}{\alpha+\beta} \right\rangle.$$

The IFP is an IFT because

$$\frac{c+\alpha}{\alpha+\beta} \geq \frac{\beta-c}{\alpha+\beta},$$

and equivalently

$$c + \alpha \geq \beta - c,$$

that is

$$2c \geq \beta - \alpha.$$

The value of the left-hand side of this inequality belongs to the interval [0, 2], while the value of the right-hand side belongs to [–2, 0]. Therefore, the inequality holds and the property (i3) is fulfilled.

(i4) For any x it holds that

$$V(x \rightarrow_{\square\square} \underline{1}) = \left\langle \frac{1-a+\alpha}{\alpha+\beta}, \frac{a-1+\beta}{\alpha+\beta} \right\rangle.$$

The IFP is an IFT because

$$\frac{1-a+\alpha}{\alpha+\beta} \geq \frac{a-1+\beta}{\alpha+\beta},$$

and equivalently

$$1 - a + \alpha \geq a - 1 + \beta,$$

that is

$$\alpha - \beta \geq 2a - 2$$

The value of the left-hand side of this inequality belongs to the interval [0, 2], while the value of the right-hand side belongs to [–2, 0]. Therefore, the inequality holds and the property (i4) is fulfilled.

(i5) It holds that

$$V(\underline{1} \rightarrow_{\square\square} \underline{0}) = \left\langle \frac{-1+\alpha}{\alpha+\beta}, \frac{1+\beta}{\alpha+\beta} \right\rangle.$$

The IFP is an IFcT because

$$\frac{1+\beta}{\alpha+\beta} \geq \frac{\alpha-1}{\alpha+\beta},$$

and equivalently

$$1 + \beta \geq \alpha - 1,$$

that is

$$\alpha - \beta \leq 2.$$

The value of the left-hand side of this inequality belongs to the interval [0, 2]. Therefore, the inequality holds and the property (i5) is fulfilled.

Thus the proof has been completed. □

The introduced implications generate negations with the following truth-values:

$$V(N_{\square\square}(x)) = \left\langle \frac{\alpha - a}{\alpha + \beta}, \frac{a + \beta}{\alpha + \beta} \right\rangle,$$

$$V(N_{\square\Diamond}(x)) = \left\langle \frac{\alpha - a}{\alpha + \beta}, \frac{a + \beta}{\alpha + \beta} \right\rangle,$$

$$V(N_{\Diamond\square}(x)) = \left\langle \frac{b + \alpha - 1}{\alpha + \beta}, \frac{1 - b + \beta}{\alpha + \beta} \right\rangle,$$

$$V(N_{\Diamond\Diamond}(x)) = \left\langle \frac{b + \alpha - 1}{\alpha + \beta}, \frac{1 - b + \beta}{\alpha + \beta} \right\rangle.$$

Remarks:

(R1) All the values $V(x \to_{\square\square} y)$, $V(x \to_{\square\Diamond} y)$, $V(x \to_{\Diamond\square} y)$, $V(x \to_{\Diamond\Diamond} y)$, $V(N_{\square\square}(x))$, $V(N_{\square\Diamond}(x))$, $V(N_{\Diamond\square}(x))$, and $V(N_{\Diamond\Diamond}(x))$ are classical fuzzy values, i.e., the sum of their membership and non-membership values is equal to 1.

(R2) $V(N_{\square\square}(x)) = V(N_{\square\Diamond}(x))$ and $V(N_{\Diamond\square}(x)) = V(N_{\Diamond\Diamond}(x))$.

(R3) $N_{\square\square}(\underline{1}) = N_{\square\Diamond}(\underline{1}) = N_{\Diamond\square}(\underline{1}) = N_{\Diamond\Diamond}(\underline{1}) = \left\langle \frac{\alpha - 1}{\alpha + \beta}, \frac{1 + \beta}{\alpha + \beta} \right\rangle \neq V(\underline{0})$, i.e., it is an IFcT, and

$N_{\square\square}(\underline{0}) = N_{\square\Diamond}(\underline{0}) = N_{\Diamond\square}(\underline{0}) = N_{\Diamond\Diamond}(\underline{0}) = \left\langle \frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta} \right\rangle \neq V(\underline{1})$, i.e., it is an IFT.

(R4) The negations $N_{\square\square}(x)$, $N_{\square\Diamond}(x)$, $N_{\Diamond\square}(x)$, and $N_{\Diamond\Diamond}(x)$ are not involutive.

(R5) The negations $N_{\square\square}(x)$, $N_{\square\Diamond}(x)$, $N_{\Diamond\square}(x)$, and $N_{\Diamond\Diamond}(x)$ do not fulfill the property: the negation of an IFT should be an IFcT and the negation of an IFcT should be an IFT. Therefore, the above negations should be carefully used in the potentially applications.

(R6) The special case of the presented above implications and negations are given by Atanassova in [5]. In this case, both parameters α and β are equal to 1.

There exist two basic rules of the inference; *Modus Ponens* (MP) and *Modus Tollens* (MT). In the IF environment they can be formulated as follows:

$$\text{if } x \text{ is an IFT and } x \Rightarrow y \text{ is an IFT, then } y \text{ is an IFT,} \quad \text{(MP)}$$

$$\text{if } x \Rightarrow y \text{ is an IFT and } y \text{ is an IFcT, then } x \text{ is an IFcT,} \quad \text{(MT)}$$

where $\Rightarrow$ is some implication.

For the WIFIs introduced in Definition 2 the following theorem holds.

Theorem 2. *The WIFIs $x \to_{\square\square} y$, $x \to_{\square\Diamond} y$, $x \to_{\Diamond\square} y$, and $x \to_{\Diamond\Diamond} y$ do not satisfy the (MP) rule and does not satisfy the (MT) rule of inference.*

Proof. We present only the counterexample for the $x \to_{\square\square} y$ implication. The counterexamples for the other implications can be created analogously.

Let $\alpha = 1$, $\beta = 1$, $V(x) = \langle a, b\rangle = \langle 0.3, 0.0 \rangle$, $V(x \to_{\square\square} y) = \langle 0.55, 0.45 \rangle$, and $V(y) = \langle c, d\rangle$.

Then $\left\langle \frac{c-a+\alpha}{\alpha+\beta}, \frac{a-c+\beta}{\alpha+\beta} \right\rangle = \left\langle \frac{c-0.3+1}{2}, \frac{0.3-c+1}{2} \right\rangle = \langle 0.55, 0.45 \rangle$ only for $c = 0.4$, but for any d. For example, for $d = 0.6$ the x is an IFT, $x \to_{\Box\Box} y$ is an IFT while y is not an IFT.

Therefore, the (MP) rule does not hold.

Let $\alpha = 3$, $\beta = 1$, $V(y) = \langle c, d \rangle = \langle 0.0, 0.4 \rangle$. In this case $V(x \to_{\Box\Box} y) = \left\langle \frac{c-a+\alpha}{\alpha+\beta}, \frac{a-c+\beta}{\alpha+\beta} \right\rangle = \left\langle \frac{3-a}{4}, \frac{a+1}{4} \right\rangle$. The $x \to_{\Box\Box} y$ is an IFT because $3 - a \geq a + 1$. Therefore, $x \to_{\Box\Box} y$ is an IFT and y is an IFcT but x is not necessarily an IFcT. For example, it can be $V(x) = \langle 0.5, 0.0 \rangle$.

Finally, the (MT) rule does not holds. □

In the literature on fuzzy implications (not necessarily intuitionistic fuzzy implications), besides (i1)–(i5), the following axioms are also postulated (see, e.g., [5]).

(i6) $V(\underline{1} \Rightarrow y) = V(y)$,

(i7) $V(x \Rightarrow x) = V(\underline{1})$,

(i8) $V(x \Rightarrow (y \Rightarrow z)) = V(y \Rightarrow (x \Rightarrow z))$

(i9) $V(x \Rightarrow y) = V(\underline{1})$ iff $V(x) \preceq V(y)$,

(i10) $V(x \Rightarrow y) = V(N(y) \Rightarrow N(x))$, where N is a negation,

where x, y, z are variables with truth-values $V(x) = \langle a, b \rangle$, $V(y) = \langle c, d \rangle$, $V(z) = \langle e, f \rangle$, and $a, b, c, d, e, f, a + b, c + d, e + f \in [0, 1]$.

Theorem 3. *The WIFIs* $\to_{\Box\Box}$, $\to_{\Box\Diamond}$, $\to_{\Diamond\Box}$ *and* $\to_{\Diamond\Diamond}$

a) *do not satisfy (i6) and (i8),*

b) *do not satisfy (i7), but* $x \to_{\Box\Box} x$, $x \to_{\Box\Diamond} x$ *and* $x \to_{\Diamond\Diamond} x$ *are IFTs,*

c) *do not satisfy (i9), but if* $V(x \to_{①②} y) = V(\underline{1})$, *then* $V(x) \preceq V(y)$, *where* $①, ② \in \{\Box, \Diamond\}$,

d) *satisfy (i10) with* $N = \neg$ *for* $\to_{\Box\Diamond}$ *and* $\to_{\Diamond\Box}$, *and does not satisfy for* $\to_{\Box\Box}$ *and* $\to_{\Diamond\Diamond}$, *but* $V(x \to_{\Box\Box} y) = V(\neg y \to_{\Diamond\Diamond} \neg x)$ *and* $V(x \to_{\Diamond\Diamond} y) = V(\neg y \to_{\Box\Box} \neg x)$,

The proof is omitted.

It is easy to check that the WIFIs $\to_{\Box\Box}$, $\to_{\Box\Diamond}$, $\to_{\Diamond\Box}$ and $\to_{\Diamond\Diamond}$ do not satisfy the classical (two-valued) logic axioms. Namely $V(\underline{0} \to_{①②} \underline{0}) = V(\underline{1} \to_{①②} \underline{1}) = \left\langle \frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta} \right\rangle \neq V(\underline{1})$, $V(\underline{1} \to_{①②} \underline{0}) = \left\langle \frac{\alpha-1}{\alpha+\beta}, \frac{\beta+1}{\alpha+\beta} \right\rangle \neq V(\underline{0})$ (except $\alpha = \beta = 1$) and $V(\underline{0} \to_{①②} \underline{1}) = \left\langle \frac{\alpha+1}{\alpha+\beta}, \frac{\beta-1}{\alpha+\beta} \right\rangle \neq V(\underline{1})$ (except $\alpha \in [1, 3]$, $\beta = 1$), where $①, ② \in \{\Box, \Diamond\}$.

But we notice that $\underline{0} \to_{①②} \underline{0}$, $\underline{1} \to_{①②} \underline{1}$ and $\underline{0} \to_{①②} \underline{1}$ are IFTs, and $\underline{1} \to_{①②} \underline{0}$ is an IFcT.

Therefore, neither of the implications $\to_{\Box\Box}$, $\to_{\Box\Diamond}$, $\to_{\Diamond\Box}$ and $\to_{\Diamond\Diamond}$ is a generalization of the classical implication.

3 Conclusion

In the paper four new modal forms of weak intuitionistic fuzzy implications with their basic properties are presented. These implications may be the subject of further research, both in terms of their properties or comparisons with other intuitionistic fuzzy implications, and possible applications.

References

[1] Atanassov, K. T. (1983). Intuitionistic fuzzy sets, VII ITKR Session, Sofia, 20-23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84), reprint and English version in: *International Journal Bioautomation*, 20, S1, 1–6.

[2] Atanassov, K. T. (2017). *Intuitionistic Fuzzy Logics*. Springer International Publishing, Cham.

[3] Atanassov, K. T., & Angelova, N. (2021). Research on intuitionistic fuzzy implications. *Notes on Intuitionistic Fuzzy Sets*, 27(2), 20–93.

[4] Atanassov, K. T., Szmidt, E., & Kacprzyk, J. (2013). On intuitionistic fuzzy pairs. *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[5] Atanassova, L. (2010). Four modal form of intuitionistic fuzzy implication $\rightarrow_{@}$ and two related intuitionistic fuzzy negations. Part 1. *Notes on Intuitionistic Fuzzy Sets*, 16(2), 31–34.

[6] Dworniczak, P. (2011). Inclusion of the intuitionistic fuzzy sets based on some weak intuitionistic fuzzy implications. *Cybernetics and Information Technologies*, 11(3), 12–22.

[7] Dworniczak, P. (2011). On some two-parametric intuitionistic fuzzy implications. *Notes on Intuitionistic Fuzzy Sets*, 17(2), 8–16.