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A new modal operator over intuitionistic fuzzy sets

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Abstract: A new operator from modal type is introduced over the intuitionistic fuzzy sets. On one hand, this operator functions by reducing the degree of membership or non-membership, and, on the other hand, by simultaneously summing it with a part of the degree of non-membership or membership, respectively. Some of its properties are studied.
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1 Introduction

In a series of papers of the two authors, a new type of intuitionistic fuzzy modal operators are introduced and some of their properties are studied. The definitions of these operattors are given in Section 2. In Section 3 a new operator from modal type is introduced and some of its bassic properties are studied. In the Conclusion an Open Problems are formulated.

2 Preliminary results

Let a set E be fixed. The Intuitionistic Fuzzy Set (IFS) A in E is defined by (see, e.g., [1]):

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Different relations and operations are introduced over the IFSs. Some of them (see, e.g. [1, 4]) are the following

$$\begin{array}{lcl}
A \subset B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \& \nu_A(x) \geq \nu_B(x)), \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)), \\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}, \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}, \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}, \\
A + B & = & \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle | x \in E\}, \\
A.B & = & \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle | x \in E\}, \\
\Box A & = & \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}, \\
\Diamond A & = & \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.
\end{array}$$

In [1, 4] the following extended modal operators are defined: Let $\alpha, \beta \in [0,1]$ and let:

$$\begin{array}{l}
F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1 \\
G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\}, \\
H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \\
H^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\}, \\
J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\}, \\
J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\},
\end{array}$$

In [1, 4] the following operator is defined

$$X_{a,b,c,d,e,f}(A) = \{\langle x, a.\mu_A(x) + b.(1 - \mu_A(x) - c.\nu_A(x)),$$

$$d.\nu_A(x) + e.(1 - f.\mu_A(x) - \nu_A(x))\rangle | x \in E\}$$

where $a, b, c, d, e, f \in [0,1]$ and there, the following two conditions are given:

$$a + e - e.f \leq 1,$$

$$b + d - b.c \leq 1.$$

In addition, in [5] it is demonstrated that it is also necesary to add the following third condition:

$$b + e \leq 1.$$

In [4] another type of modal operators are described.

The following are the first two operators of modal type, which are similar to the standard modal operators $\Box$ and $\Diamond$:

$$\boxplus A = \{\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\rangle | x \in E\},$$

$$\boxtimes A = \{\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\rangle | x \in E\}.$$

For a given real number $\alpha \in [0, 1]$ and IFS A, the above operators are extended to the forms:

$$\boxplus_\alpha A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + 1 - \alpha\rangle | x \in E\},$$

$$\boxtimes_\alpha A = \{\langle x, \alpha.\mu_A(x) + 1 - \alpha, \alpha.\nu_A(x)\rangle | x \in E\}.$$

The second extension was introduced by Katerina Dencheva in [8]. She extended the last two operators to the forms:

$$\boxplus_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + \beta\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x) + \beta, \alpha.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \alpha + \beta \in [0, 1]$.

The third extension of the above operators gives the following operators (see [4]):

$$\boxplus_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x) + \gamma\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \leq 1$.

In [2, 4], the idea for extending the last two operators naturally produced the operator

$$\boxdot_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\max(\alpha, \beta) + \gamma + \delta \leq 1$.

In 2007in [6], Gökhan Çuvalcıoğlu introduced operator $E_{\alpha,\beta}$ by

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha.\mu_A(x) + 1 - \alpha), \alpha(\beta.\nu_A(x) + 1 - \beta)\rangle | x \in E\},$$

where $\alpha, \beta \in [0, 1]$, and he studied some of its properties. Obviously,

$$E_{\alpha,\beta}(A) = \boxdot_{\alpha\beta,\alpha\beta,(1-\alpha)\beta,(1-\beta)\alpha} A.$$

In 2010, he extended the previous operator to the form:

$$Z^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta)\rangle | x \in X\},$$

where $\omega, \alpha, \beta \in [0, 1]$ (see [7]).

A new (and probably final?) extension of the above operators is the operator

$$\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A = \{\langle x, \alpha.\mu_A(x) - \varepsilon.\nu_A(x) + \gamma,$$

$$\beta.\nu_A(x) - \zeta.\mu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ and

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1,$$

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0$$

(see [3, 4]).

Theorem [4]. Operators $X_{a,b,c,d,e,f}$ and $\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ are equivalent.

Finally, we construct the following Figure 1 in which "$X \longrightarrow Y$" denotes that operator X represents operator Y, while the reverse is not valid.

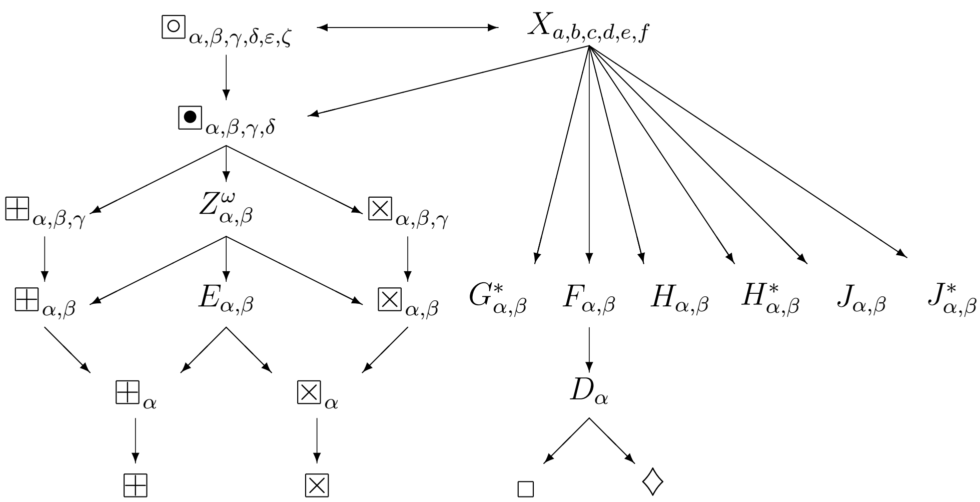

Figure 1.

3 Main results

Here, we introduce the following new operator from modal type, that is a modification of the above discussed operators. It has the form

$$\otimes_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha.\mu_A(x) + \gamma.\nu_A(x), \beta.\mu_A(x) + \delta.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\alpha + \beta \leq 1, \gamma + \delta \leq 1$.

According to this definition, on one hand, the operator reduces by α the degree of membership $\mu_A(x)$ original IFS A's and sums it up with a part of the degree of non-membership $(\gamma.\nu_A(x))$, and in the same time it reduces the original A's degree of non-membership ($\nu_A(x)$) by δ and sums it up with a part of the degree of membership ($\beta.\mu_A(x)$).

It is easy to see that the operator

$$\otimes_{1,0,0,1} A = A,$$

and

$$\otimes_{0,1,1,0} A = \neg A.$$

Therefore, this operator gives the possibility to express the operation identity and the operation "classical negation". In this way, by varying the values of the variables $\alpha, \beta, \gamma, \delta$ in the $[0; 1]$ range, we can obtain the whole continuity of sets existing between a given set A and its classical negation $\neg A$.

Let us study the basic properties of the new operator.

First, we check that the new set is an IFS. Really,

$$0 \leq \alpha.\mu_A(x) + \gamma.\nu_A(x) \leq \mu_A(x) + \nu_A(x) \leq 1,$$

$$0 \leq \beta.\mu_A(x) + \delta.\nu_A(x) \leq \mu_A(x) + \nu_A(x) \leq 1$$

and

$$\begin{aligned} 0 \leq \alpha.\mu_A(x) + \gamma.\nu_A(x), \beta.\mu_A(x) + \delta.\nu_A(x) \\ = (\alpha + \beta).\mu_A(x) + (\gamma + \delta).\nu_A(x) \\ \leq \mu_A(x) + \nu_A(x) \leq 1. \end{aligned}$$

Theorem 1: For every IFS A and for every four real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$

$$\neg \otimes_{\alpha,\beta,\gamma,\delta} \neg A = \otimes_{\delta,\gamma,\beta,\alpha} A.$$

Proof: We obtain sequentially that

$$\begin{aligned} &\neg \otimes_{\alpha,\beta,\gamma,\delta} \neg A \\ &= \neg \otimes_{\alpha,\beta,\gamma,\delta} \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\} \\ &= \neg\{\langle x, \alpha.\nu_A(x) + \gamma.\mu_A(x), \beta.\nu_A(x) + \delta.\mu_A(x)\rangle | x \in E\} \\ &= \{\langle x, \beta.\nu_A(x) + \delta.\mu_A(x), \alpha.\nu_A(x) + \gamma.\mu_A(x)\rangle | x \in E\} \\ &= \otimes_{\delta,\gamma,\beta,\alpha} A. \end{aligned}$$

This completes the proof. □

Theorem 2: For every two IFSs A and B and for every four real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$, it holds that

(a) $\otimes_{\alpha,\beta,\gamma,\delta} (A \cup B) = \otimes_{\alpha,\beta,\gamma,\delta} A \cup \otimes_{\alpha,\beta,\gamma,\delta} B$,
(b) $\otimes_{\alpha,\beta,\gamma,\delta} (A \cap B) = \otimes_{\alpha,\beta,\gamma,\delta} A \cap \otimes_{\alpha,\beta,\gamma,\delta} B$,
(c) $\otimes_{\alpha,\beta,\gamma,\delta} (A + B) = \otimes_{\alpha,\beta,\gamma,\delta} A + \otimes_{\alpha,\beta,\gamma,\delta} B$,
(d) $\otimes_{\alpha,\beta,\gamma,\delta} (A.B) = \otimes_{\alpha,\beta,\gamma,\delta} A. \otimes_{\alpha,\beta,\gamma,\delta} B$.

Proof: For (a), first, we obtain that

$$\otimes_{\alpha,\beta,\gamma,\delta}(A \cup B)$$
$$= \otimes_{\alpha,\beta,\gamma,\delta}\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}$$
$$= \{\langle x, \alpha.\max(\mu_A(x), \mu_B(x)) + \gamma.\min(\nu_A(x), \nu_B(x)),$$
$$\beta.\max(\mu_A(x), \mu_B(x)) + \delta.\min(\nu_A(x), \nu_B(x))\rangle | x \in E\}.$$

Second, we obtain that

$$\otimes_{\alpha,\beta,\gamma,\delta}A \cup \otimes_{\alpha,\beta,\gamma,\delta}B$$
$$= \{\langle x, \alpha.\mu_A(x) + \gamma.\nu_A(x), \beta.\mu_A(x) + \delta.\nu_A(x)\rangle | x \in E\}$$
$$\cup\{\langle x, \alpha.\mu_B(x) + \gamma.\nu_B(x), \beta.\mu_B(x) + \delta.\nu_B(x)\rangle | x \in E\}$$
$$= \{\langle x, \max(\alpha.\mu_A(x) + \gamma.\nu_A(x), \alpha.\mu_B(x) + \gamma.\nu_B(x)),$$
$$\min(\beta.\mu_A(x) + \delta.\nu_A(x), \beta.\mu_B(x) + \delta.\nu_B(x))\rangle | x \in E\}$$

Let

$$X \equiv \max(\alpha.\mu_A(x) + \gamma.\nu_A(x), \alpha.\mu_B(x) + \gamma.\nu_B(x))$$
$$-\alpha.\max(\mu_A(x), \mu_B(x)) - \gamma.\min(\nu_A(x), \nu_B(x)).$$

Now, for $\mu_A(x), \mu_B(x), \nu_A(x), \nu_B(x)$ we must study the following four cases.

Case 1: $\mu_A(x) \geq \mu_B(x), \nu_A(x) \geq \nu_B(x)$:

$$X = \max(\alpha.\mu_A(x) + \gamma.\nu_A(x), \alpha.\mu_B(x) + \gamma.\nu_B(x)) - \alpha.\mu_A(x) - \gamma.\nu_B(x)$$
$$\geq \alpha.\mu_A(x) + \gamma.\nu_A(x) - \alpha.\mu_A(x) - \gamma.\nu_B(x) \geq 0.$$

Case 2: $\mu_A(x) \geq \mu_B(x), \nu_A(x) < \nu_B(x)$:

$$X = \max(\alpha.\mu_A(x) + \gamma.\nu_A(x), \alpha.\mu_B(x) + \gamma.\nu_B(x)) - \alpha.\mu_A(x) - \gamma.\nu_A(x) \geq 0.$$

Case 3: $\mu_A(x) < \mu_B(x), \nu_A(x) \geq \nu_B(x)$:

$$X = \max(\alpha.\mu_A(x) + \gamma.\nu_A(x), \alpha.\mu_B(x) + \gamma.\nu_B(x)) - \alpha.\mu_B(x) - \gamma.\nu_B(x) \geq 0.$$

Case 4: $\mu_A(x) < \mu_B(x), \nu_A(x) < \nu_B(x)$:

$$X = \max(\alpha.\mu_A(x) + \gamma.\nu_A(x), \alpha.\mu_B(x) + \gamma.\nu_B(x)) - \alpha.\mu_B(x)) - \gamma.\nu_A(x)$$
$$\geq \alpha.\mu_B(x) + \gamma.\nu_B(x) - \alpha.\mu_B(x)) - \gamma.\nu_A(x) \geq 0.$$

Asserions (b), (c) and (d) are proved analogously. The same is valid for the proofs of the next theorem. □

Theorem 3: For every IFS A and for every four real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha+\beta \leq 1, \gamma + \delta \leq 1$

(a) $\Box \otimes_{\alpha,\beta,\gamma,\delta} A \subset \otimes_{\alpha,\beta,\gamma,\delta} \Box A$,
(b) $\otimes_{\alpha,\beta,\gamma,\delta} \Diamond A \subset \Diamond \otimes_{\alpha,\beta,\gamma,\delta} A$.

Now, Figure 1 from above is modified, as illustrated on Figure 2.

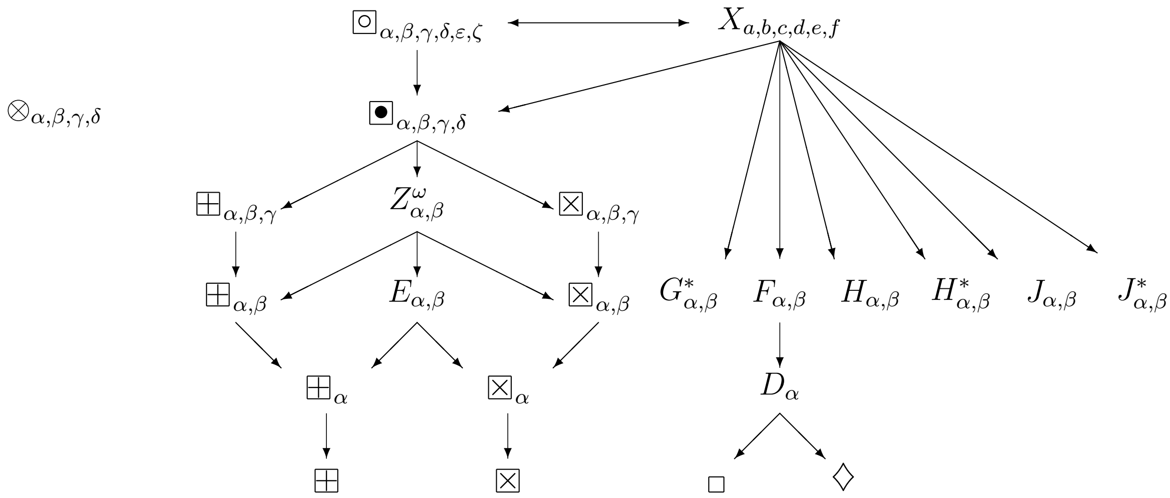

Figure 2.

4 Conclusion

In the present paper, a new modal operator is introduced. It is different from the rest modal operators, defined over IFSs. It arises some open problems, as the following ones.

Open Problem 1: Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be represented by the extended modal operators?

Open Problem 2: Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be represented by the modal operators from Section 2?

Open Problem 3: Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be used for representation of some type of modal operators?

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