25 years of intuitionistic fuzzy sets, or:

The most important results and mistakes of mine

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R_1

Let us have a fixed universe E and its subset A.

The set

$$A = \{<x, \mu_A(x), \nu_A(x)> \mid x \in E\},$$

where

$$0 = \mu_A(x) + \nu_A(x) = 1$$

is called intuitionistic fuzzy set and functions $\mu_A: E \to [0;1]$ and $\nu_A: E \to [0;1]$ represent degree of membership (validity, etc.) and non-membership (non-validity, etc.).

Now, we can define also function $\pi_A: E \to [0;1]$ through

$$\pi(x) = 1 - \mu(x) - \nu(x) .$$

Obviously, for every ordinary fuzzy set A : $\pi_A(x) = 0$ for each $x \in E$ and these sets have the form $\{<x, \mu_A(x), 1 - \mu_A(x) > \mid x \in E\}$.

M_1

For many years I have been thinking that the above definition can be changed, but I had never found strength to do this. Probably, 25 years are enough.

Here I will re-write the above set to the form:

$$A = \{<x, \mu_A(x), \nu_A(x)> \ | \ x \in E \ \& \ 0 = \mu A(x) + \nu A(x) = 1\}.$$

This notation of the concept is more precise from the set-theoretical point of view, but a longer and more difficult to work with.

So, I think, the shorter form should be kept in future, but let exactly me be the one to show the more correct record.

As it is well-known, in the beginning of the last century L. Brouwer introduced the concept of the intuitionism.

He invited the mathematicians to remove Aristoteles' law of excluded middle. In his Cambridge Lectures on Intuitionism he wrote:

“An immediate consequence was that for a mathematical assertion the two cases of truth and falsehood, formerly exclusively admitted, were replaced by the following three:
(1) has been proved to be true
(2) has been proved to be absurd;
(3) has neither been proved to be true nor to be absurd, nor do we know a finite algorithm leading to the statement either that is true or that is absurd.”

Therefore, if we have a proposition A, we can state that either A is true, or A is false, or that we do not know whether A is true or false.

On the level of first order logic, the proposition $A \vee \neg A$ is always valid. In the framework of a G. Boole's algebra this expression has truth value “true" (or 1).

In the ordinary fuzzy logic of L. Zadeh, as well as in many-valued logics (starting with that of J. Lukasiewicz) the above expression can possess value smaller than 1.

The same is true in the case of IFS, but here this situation occurs on semantic as well as on estimations' level. Practically, we fuzzify our estimation in Brouwer's sense, accounting for the three possibilities. This was Gargov's reason to offer the name *intuitionistic fuzzy sets*.

M_2

Up to now I have not researched in details the connections between the IFS theory and Brouwer's intuitionism.

I already see that serious research on this theme is very necessary, even late.

R_2

In March 1983 it turned out that the new sets allow the definition of operators which are, in a sense, analogous to the modal ones (in the case of ordinary fuzzy sets such operators are meaningless, since they reduce to identity).

It was then when I realized that I had found a promising new direction of research and published the results.

M_3 **R_3**

Circa 1986 I saw for a first time the concept of *"Interval-Valued Fuzzy Set"* (IVFS), but my information did not contain the original authors of the research and the results achieved before 1984.

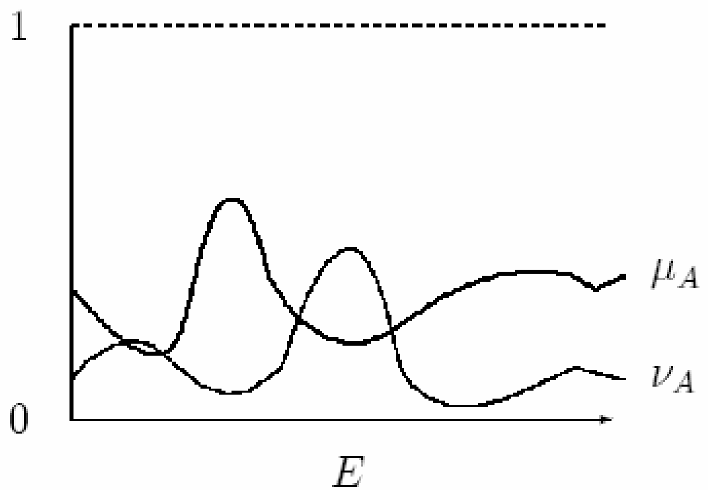
Later, together with G. Gargov, we discussed the equipolence of this concept with IFS. From our construction it is seen that each IFS can be represented by an IVFS and each IVFS can be represented by an IFS.

I specify these years to emphasize that then I believed IFS were defined prior to IVFS. Now, I know (merely as a fact, without having seen the original texts) that IVFS are essentially older.

I am preparing a detailed comparison between both concepts, that will be published soon.

R_4

IFS have different geometrical interpretations. The first of them is a trivial modification of the fuzzy set one and was constructed in the times when I started working on IFS.



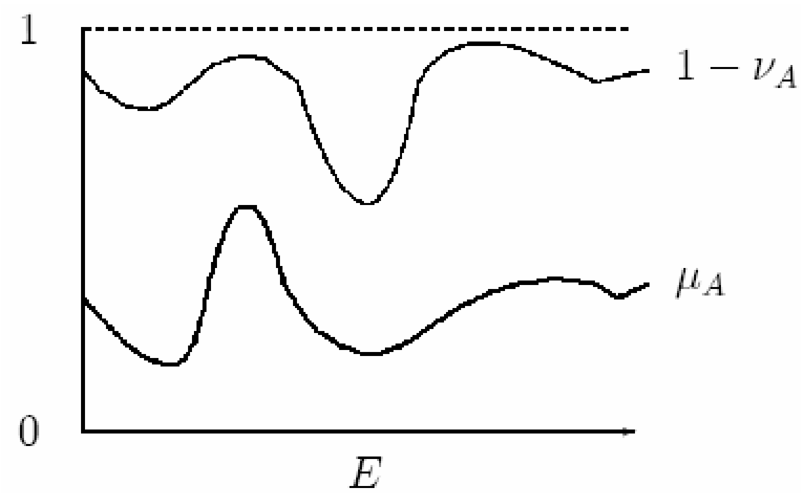
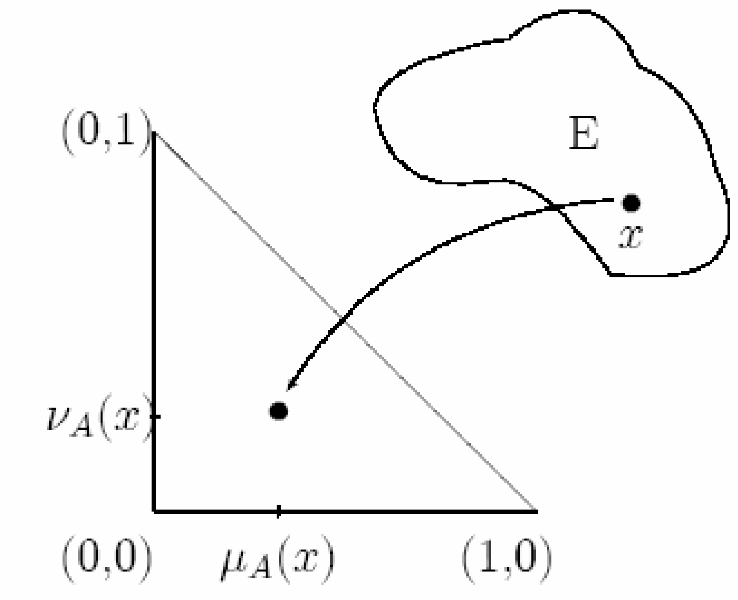


Figure 1 ▲

Figure 2 ▶

M_4

Another interpretation, which in my opinion played very important role in the development of IFS theory is the following:



◀ Figure 3

A big omission of mine is that up to now transformation formulae between both geometrical interpretations (from Figs. 2 and 3) have not been searched.

R_5

Similarly to the fuzzy set theory, a large number of relations and operations over IFSs have been defined.

However, more interesting are the modal operators that can be defined over the IFSs. They do not have analogues in fuzzy set theory.

First, we shall mention some relations and operations. For every two IFSs A and B we can define:

$$A \subset B \quad iff \quad (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \& \nu_A(x) \geq \nu_B(x));$$

$$A = B \quad iff \quad (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x));$$

$$\begin{aligned}
\overline{A} &= \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\};\\
A \cap B &= \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle | x \in E\};\\
A \cup B &= \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x))\rangle | x \in E\};
\end{aligned}$$

$$\begin{aligned}
A + B &= \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle \mid x \in E\};\\
A.B &= \{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle \mid x \in E\};\\
A@B &= \{\langle x, (\frac{\mu_A(x) + \mu_B(x))}{2}, \frac{(\nu_A(x) + \nu_B(x))}{2}\rangle | x \in E\}.
\end{aligned}$$

The algebraic properties of these (and other) operations were studied.

M_5 **R_6**

Let A be a fixed IFS. The first form of operation “negation", introduced in 1983, was

$$\neg_1 A = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\}.$$

One of my mistakes is that for a long time I made use of just the simplest form of negation. In its case the equality $\neg\neg A = A$ holds, which may resemble classical logic. Now, a series of new negations were constructed.

$$\neg^2 A = \{\langle x, 1 - \text{sg}(\mu_A(x)), \text{sg}(\mu_A(x))\rangle | x \in E\},$$

$$\neg^3 A = \{\langle x, \nu_A(x), \mu_A(x)\nu_A(x) + \mu_A(x)^2\rangle | x \in E\},$$

$$\neg^4 A = \{\langle x, \nu_A(x), 1 - \nu_A(x)\rangle | x \in E\},$$

$$\neg^5 A = \{\langle x, \overline{sg}(1 - \nu_A(x)), \text{sg}(1 - \nu_A(x))\rangle | x \in E\},$$

where

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

The last four negations strongly satisfy the intuitionistic properties.

In a series of papers with my participation a lot of new negations were constructed and their properties were studied.

For example, it is checked that $\neg_1$ satisfies all three properties below, while the rest negations satisfy only the first and the third of them:

Property P1: $A \rightarrow \neg\neg A$

Property P2: $\neg\neg A \rightarrow A$

Property P3: $\neg\neg\neg A = \neg A$.

We can show that negations $\neg_2, \ldots, \neg_5$ do not satisfy LEM ($P \vee \neg P$, where P is a propositional form) and they satisfy some of its modifications (e.g., $\neg\neg P \vee \neg P$).

It is shown that the same negations do not satisfy De Morgan's Laws, but they satisfy some of their modifications.

R_7

Independently on or in relation with some negation, a lot of implications were defined over IFSs. Initially, in a series of papers they were introduced in the frames of the intuitionistic fuzzy logic, but in June 2006 they obtained the following IFS-analogues.

Let A and B be two given IFSs and let

$$X_i = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \rightarrow_i \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}.$$

A long list of IFS-implications have been defined:

$$X_1 = \{\langle x, \max(\nu_A(x), \min(\mu_A(x), \mu_B(x))), \min(\mu_A(x), \nu_B(x))\rangle | x \in E\},$$

$$X_2 = \{\langle x, 1 - \mathrm{sg}(\mu_A(x) - \mu_B(x)), \nu_B(x).\mathrm{sg}(\mu_A(x) - \mu_B(x))\rangle | x \in E\},$$

$$X_3 = \{\langle x, 1-(1-\mu_B(x)).\mathrm{sg}(\mu_A(x)-\mu_B(x)), \nu_B(x).\mathrm{sg}(\mu_A(x)-\mu_B(x))\rangle | x \in E\},$$

$$X_4 = \{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle | x \in E\},$$

$$X_5 = \{\langle x, \min(1, \nu_A(x) + \mu_B(x)), \max(0, \mu_A(x) + \nu_B(x) - 1)\rangle | x \in E\},$$

$$X_6 = \{\langle x, \nu_A(x) + \mu_A(x).\mu_B(x), \mu_A(x).\nu_B(x)\rangle | x \in E\},$$

$$X_7 = \{\langle x, \min(\max(\nu_A(x), \mu_B(x)), \max(\mu_A(x), \nu_A(x)), \max(\mu_B(x), \nu_B(x))),$$
$$\max(\min(\mu_A(x), \nu_B(x)), \min(\mu_A(x), \nu_A(x)), \min(\mu_B(x), \nu_B(x)))\rangle | x \in E\},$$

$$X_8 = \{\langle x, 1 - (1 - \min(\nu_A(x), \mu_B(x))).\mathrm{sg}(\mu_A(x) - \mu_B(x)),$$
$$\max(\mu_A(x), \nu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x)).\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle | x \in E\},$$

$$X_9 = \{\langle x, \nu_A(x) + \mu_A(x)^2\mu_B(x), \mu_A(x)\nu_A(x) + \mu_A(x)^2\nu_B(x)\rangle | x \in E\},$$

$$X_{10} = \{\langle x, \mu_B(x).\overline{sg}(1-\mu_A(x)) + \mathrm{sg}(1-\mu_A(x)).(\overline{sg}(1-\mu_B(x)) + \nu_A(x).\mathrm{sg}(1-\mu_B(x))),$$

$$\nu_B(x).\overline{sg}(1 - \mu_A(x)) + \mu_A(x).\mathrm{sg}(1 - \mu_A(x)).\mathrm{sg}(1 - \mu_B(x))\rangle | x \in E\},$$

$$X_{11} = \{\langle x, 1 - (1 - \mu_B(x)).\mathrm{sg}(\mu_A(x) - \mu_B(x)),$$

$$\nu_B(x).\mathrm{sg}(\mu_A(x) - \mu_B(x)).\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle | x \in E\},$$

$$X_{12} = \{\langle x, \max(\nu_A(x), \mu_B(x)), 1 - \max(\nu_A(x), \mu_B(x))\rangle | x \in E\},$$

$$X_{13} = \{\langle x, \nu_A(x) + \mu_B(x) - \nu_A(x).\mu_B(x), \mu_A(x).\nu_B(x)\rangle | x \in E\},$$

$$X_{14} = \{\langle x, 1-(1-\mu_B(x)).\mathrm{sg}(\mu_A(x)-\mu_B(x)) - \nu_B(x).\overline{sg}(\mu_A(x)-\mu_B(x)).\mathrm{sg}(\nu_B(x)-\nu_A(x)),$$

$$\nu_B(x).\mathrm{sg}(\nu_B(x) - \nu_A(x))\rangle | x \in E\},$$

Other 149 implications were introduced, some of them coinciding.

One of the actual problems is to determine the significantly different implications. The above ones are examples of them, but there are about 100 other that are different then the rest.

For all the 174 implications that I proved that in some sense they are extensions of the classical first order logic implication, because

if

$$\overline{O} = \{\langle x, 0, 1\rangle | x \in E\},$$

$$\overline{E} = \{\langle x, 1, 0\rangle | x \in E\},$$

then for the i-th implication ($1 \leq i \leq 25$) equalities

$$\overline{O} \to_i A = \overline{E},$$

$$A \to_i \overline{E} = \overline{E},$$

$$\overline{E} \to_i \overline{O} = \overline{O}$$

hold for every IFS A.

M_6

I constructed implications X_4 and X_{11} more than 20 years ago, but the second implication stressed me with its complexity at least compared to the first one.

By this reason, for a long time I used only implication X4. Even some years ago, reading Klir and Yuan's *"Fuzzy Sets and Fuzzy Logic"* I understood that its fuzzy form is called *"Kleene-Dienes implication"*.

Many discussions related to the name *"intuitionistic fuzzy sets"* would have been meaningless if 20 years ago I had started using implication X_{11} and generated negation $\neg_2$ on its basis.

R_8

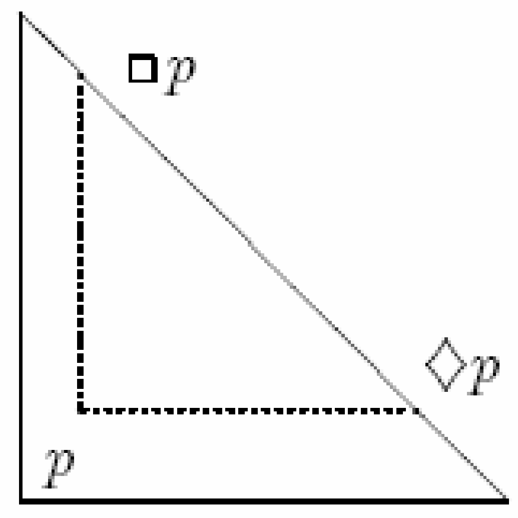
Back in 1983, I defined the modal operators over IFS, the simplest of which are:

$$\square A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\};$$

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.$$

They are analogous to the modal logic operators *"necessity"* and *"possibility"*.

Figure 4 ▶



In the frames of the IFSs theory we can extend these operators in a step by step manner. The first group of extended modal operators are:

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\},$$

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1,$$

$$G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$

$$H_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},$$

$$H^*_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\},$$

$$J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$

$$J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\}.$$

where $\alpha, \beta \in [0, 1]$ are fixed numbers. The geometrical interpretations of the seven operators are given on Fig. 5 - 11.

For a fuzzy set A (whose special case is, e.g., Takeuti and Titani's sets), the modal operators □ and ◊ would satisfy

$$\Box A = A = \Diamond A$$

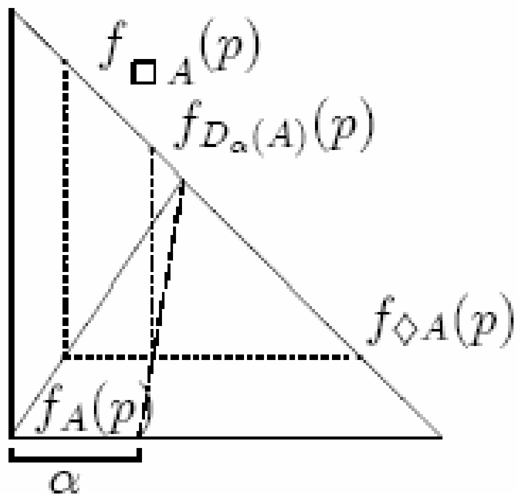
while for a proper IFS A

$$\Box A \subset A \subset \Diamond A$$

and

$$\Box A \neq A \neq \Diamond A$$

This example shows that IFSs are essential extensions of fuzzy sets.



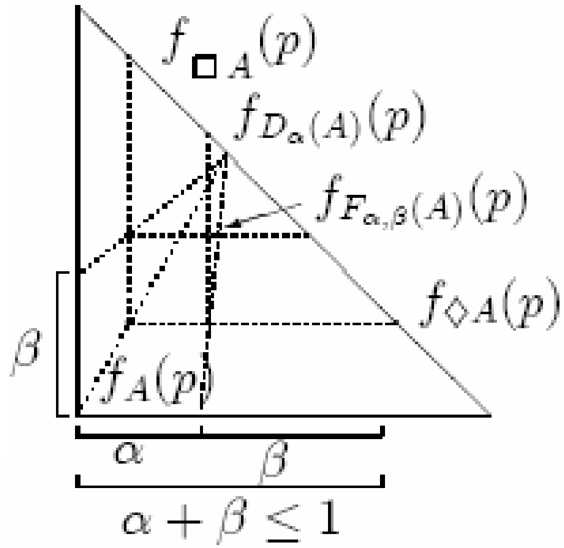
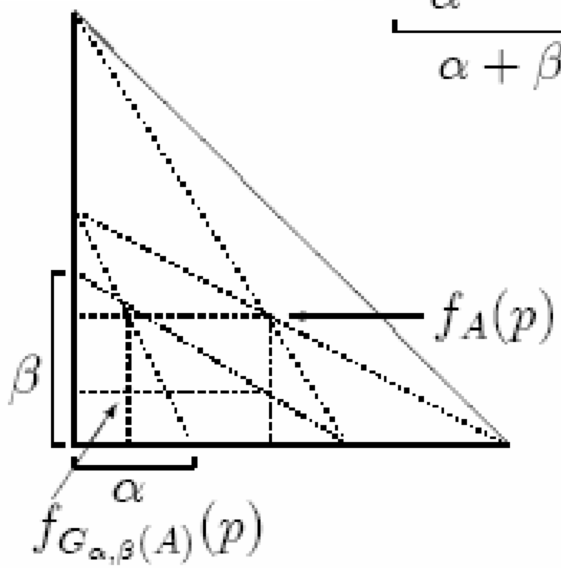
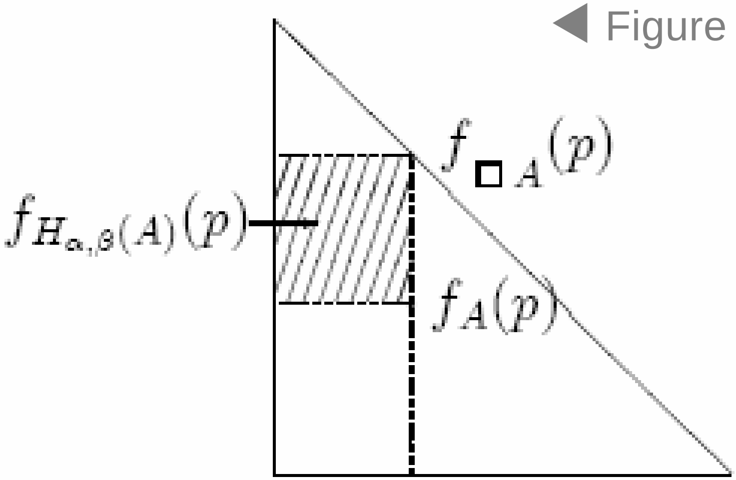


Figure 5 ▲

Figure 6 ▶▶

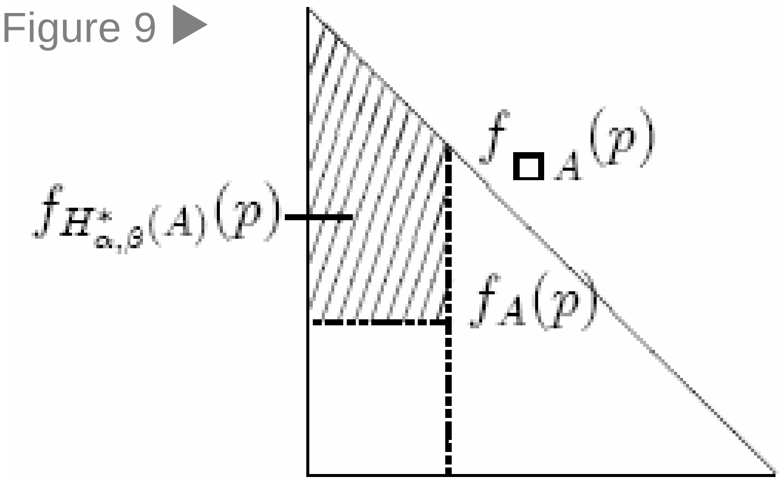
Figure 7 ▶

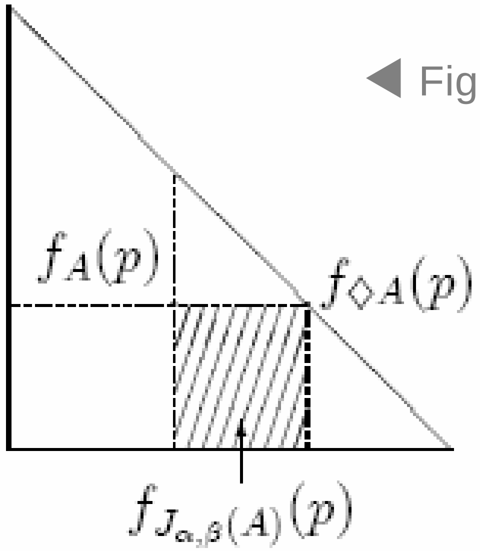




◀ Figure 8

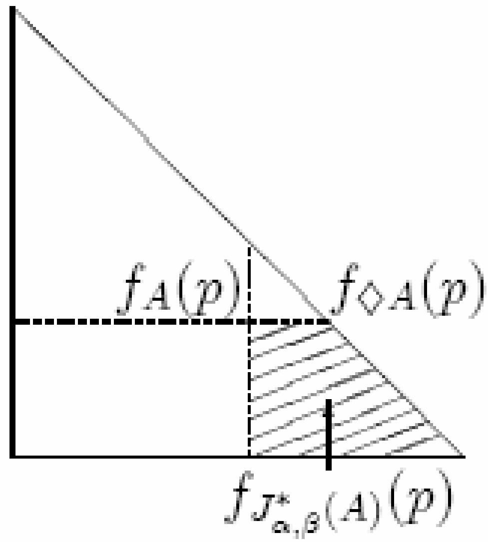
Figure 9 ▶





◀ Figure 10

Figure 11 ▶



These operators are extended to the operators:

$$F_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle \mid x \in E\},$$

$$G_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_B(x).\nu_A(x)\rangle \mid x \in E\},$$

$$H_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle \mid x \in E\},$$

$$H^*_B(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_B(x).(1 - \mu_B(x).\mu_A(x) - \nu_A(x))\rangle \mid x \in E\},$$

$$J_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_B(x).\nu_A(x)\rangle \mid x \in E\},$$

$$J^*_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).(1 - \mu_A(x) - \nu_B(x).\nu_A(x)), \nu_B(x).\nu_A(x)\rangle \mid x \in E\},$$

where B is a given IFS; and modified to two other groups of operators.

R_9

Four pairs of modal-like operators were introduced several years ago. The simplest of them are:

$$\boxplus A = \{\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\rangle | x \in E\},$$

$$\boxtimes A = \{\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\rangle | x \in E\}.$$

Let $\alpha \in [0, 1]$ and let A be an IFS. Then we can define the first extension:

$$\boxplus_\alpha A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + 1 - \alpha\rangle | x \in E\},$$

$$\boxtimes_\alpha A = \{\langle x, \alpha.\mu_A(x) + 1 - \alpha, \alpha.\nu_A(x)\rangle | x \in E\}.$$

The second extension of these operators was introduced by Katerina Dencheva. She extended the last two operators to the forms:

$$\boxplus_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x), \alpha.\nu_A(x) + \beta\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha.\mu_A(x) + \beta, \alpha.\nu_A(x)\rangle | x \in E\},$$

where α, β, $\alpha + \beta \in [0, 1]$.

The third extension of the above operators has the form:

$$\boxplus_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x) + \gamma\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta,\gamma} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x)\rangle | x \in E\},$$

where α, β, $\gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \leq 1$.

Gökhan Cuvalcioglu introduced operator $E_{a,ß}$ by:

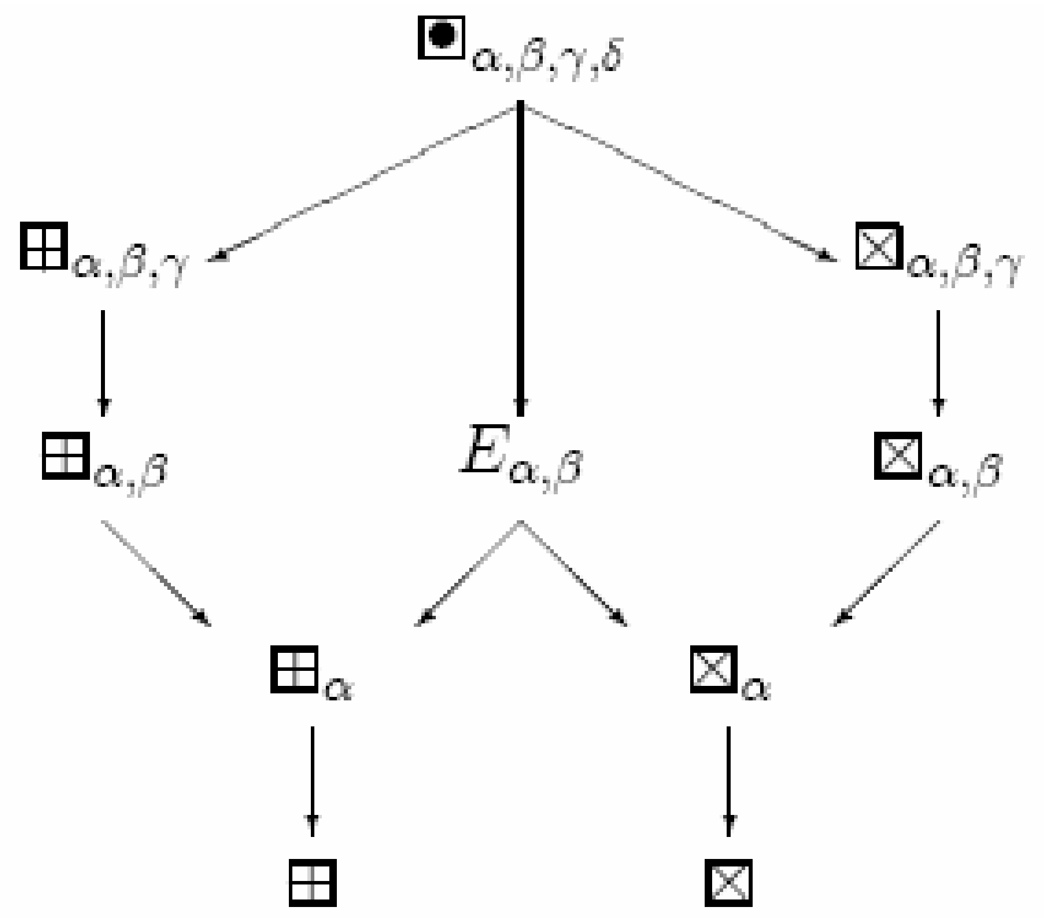
$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha.\mu_A(x) + 1 - \alpha), \alpha(\beta.\nu_A(x) + 1 - \beta)\rangle | x \in E\},$$

where $\alpha, \beta \in [0, 1]$
and studied some of its properties.

A natural extension of the three later operators is the operator

$$\boxdot_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha.\mu_A(x) + \gamma, \beta.\nu_A(x) + \delta\rangle | x \in E\},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\max(\alpha, \beta) + \gamma + \delta \leq 1$.


$\boxdot_{\alpha,\beta,\gamma,\delta}$
$\boxplus_{\alpha,\beta,\gamma}$
$\boxtimes_{\alpha,\beta,\gamma}$
$\boxplus_{\alpha,\beta}$
$E_{\alpha,\beta}$
$\boxtimes_{\alpha,\beta}$
$\boxplus_{\alpha}$
$\boxtimes_{\alpha}$
$\boxplus$
$\boxtimes$

R_{10}

Two analogues of the topological operators were defined over the IFSs, too: operator *“closure"* **C** and operator *“interior“* **I** :

$$C(A) = \{\langle x, \sup_{y\in E} \mu_A(y), \inf_{y\in E} \nu_A(y)\rangle | x \in E\},$$

$$I(A) = \{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}.$$

It is very interesting to note that the IFS-interpretations of both operators coincide, respectively, with the IFS-interpretations of the logic quantifiers $ and " .

M_7

Already ten years I cannot finish and prepare in complete form my research in the area of intuitionistic fuzzy logics.

I believe that the systematic research in this direction will be very important, so that intuitionistic fuzzy logics takes its place not only in the frames of the fuzzy set theory, but also in the frames of intuitionistic mathematics.

R_{11}

During the recent years, many extensions of the concept of intuitionistic fuzzy sets have been developed:

- Interval-valued IFS *(George Gargov and me)*
- Intuitionistic L-fuzzy sets *(Stefka Stoeva and me)*
- IFS over different universes *(me)*
- IFS of type 2 and type *n* *(Parvathi Rangasamy, Peter Vassilev and me)*
- Temporal IFS *(me)*
- Intuitionistic fuzzy multidimensional sets *(Eulalia Szmidt, Janusz Kacprzyk and me)*

Thank you for your attention!