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INTERVAL VALUED INTUITIONISTIC FUZZY SETS

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Abstract: A generalization of the notion of intuitionistic fuzzy set is given in the spirit of ordinary interval valued fuzzy sets. The new notion is called interval valued intuitionistic fuzzy set (IVIFS). Here we present the basic preliminaries of IVIFS theory.

Keywords: Fuzzy set; interval valued fuzzy set; interval valued intuitionistic fuzzy set; intuitionistic fuzzy set; modal operator.

1. Introduction

An intuitionistic fuzzy set (IFS) A, for a given underlying set E is represented by a pair $\langle \mu_A, \nu_A \rangle$ of functions $E \to [0, 1]$. For $x \in E$, $\mu_A(x)$ gives the degree of membership to A, $\nu_A(x)$ gives the degree of non-membership. This interpretation entails the natural restriction

$$\mu_A(x) + \nu_A(x) \leq 1.$$

Clearly ordinary fuzzy sets (FS) over E may be viewed as special cases of IFS's – here the degree of membership is the only necessary data, i.e. ordinary fuzzy sets are to be considered as IFS's with the additional condition

$$\nu_A(x) = 1 - \mu_A(x).$$

The theory of IFS's is developed in [1]. In the present paper we begin the investigation of a generalization of this notion – the interval valued IFS's (IVIFS's), but we first consider the relationship between IFS's and another generalization of FS's – the interval valued FS's (IVFS's, cf. e.g. [2]).

2. IVFS and IFS

A generalization of the notion of fuzzy set, proposed by some researchers in the area in the seventies (cf. e.g. [2]), is the so-called interval valued fuzzy set. Here we give a definition and establish that in a sense interval valued fuzzy sets are a version of the IFS (or if you like – the other way around!).

Definition 1. An interval valued fuzzy set A (over a basic set E) is given by a function $M_A(x)$, where $M_A : E \to \mathrm{INT}([0, 1])$, the set of all subintervals of the unit interval, i.e. for every $x \in E$, $M_A(x)$ is an interval within $[0, 1]$.

The justification of this generalization lies in the following observation: sometimes it is not appropriate to assume that the degrees of membership for certain elements of E are exactly defined, so we admit a kind of further uncertainty – the value of M_A is not a number anymore, but a whole interval. Let us denote such objects by IVFS.

Definition 2. (a) The map f assigns to every IVFS A an IFS

$$B = f(A)$$

given by

$$\mu_B(x) = \inf M_A(x), \qquad \nu_B(x) = \sup M_A(x).$$

(b) The map g assigns to every IFS B an IVFS

$$A = g(B)$$

given by

$$M_A(x) = [\mu_B(x), 1 - \nu_B(x)].$$

The relationship between the two generalizations mentioned above – the IFS and the IVFS – is given in the following lemma.

Lemma 1. (a) *For every IVFS A, $g(f(A)) = A$.*
(b) *For every IFS B, $f(g(B)) = B$.*

Proof. (a) Let A be an IVFS. Then for every $x \in E$,

$$\begin{aligned} M_{g(f(A))}(x) &= [\mu_{f(A)}(x), 1 - \nu_{f(A)}(x)] \\ &= [\inf M_A(x), 1 - 1 + \sup M_A(x)] \\ &= M_A(x), \end{aligned}$$

since $M_A(x)$ is an interval.
(b) Let B be an IFS. Then for every $x \in E$,

$$\begin{aligned} \mu_{f(g(B))}(x) &= \inf M_{g(B)}(x) \\ &= \inf[\mu_B(x), 1 - \nu_B(x)] \\ &= \mu_B(x), \\ \nu_{f(g(B))}(x) &= 1 - \sup M_{g(B)}(x) \\ &= 1 - \sup[\mu_B(x), 1 - \nu_B(x)] \\ &= \nu_B(x). \end{aligned}$$

This shows IFS and IVFS to be equipollent generalizations of the notion of FS. But the definition of IFS allows a further generalization – to be considered in the next section.

3. The basic theory of IVIFS

Let a set E be fixed. An interval valued intuitionistic fuzzy sets (IVIFS) A over E is an object having the form

$$A = \{\langle x, M_A(x), N_A(x)\rangle : x \in E\},$$

where $M_A(x) \subset [0, 1]$ and $N_A(x) \subset [0, 1]$ are intervals and for every $x \in E$,

$$\sup M_A(x) + \sup N_A(x) \leqslant 1.$$

For every two IVIFS's A and B the following relations, operations and operators are valid (by analogy from [1]):

$$A \subset B \text{ iff } (\forall x \in E)(\sup M_A(x) \leqslant \sup M_B(x) \;\&\; \inf M_A(x) \leqslant \inf M_B(x)$$
$$\& \sup N_A(x) \geqslant \sup N_B(x) \;\&\; \inf N_A(x) \geqslant \inf N_B(x));$$
$$A = B \text{ iff } A \subset B \;\&\; B \subset A;$$
$$\bar{A} = \{\langle x, N_A(x), M_A(x)\rangle : x \in E\};$$
$$A \cap B = \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))],$$
$$[\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))]\rangle : x \in E\};$$
$$A \cup B = \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))],$$
$$[\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle : x \in E\};$$
$$\Box A = \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle : x \in E\};$$
$$\Diamond A = \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle : x \in E\}.$$

Theorem 1. *For every IVIFS A*:
(a) $(\Box \bar{A})^- = \Diamond A$;
(b) $(\Diamond \bar{A})^- = \Box A$;
(c) $\Box A \subset A \subset \Diamond A$;
(d) $\Box\Box A = \Box A$;
(e) $\Box\Diamond A = \Diamond A$;
(f) $\Diamond\Box A = \Box A$;
(g) $\Diamond\Diamond A = \Diamond A$.

Proof. (a):

$$\begin{aligned}(\Box \bar{A})^- &= (\Box\{\langle x, N_A(x), M_A(x)\rangle ; x \in E\})^- \\ &= \{\langle x, N_A(x), [\inf M_A(x), 1 - \sup N_A(x)]\rangle ; x \in E\}^- \\ &= \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle ; x \in E\} \\ &= \Diamond A;\end{aligned}$$

(b): is proved analogically;

(c): From $\sup N_A(x) \leqslant 1 - \sup M_A(x)$ it follows that $\Box A \subset A$. The other inclusion follows analogically:

(d):

$$\begin{aligned}\Box\Box A &= \Box\{\langle x, M_A(x), [\inf N_A(x), 1-\sup M_A(x)]\rangle : x \in E\}\\ &= \{\langle x, M_A(x), [\inf[\inf N_A(x), 1-\sup M_A(x)], 1-\sup M_A(x)]\rangle : x \in E\}\\ &= \{\langle x, M_A(x), [\inf N_A(x), 1-\sup M_A(x)]\rangle : x \in E\}\\ &= \Box A;\end{aligned}$$

(e)–(g) are proved analogically.

Theorem 2. *For every two IVIFS's A and B:*
(a) $(\bar{A} \cap \bar{B})^- = A \cup B$;
(b) $(\bar{A} \cup \bar{B})^- = A \cap B$.

Proof. (a):

$$\begin{aligned}(\bar{A} \cap \bar{B})^- &= \{\langle x, N_A(x), M_A(x)\rangle : x \in E\} \cap \{\langle x, N_B(x), M_B(x)\rangle : x \in E\}^-\\ &= \{\langle x, [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))],\\ &\qquad [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))]; x \in E\}^-\\ &= \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))],\\ &\qquad [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle : x \in E\}\\ &= A \cup B;\end{aligned}$$

(b) is proved analogically.

Theorem 3. *For every two IVIFS's A and B:*
(a) $\Box(A \cup B) = \Box A \cup \Box B$;
(b) $\Box(A \cap B) = \Box A \cap \Box B$;
(c) $\Diamond(A \cup B) = \Diamond A \cup \Diamond B$;
(d) $\Diamond(A \cap B) = \Diamond A \cap \Diamond B$.

Proof. (a):

$$\begin{aligned}\Box(A \cup B) &= \Box\{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))],\\ &\qquad [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle : x \in E\}\\ &= \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))],\\ &\qquad [\inf[\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))],\\ &\qquad 1-\sup[\max(\inf M_A(x), \inf M_B(x)),\\ &\qquad\qquad \max(\sup M_A(x), \sup M_B(x))]]\rangle : x \in E\}\\ &= \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))],\\ &\qquad [\min(\inf N_A(x), \inf N_B(x)), 1-\max(\sup M_A(x), \sup M_B(x))]\rangle : x \in E\}\\ &= \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))],\\ &\qquad [\min(\inf N_A(x), \inf N_B(x)),\\ &\qquad\qquad \min(1-\sup M_A(x), 1-\sup M_B(x))]\rangle : x \in E\}\end{aligned}$$

$$= \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle : x \in E\}$$
$$\cup\ \{\langle x, M_B(x), [\inf N_B(x), 1 - \sup M_B(x)]\rangle : x \in E\}$$
$$= \Box A \cup \Box B;$$

(b) is proved analogically;
(d): From (a) and from Theorems 1(a, b) and 2 follows that:

$$\Diamond(A \cap B) = (\Box(A \cap B)^-)^- = (\Box(\bar{A} \cup \bar{B}))^-$$
$$= (\Box\bar{A} \cup \Box\bar{B})^- = (\Box\bar{A})^- \cap (\Box\bar{B})^- = \Diamond A \cap \Diamond B;$$

(c) is proved analogically.

An operator which associates to every IVIFS and IFS can be defined. Let A be an IVIFS. Then we set

$$*A = \{\langle x, \inf M_A(x), \inf N_A(x)\rangle : x \in E\}.$$

Theorem 4. *For every IVIFS A*:
(a) $*\Box A = *A$;
(b) $*\Diamond A = *A$;
(c) $*\bar{A} = \overline{*A}$.

Proof. (a):

$$*\Box A = *[\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle : x \in E\}$$
$$= \{\langle x, \inf M_A(x), \inf N_A(x)\rangle : x \in E\}$$
$$= *A;$$

(b) is proved analogically;
(c):

$$*\bar{A} = *\{\langle x, N_A(x), M_A(x)\rangle : x \in E\}$$
$$= \{\langle x, \inf N_A(x), \inf M_A(x)\rangle : x \in E\}$$
$$= \{\langle x, \inf M_A(x), \inf N_A(x)\rangle : x \in E\}^-$$
$$= \overline{*A}.$$

Theorem 5. *For every two IVIFS's A and B*:
(a) $*(A \cup B) = *A \cup *B$;
(b) $*(A \cap B) = *A \cap *B$.

Proof. (a):

$$*(A \cup B) = *\{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x), \sup M_B(x))],$$
$$[\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle : x \in E\}$$
$$= \{\langle x, \max(\inf M_A(x), \inf M_B(x)), \min(\inf N_A(x), \inf N_B(x))\rangle : x \in E\}$$
$$= \{\langle x, \inf M_A(x), \inf N_A(x)\rangle : x \in E\} \cup \{\langle x, \inf M_B(x)), \inf N_B(x)\rangle : x \in E\}$$
$$= *A \cup *B.$$

Definition 3. Let A be an IFS. We shall define:

$$\begin{aligned}\#(A) = \{B\colon B = \{\langle x, M_B(x), N_B(x)\rangle : x \in E\}\\ \& (\forall x \in E)(\sup M_B(x) + \sup N_B(x) \leq 1)\\ \& (\forall x \in E)(\inf M_B(x) \geq \mu_A(x) \;\&\; \sup N_B(x) \leq \nu_A(x))\},\end{aligned}$$

$$\begin{aligned}\nabla(A) = \{B\colon B = \{\langle x, M_B(x), N_B(x)\rangle : x \in E\}\\ \& (\forall x \in E)(\sup M_B(x) + \sup N_B(x) \leq 1)\\ \& (\forall x \in E)(\sup M_B(x) \leq \mu_A(x) \;\&\; \inf N_B(x) \geq \nu_A(x))\}\end{aligned}$$

Theorem 6. *For every IFS A:*
(a) $\#(A) = \{B\colon A \subset *B\}$;
(b) $\nabla(A) = \{B\colon *B \subset A\}$.

Proof. (a) Let $C \in \#(A)$. Then

$$C = \{\langle x, M_C(x), N_C(x)\rangle : x \in E\}$$

and

$$\begin{aligned}(\forall x \in E)(\sup M_C(x) + \sup N_C(x) \leq 1)\\ \& (\forall x \in E)(\inf M_C(x) \geq \mu_A(x) \;\&\; \sup N_C(x) \leq \nu_A(x))\}.\end{aligned}$$

Then C is an IVIFS and

$$*C = \{\langle x, \inf M_C(x), \sup N_C(x)\rangle : x \in E\}$$

and $A \subset *C$, i.e. $C \in \{B\colon A \subset *B\}$.
In the opposite direction we argue analogically.
(b) is proved in the same manner.

Definition 4. A non-empty set X of subsets of a certain set is called a filter if it has the following properties:
(1) if $a \in X$ and $a \subset b$, then $b \in X$;
(2) $a, b \in X$ implies $a \cap b \in X$.

Definition 5. A non-empty set X of subsets of a certain set is called an ideal if it has the following properties:
(1) if $a \in X$ and $b \subset a$, then $b \in X$;
(2) $a, b \in X$ implies $a \cup b \in X$.

Theorem 7. *For every IFS A:*
(a) *$\#(A)$ is a filter,*
(b) *$\nabla(A)$ is an ideal.*

Proof. (a) Obviously $A \in \#(A)$, i.e. $\#(A)$ is non-empty. Let $C \in \#(A)$ and

$C \subset D$. Then

$$(\forall x \in E)(\inf M_C(x) \leqslant \inf M_D(x) \ \& \ \sup M_C(x) \leqslant \sup M_D(x)$$
$$\& \ \inf N_C(x) \geqslant \inf N_D(x) \ \& \ \sup N_C(x) \geqslant \sup N_D(x)),$$

i.e.

$$(\forall x \in E)(\mu_A(x) \leqslant \inf M_D(x) \ \& \ \sup N_D(x) \geqslant \nu_A(x)).$$

Hence $D \in \#(A)$.

Let $C, D \in \#(A)$. Then

$$C \cap D = \{\langle x, [\min(\inf M_C(x), \inf M_D(x)), \min(\sup M_C(x), \sup M_D(x))], [\max(\inf N_C(x), \inf N_D(x)), \max(\sup N_C(x), \sup N_D(x))]\rangle : x \in E\},$$

from which

$$*(C \cap D) = \{\langle x, \min(\inf M_C(x), \inf M_D(x)), \max(\inf N_C(x), \inf N_D(x))\rangle : x \in E\}.$$

From

$$\mu_A(x) \leqslant \min(\inf M_C(x), \inf M_D(x)), \qquad \nu_A(x) \geqslant \max(\sup N_C(x), \sup N_D(x))$$

it follows that $A \subset *(C \cap D)$, i.e. $C \cap D \in \#(A)$. Therefore $\#(A)$ is a filter.

(b) is proved analogically.

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