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Intuitionistic fuzzy implications revisited. Part 2

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Abstract: New conditions for correctness of the intuitionistic fuzzy implications are formulated and they are checked for the separate implications.
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1 Introduction

The paper is a continuation of [3, 12], in which was mentioned that there have been 191 different intuitionistic fuzzy implications (see [5–7, 9–11, 13]).

In the present research, we will formulate new conditions for correctness of the intuitionistic fuzzy implications and will check the properties of the implications $\rightarrow_{186}, ..., \rightarrow_{191}$ that are checked for the first 185 implications [5–7, 9–11, 13]. Definitions of these 6 implications are given below. In [15] it was shown that implications $\rightarrow_{40}$ and $\rightarrow_{173}$ coincide.

2 Main results

Let $A = \langle a, b\rangle$ and $B = \langle c, d\rangle$, where $a, b, a + b, c, d, c + d \in [0, 1]$, be intuitionistic fuzzy pairs (cf. [8]). The definitions of the 6 implications are:

$$\begin{aligned}
A \rightarrow_{186} B &= \langle \overline{\mathrm{sg}}(d-b) + \mathrm{sg}(d-b)\max(b,c), \mathrm{sg}(d-b)\min(a,d)\rangle,\\
A \rightarrow_{187} B &= \langle \max(b,c), ad\rangle,\\
A \rightarrow_{188} B &= \langle \min(b,c), ad\rangle,\\
A \rightarrow_{189} B &= \langle bc, ad\rangle,\\
A \rightarrow_{190} B &= \langle \frac{\overline{\mathrm{sg}}(a-c) + \overline{\mathrm{sg}}(d-b)}{2}, \frac{\mathrm{sg}(a-c) + \mathrm{sg}(d-b)}{2}\rangle,\\
A \rightarrow_{191} B &= \langle \overline{\mathrm{sg}}(a-c)\,\overline{\mathrm{sg}}(d-b), \mathrm{sg}(a-c)\,\mathrm{sg}(d-b)\rangle,
\end{aligned}$$

where for each real number x:

$$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \quad \overline{\mathrm{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

These implications generate the following negations:

In the first part of our research, we will check which implications satisfy the following two well-known formulas

$$((A \rightarrow \neg A) \rightarrow A) \rightarrow A, \tag{1}$$

$$(\neg A \rightarrow A) \rightarrow A, \tag{2}$$

that in first order logic (see, e.g., [14]) are tautologies.

Theorem 1. Implication $\rightarrow_i$ satisfies (1) as a tautology for i = 20, 23, 42, 74, 77, 88, 90.

Proof. We will check Theorem 1 for $i = 20$. The rest checks are analogous. Implication $\rightarrow_{20}$ and the generated by it negation $\neg_2$ have the form: $A \rightarrow_{20} B = \langle \max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c)), \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(c))\rangle$, where $B = \langle c, d\rangle$ and $\neg_2 A = \langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle$. Let

$$\begin{aligned}
Z &\equiv ((A \rightarrow_{20} \neg_2 A) \rightarrow_{20} A) \rightarrow_{20} A\\
&= (((\langle a,b\rangle \rightarrow_{20} \neg_2\langle a,b\rangle) \rightarrow_{20} \langle a,b\rangle) \rightarrow_{20} \langle a,b\rangle\\
&= (((\langle a,b\rangle \rightarrow_{20} \neg_2\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle) \rightarrow_{20} \langle a,b\rangle) \rightarrow_{20} \langle a,b\rangle\\
&= (((\langle \max(\overline{\mathrm{sg}}(a), \mathrm{sg}(\overline{\mathrm{sg}}(a))), \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)))\rangle) \rightarrow_{20} \langle a,b\rangle) \rightarrow_{20} \langle a,b\rangle\\
&\quad \text{(because it is checked directly that } \mathrm{sg}(\overline{\mathrm{sg}}(a)) = \overline{\mathrm{sg}}(a) \text{ and } \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)) = \mathrm{sg}(a))\\
&= ((\langle \max(\overline{\mathrm{sg}}(a), \overline{\mathrm{sg}}(a)), \min(\mathrm{sg}(a), \mathrm{sg}(a))\rangle \rightarrow_{20} \langle a,b\rangle) \rightarrow_{20} \langle a,b\rangle\\
&= ((\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle \rightarrow_{20} \langle a,b\rangle) \rightarrow_{20} \langle a,b\rangle\\
&= \langle \max(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)), \mathrm{sg}(a)), \min(\mathrm{sg}(\mathrm{sg}(a)), \overline{\mathrm{sg}}(a))\rangle \rightarrow_{20} \langle a,b\rangle\\
&\quad \text{(because it is checked directly that } \mathrm{sg}(\mathrm{sg}(a)) = \mathrm{sg}(a))\\
&= \langle \max(\mathrm{sg}(a), \mathrm{sg}(a)), \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(a))\rangle \rightarrow_{20} \langle a,b\rangle\\
&= \langle \mathrm{sg}(a), \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(a))\rangle \rightarrow_{20} \langle a,b\rangle
\end{aligned}$$

$= \langle \max(\overline{\text{sg}}(\text{sg}(a)), \text{sg}(a)), \min(\text{sg}(\text{sg}(a)), \overline{\text{sg}}(a))\rangle$

(because it is checked directly that $\overline{\text{sg}}(\text{sg}(a)) = \overline{\text{sg}}(a)$)

$= \langle \max(\overline{\text{sg}}(a), \text{sg}(a)), \min(\text{sg}(a), \overline{\text{sg}}(a))\rangle$

$= \langle 1, 0\rangle$,

(because, if $a = 0$, then $\text{sg}(a) = 0$ and $\overline{\text{sg}}(a) = 1$, while, if $a = 1$, then $\text{sg}(a) = 1$ and $\overline{\text{sg}}(a) = 0$.)

Now, we will give an example in which, e.g., implication $A \rightarrow_{35} B = \langle 1 - ad, ad\rangle$ and the generated by it negation $\neg_8 = \langle 1 - a, a\rangle$ do not satisfy (1). Let

$$\begin{aligned} Z &\equiv ((A \rightarrow_{35} \neg_8 A) \rightarrow_{35} A) \rightarrow_{35} A \\ &= (((\langle a, b\rangle \rightarrow_{35} \neg_8\langle a, b\rangle) \rightarrow_{35} \langle a, b\rangle) \rightarrow_{35} \langle a, b\rangle \\ &= (((\langle a, b\rangle \rightarrow_{35} \langle 1 - a, a\rangle) \rightarrow_{35} \langle a, b\rangle) \rightarrow_{35} \langle a, b\rangle \\ &= ((\langle 1 - a^2, a^2\rangle \rightarrow_{35} \langle a, b\rangle) \rightarrow_{35} \langle a, b\rangle \\ &= \langle 1 - (1 - a^2)b, (1 - a^2)b\rangle \rightarrow_{35} \langle a, b\rangle \\ &= \langle 1 - (1 - (1 - a^2)b)b, (1 - (1 - a^2)b)b\rangle. \end{aligned}$$

Obviously, e.g., for $a = b = 0.5$ we obtain that $Z = \langle 1 - (1 - (1 - a^2)b)b, (1 - (1 - a^2)b)b\rangle = \langle 0.6875, 0, 3125\rangle$, which is not a tautology.

This completes the proof. □

Theorem 2. Implication $\rightarrow_i$ satisfies (1) as an intuitionistic fuzzy tautology for i = 1, 4, 5, 6, 7, 9, 13, 17, 18, 20, 21, 22, 23, 25, 27, 28, 29, 30, 33, 34, 35, 36, 38, 42, 44, 45, 61, 64, 66, 71, 72, 74, 75, 76, 77, 79, 80, 81, 82, 85, 88, 89, 90, 100, 101, 102, 103, 104, 105, 109, 110, 111, 112, 113, 114, 115, 116, 117, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 166, 167, 168, 169, 170, 181, 182, 183, 185, 186, 187.

Proof. We will check Theorem 2 for $i = 18$. The rest checks are analogous. Implication $\rightarrow_{18}$ and the generated by it negation $\neg_4$ have the form: $A \rightarrow_{18} B = \langle \max(b, c), \min(1 - b, d)\rangle$, where B has the above form, and $\neg_4 A = \langle b, 1 - b\rangle$.

Let

$$\begin{aligned} Z &\equiv ((A \rightarrow_{18} \neg_4 A) \rightarrow_{18} A) \rightarrow_{18} A \\ &= (((\langle a, b\rangle \rightarrow_{18} \neg_4\langle a, b\rangle) \rightarrow_{18} \langle a, b\rangle) \rightarrow_{18} \langle a, b\rangle \\ &= (((\langle a, b\rangle \rightarrow_{18} \neg_4\langle b, 1 - b\rangle) \rightarrow_{18} \langle a, b\rangle) \rightarrow_{18} \langle a, b\rangle \\ &= ((\langle \max(b, b), \min(1 - b, 1 - b)\rangle \rightarrow_{18} \langle a, b\rangle) \rightarrow_{18} \langle a, b\rangle \\ &= ((\langle b, 1 - b\rangle \rightarrow_{18} \langle a, b\rangle) \rightarrow_{18} \langle a, b\rangle \\ &= \langle \max(1 - b, a), \min(1 - (1 - b), b)\rangle \rightarrow_{18} \langle a, b\rangle \qquad \text{(because } a \leq 1 - b\text{)} \\ &= \langle 1 - b, b\rangle \rightarrow_{18} \langle a, b\rangle \\ &= \langle \max(a, b), \min(1 - b, b)\rangle. \end{aligned}$$

From $\max(a, b) \geq b \geq \min(1 - b, b)$ it follows that (1) is an IFT for implication $\rightarrow_{18}$.

Now, we will give an example in which, e.g., implication

$$A \to_{12} B = \langle \max(b,c), 1 - \max(b,c) \rangle$$

and the generated by it negation $\neg_4$ do not satisfy (1). Let

$$\begin{aligned}
Z &\equiv ((A \to_{12} \neg_4 A) \to_{12} A) \to_{12} A \\
&= (((\langle a,b \rangle \to_{12} \neg_4 \langle a,b \rangle) \to_{12} \langle a,b \rangle) \to_{12} \langle a,b \rangle \\
&= (((\langle a,b \rangle \to_{12} \langle b, 1-b \rangle) \to_{12} \langle a,b \rangle) \to_{12} \langle a,b \rangle \\
&= ((\langle \max(b,b), 1 - \max(b,b) \rangle \to_{12} \langle a,b \rangle) \to_{12} \langle a,b \rangle \\
&= ((\langle b, 1-b \rangle \to_{12} \langle a,b \rangle) \to_{12} \langle a,b \rangle \\
&= \langle \max(1-b,a), 1 - \max(1-b,a) \rangle \to_{12} \langle a,b \rangle \\
&= \langle 1-b, 1-(1-b) \rangle \to_{12} \langle a,b \rangle \\
&= \langle 1-b, b \rangle \to_{12} \langle a,b \rangle \\
&= \langle \max(a,b), 1 - \max(a,b) \rangle.
\end{aligned}$$

Obviously, for $a = b = 0$, $Z = \langle 0, 1 \rangle$ that is not an IFT. □

Theorem 3. Implication $\to_i$ satisfies (2) as a tautology for $i = 20, 23, 42, 74, 77, 88$.

Proof. We will check Theorem 3 again for $i = 20$. The rest checks are analogous. Let

$$\begin{aligned}
Z &\equiv (\neg_2 A \to_{20} A) \to_{20} A \\
&= (\neg_2 \langle a,b \rangle \to_{20} \langle a,b \rangle) \to_{20} \langle a,b \rangle \\
&= ((\langle \overline{\text{sg}}(a), \text{sg}(a) \rangle \to_{20} \langle a,b \rangle) \to_{20} \langle a,b \rangle \\
&= \langle \max(\overline{\text{sg}}(\overline{\text{sg}}(a)), \text{sg}(a)), \min(\text{sg}(\overline{\text{sg}}(a)), \overline{\text{sg}}(a)) \rangle \to_{20} \langle a,b \rangle \\
&= \langle \max(\text{sg}(a)), \text{sg}(a)), \min(\overline{\text{sg}}(a), \overline{\text{sg}}(a)) \rangle \to_{20} \langle a,b \rangle \\
&= \langle \text{sg}(a), \overline{\text{sg}}(a) \rangle \to_{20} \langle a,b \rangle \\
&= \langle \max(\overline{\text{sg}}(\text{sg}(a)), \text{sg}(a)), \min(\text{sg}(\text{sg}(a)), \overline{\text{sg}}(a)) \rangle \\
&= \langle \max(\overline{\text{sg}}(a), \text{sg}(a)), \min(\text{sg}(a), \overline{\text{sg}}(a)) \rangle = \langle 1, 0 \rangle.
\end{aligned}$$

Now, we will give an example in which, e.g., implication $\to_{35}$ and the generated by it negation $\neg_8$ do not satisfy (2). Let

$$\begin{aligned}
Z &\equiv (\neg_8 A \to_{35} A) \to_{35} A \\
&= (\neg_8 \langle a,b \rangle \to_{35} \langle a,b \rangle) \to_{35} \langle a,b \rangle \\
&= ((\langle 1-a, a \rangle \to_{35} \langle a,b \rangle) \to_{35} \langle a,b \rangle \\
&= \langle 1-(1-a)b, (1-a)b \rangle \to_{35} \langle a,b \rangle \\
&= (\langle 1-(1-(1-a)b)b, (1-(1-a)b)b \rangle.
\end{aligned}$$

Obviously, for $a = b = 0.5$, $Z = \langle 0.625, 0.325 \rangle$ that is not a tautology.

This completes the proof. □

Theorem 4. Implication $\to_i$ satisfies (2) as an intuitionistic fuzzy tautology for $i = 1$, 4, 5, 6, 7, 9, 13, 17, 18, 20, 21, 22, 23, 25, 27, 28, 29, 30, 33, 34, 35, 36, 38, 42, 45, 61, 64, 66, 71, 72, 74, 75, 76, 77, 79, 80, 81, 82, 85, 88, 89, 100, 101, 102, 103, 104, 105, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 166, 167, 168, 169, 170, 181, 182, 183, 185, 186, 187.

Proof. We will check Theorem 4 for $i = 4$. The rest checks are analogous.

Implication $\to_4$ and the generated by it negation $\neg_1$ have the forms:

$$A \to_4 B = \langle \max(b, c), \min(a, d)\rangle, \qquad \neg_1 A = \langle b, a\rangle.$$

Let

$$\begin{aligned} Z &\equiv (\neg_1 A \to_4 A) \to_4 A \\ &= (\neg_1\langle a, b\rangle \to_4 \langle a, b\rangle) \to_4 \langle a, b\rangle \\ &= (\langle b, a\rangle \to_4 \langle a, b\rangle) \to_4 \langle a, b\rangle \\ &= \langle \max(a, a), \min(b, b)\rangle \to_4 \langle a, b\rangle \\ &= \langle a, b\rangle \to_4 \langle a, b\rangle \\ &= \langle \max(a, b), \min(a, b)\rangle, \end{aligned}$$

that, obviously, is an IFT.

Now, we will give an example in which, e.g., implication $\to_2$ and the generated by it negation $\neg_2$ do not satisfy (2). They have the forms, respectively, $A \to_2 B = \langle \overline{\text{sg}}(a - c), d\,\text{sg}(a - c)\rangle$, and $\neg_2 A = \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle$. Let

$$\begin{aligned} Z &\equiv (\neg_2 A \to_2 A) \to_2 A \\ &= (\neg_2\langle a, b\rangle \to_2 \langle a, b\rangle) \to_2 \langle a, b\rangle \\ &= (\langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \to_2 \langle a, b\rangle) \to_2 \langle a, b\rangle \\ &= \langle \overline{\text{sg}}(\overline{\text{sg}}(a) - a), b\,\text{sg}(\overline{\text{sg}}(a) - a)\rangle \to_2 \langle a, b\rangle) \to_2 \langle a, b\rangle. \end{aligned}$$

If $a = 0$, then

$$\overline{\text{sg}}(\overline{\text{sg}}(a) - a) = \overline{\text{sg}}(1 - 0) = \overline{\text{sg}}(1) = 0,$$
$$\text{sg}(\overline{\text{sg}}(a) - a) = \text{sg}(1 - 0) = \text{sg}(1) = 1.$$

If $a > 0$, then

$$\overline{\text{sg}}(\overline{\text{sg}}(a) - a) = \overline{\text{sg}}(0 - a) = 1,$$
$$\text{sg}(\overline{\text{sg}}(a) - a) = \text{sg}(0 - a) = 0,$$

i.e., $\text{sg}(\overline{\text{sg}}(a)) = 1 - \overline{\text{sg}}(a)$.

Therefore, $\overline{\text{sg}}(\overline{\text{sg}}(a) - a) = \text{sg}(a)$ and

$$Z = (\langle \text{sg}(a), b(1 - \overline{\text{sg}}(a))\rangle \to_2 \langle a, b\rangle) \to_2 \langle a, b\rangle.$$

$$Z = \langle \overline{\text{sg}}(\text{sg}(a) - a), b\,\text{sg}(\text{sg}(a) - a)\rangle \to_2 \langle a, b\rangle.$$

If $a = 0$, then

$$\overline{\text{sg}}(\text{sg}(a) - a) = \overline{\text{sg}}(0 - 0) = \overline{\text{sg}}(0) = 1,$$
$$\text{sg}(\text{sg}(a) - a) = \text{sg}(0 - 0) = \text{sg}(0) = 0,$$

and

$$\begin{aligned} Z &= \langle 1, 0 \rangle \to_2 \langle a, b \rangle \\ &= \langle \overline{\text{sg}}(1 - a), b \, \text{sg}(1 - a) \rangle \\ &= \langle \overline{\text{sg}}(1), b \, \text{sg}(1) \rangle = \langle 0, b \rangle. \end{aligned}$$

If $1 > a > 0$, then

$$\overline{\text{sg}}(\text{sg}(a) - a) = \overline{\text{sg}}(1 - a) = 0,$$

$$\text{sg}(\text{sg}(a) - a) = \text{sg}(1 - a) = 1,$$

and

$$\begin{aligned} Z &= \langle 0, b \rangle \to_2 \langle a, b \rangle \\ &= \langle \overline{\text{sg}}(0 - a), b \, \text{sg}(0 - c) \rangle = \langle 1, 0 \rangle. \end{aligned}$$

If $a = 1$, then

$$\overline{\text{sg}}(\text{sg}(a) - a) = \overline{\text{sg}}(1 - 1) = \overline{\text{sg}}(0) = 1,$$

$$\text{sg}(\text{sg}(a) - a) = \text{sg}(1 - 1) = \text{sg}(0) = 0,$$

and

$$Z = \langle 1, 0 \rangle \to_2 \langle a, b \rangle = \langle 0, b \rangle.$$

Therefore, for $a \in \{0, 1\}$ formula (2) is not an IFT.

This completes the proof. □

Now, following [2], we will check the validity of other formulas, discussed in [1,2,4] for the cases of implications $\to_i$ for $i = 186, ..., 191$. The proofs are similar to the above ones and we will omit them.

The axioms of Kolmogorov comprise:

(K1) $A \to (B \to A)$,

(K2) $(A \to (A \to B)) \to (A \to B))$,

(K3) $(A \to (B \to C)) \to (B \to (A \to C))$,

(K4) $(B \to C) \to ((A \to B) \to (A \to C))$,

(K5) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

Theorem 5. Implication $\to_i$ satisfies as a tautology the Kolmogorov axiom:

(K1) for $i = 185, 186$;

(K2) for none of $i = 185, 186, 187, 188, 189, 190$;

(K3) for $i = 185$;

(K4) for $i = 185, 191$;

(K5) for none of $i = 185, 186, 187, 188, 189, 190$.

Theorem 6. Implication $\rightarrow_i$ satisfies as an intuitionistic fuzzy tautology the Kolmogorov axiom:
(K1) for $i = 185, 186, 187$;
(K2) for $i = 185, 186, 187, 191$;
(K3) for $i = 186, 187, 188, 189$;
(K4) for $i = 186, 187, 188, 189, 191$;
(K5) for $i = 185, 187, 188, 189$.

The axioms of Łukasiewicz and Tarski are
(LT1) $A \rightarrow (B \rightarrow A)$,
(LT2) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$,
(LT3) $\neg A \rightarrow (\neg B \rightarrow (B \rightarrow A))$,
(LT4) $((A \rightarrow \neg A) \rightarrow A) \rightarrow A$.

Theorem 7. Implication $\rightarrow_i$ satisfies as a tautology the Łukasiewicz and Tarski axiom:
(LT1) for $i = 186$;
(LT2) for $i = 191$;
(LT3) for $i = 186, 190, 191$;
(LT4) for none of $i = 185, 186, 187, 188, 189, 190, 191$.

Theorem 8. Implication $\rightarrow_i$ satisfies as an intuitionistic fuzzy tautology the Łukasiewicz and Tarski axiom:
(LT1) for $i = 186, 187$;
(LT2) for $i = 186, 187, 188, 189, 191$;
(LT3) for $i = 186, 187, 188, 189, 190, 191$;
(LT4) for $i = 186, 187$.

We will leave the proofs of Theorems 5–8 to the interested reader.

3 Conclusion

In the third part of the present research, we will determine which intuitionistic fuzzy implications satisfy some other systems of axioms as tautologies or as IFTs. The implications that satisfy the most important axioms will be shown and other of their properties willl be discussed.

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References

[1] Angelova, N., & Atanassov, K. (2016). Intuitionistic Fuzzy Implications and Klir-Yuan's Axioms. *Novel Developments in Uncertainty Representation and Processing. Advances in Intuitionistic Fuzzy Sets and Generalized Nets. (Atanassov, K.T., Castillo, O., et al., Eds.)*, 401, 97–110.

[2] Atanassov, K. (2017). *Intuitionistic Fuzzy Logics.* Springer, Cham.

[3] Atanassov, K. (2019). On the new intuitionistic fuzzy implication $\rightarrow_{191}$. *Notes on Intuitionistic Fuzzy Sets*, 25 (4), 1–6.

[4] Atanassov, K., & Angelova, N. (2016). Properties of the intuitionistic fuzzy implications and negations. *Notes on Intuitionistic Fuzzy Sets*, 22 (3), 25–33.

[5] Atanassov, K., Angelova, N., Szmidt, E., & Kacprzyk, J. (2016). Properties of the intuitionistic fuzzy implication $\rightarrow_{186}$. *Notes on Intuitionistic Fuzzy Sets*, 22 (4), 6–12.

[6] Atanassov, K., Ribagin, S., Doukovska, L., & Atanassova, V. (2017). Intuitionistic fuzzy implication $\rightarrow_{190}$. *Notes on Intuitionistic Fuzzy Sets*, 23 (4), 79–83.

[7] Atanassov, K., Szmidt, E., & Angelova, N. (2017). Properties of the intuitionistic fuzzy implication $\rightarrow_{187}$. *Notes on Intuitionistic Fuzzy Sets*, 23 (3), 3–8.

[8] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013). On intuitionistic fuzzy pairs, *Notes on Intuitionistic Fuzzy Sets*, 19 (3), 1–13.

[9] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2016). New Fodor's type of intuitionistic fuzzy implication and negation. *Notes on Intuitionistic Fuzzy Sets*, 22 (3), 1–8.

[10] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2017). Intuitionistic fuzzy implication $\rightarrow_{188}$. Notes on Intuitionistic Fuzzy Sets, 23 (1), 6–13.

[11] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2017). Intuitionistic fuzzy implication $\rightarrow_{187}$. *Notes on Intuitionistic Fuzzy Sets*, 23 (2), 37–43.

[12] Atanassov, K., Szmidt, E., Kacprzyk, J., & Angelova, N. (2019). Intuitionistic fuzzy implications revisited. Part 1. *Notes on Intuitionistic Fuzzy Sets*, 25 (3), 71–78.

[13] Atanassova, L. (2017). Intuitionistic fuzzy implication $\rightarrow_{189}$. *Notes on Intuitionistic Fuzzy Sets*, 23 (1), 14–20.

[14] Mendelson, E. (1964). I*ntroduction to Mathematical Logic*, Princeton, NJ: D. Van Nostrand.

[15] Vassilev, P., Ribagin, S., & Kacprzyk, J. (2018). A remark on intuitionistic fuzzy implications. Notes on Intuitionistic Fuzzy Sets, 24 (2), 1–7.