Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2024, Volume 30, Number 4, 309-322
DOI: 10.7546/nifs.2024.30.4.309-322

Intuitionistic fuzzy Nakayama's Lemma

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Received: 14 October 2024 **Revised:** 8 November 2024
Accepted: 5 December 2024 **Online First:** 13 December 2024

Abstract: Drawing upon Atanassov's pioneering work on intuitionistic fuzzy sets [1], this paper presents the concept of intuitionistic fuzzy Nakayama's Lemma, offering a natural extension to Rajesh Kumar et al.'s fuzzy Nakayama's Lemma [8]. Additionally, we define the intuitionistic fuzzy Jacobson radical and explore the product of an intuitionistic fuzzy ideal and intuitionistic fuzzy submodule. Finally, we conclude by introducing the sum of two intuitionistic fuzzy submodules.

Keywords: Intuitionistic fuzzy ideal, Intuitionistic fuzzy submodule, Intuitionistic fuzzy Jacobson radical, Intuitionistic fuzzy Nakayama's Lemma.
2020 Mathematics Subject Classification: 03F55, 05C25, 13B02.

1 Introduction

Nakayama's Lemma is a fundamental theorem in commutative algebra with far-reaching applications in various branches of mathematics. Understanding this lemma is crucial for anyone delving into algebraic geometry, commutative algebra, or algebraic number theory. Named after the Japanese mathematician Tadasi Nakayama, this lemma provides powerful insights into the structure of certain algebraic objects, shedding light on their properties and behavior. In this work, we started by defining the product of an intuitionistic fuzzy ideal with an intuitionistic fuzzy submodule, and showing that it's an intuitionistic fuzzy submodule, this is this is what allowed us to extend the Nakayama's Lemma into intuitionistic fuzzy case. In addition, we define the sum of two intuitionistic fuzzy submodules. Additionally, we present a variant of the previous lemma.

2 Preliminary concepts

In this section, we provide an overview of a number of important concepts and theorems that will be cited in later sections.

We denote X the universe. As an extension of Zadeh's fuzzy set theory [10], we first provide the notion of intuitionistic fuzzy subset as defined by Atanassov.

Definition 1. [1,2]*The intuitionistic fuzzy sets (in shorts IFS) defined on X as objects having the form*

$$A = \{\langle a, \gamma(a), \zeta(a)\rangle : a \in X\},$$

where the functions $\gamma : X \to [0,1]$ and $\zeta : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $a \in X$ to the set A, respectively, and $0 \leq \gamma(a) + \zeta(a) \leq 1$ for all $a \in X$.

Definition 2. [3] *Let A be an intuitionistic fuzzy set of X. Then $(\alpha, \beta)-$cut of A is a crisp subset $A_{(\alpha,\beta)}$ of the **IFS** A is given by:*

$$A_{(\alpha,\beta)} = \{a : a \in X \textit{ such that } \gamma_A(a) \geq \alpha, \zeta_A(a) \leq \beta\},$$

where $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$

Definition 3. [6] *Let $\mathfrak{R}$ be a ring and $A = \{\langle a, \gamma(a), \zeta(a)\rangle : a \in \mathfrak{R}\}$ be an IFS of $\mathfrak{R}$. then A is called an intuitionistic fuzzy subring of $\mathfrak{R}$ if the following conditions are satisfied:*

i) $\gamma(a - b) \geq \gamma(a) \wedge \gamma(b)$, $\forall a, b \in \mathfrak{R}$.

ii) $\zeta(a - b) \leq \zeta(a) \vee \zeta(b)$, $\forall a, b \in \mathfrak{R}$.

iii) $\gamma(ab) \geq \gamma(a) \wedge \gamma(b)$, $\forall a, b \in \mathfrak{R}$.

iv) $\zeta(ab) \leq \zeta(a) \vee \zeta(b)$, $\forall a, b \in \mathfrak{R}$.

Definition 4. [6] *Let $\mathfrak{R}$ be a ring. Then an IFS $I = \{\langle a, \gamma_I(a), \zeta_I(a)\rangle : a \in \mathfrak{R}\}$ of $\mathfrak{R}$ is said to be an intuitionistic fuzzy ideal (in short IFI) of $\mathfrak{R}$ if:*

i) $\gamma_I(a-b) \geq \gamma_I(a) \wedge \gamma_I(b) \quad \forall a, b \in I,$

ii) $\zeta_I(a-b) \leq \zeta_I(a) \vee \zeta_I(b) \quad \forall a, b \in I,$

iii) $\gamma_I(ab) \geq \gamma_I(a) \vee \gamma_I(b) \quad \forall a, b \in \mathfrak{R},$

iv) $\zeta_I(ab) \leq \zeta_I(a) \wedge \zeta_I(b) \quad \forall a, b \in \mathfrak{R}.$

Definition 5. [7] *Let M be a module over a ring $\mathfrak{R}$. An IFS $A = \{\langle a, \gamma(a), \zeta(a)\rangle : a \in X\}$ in M is called an intuitionistic fuzzy submodule (IFSM) of M if:*

i) $\gamma_A(0) = 1$, $\zeta_A(0) = 0$

ii) $\gamma_A(a+b) \geq \gamma_A(a) \wedge \gamma_A(b)$, $\zeta_A(a+b) \leq \zeta_A(a) \vee \zeta_A(b)\ \forall a, b \in \mathrm{M},$

iii) $\gamma_A(ra) \geq \gamma_A(a)$, $\zeta_A(ra) \leq \zeta_A(a)\ \forall a \in \mathrm{M}, \forall r \in \mathfrak{R},$

3 Intuitionistic fuzzy maximal ideal and Jacobson radical

The primary aim of this section is to define the intuitionistic fuzzy maximal ideal and Jacobson radical. Let $I_* := \{x \in \mathfrak{R} | I(x) = I(0)\}$

Definition 6. *An intuitionistic fuzzy non-constant ideal $I = (\gamma_I, \zeta_I)$ of the ring $\mathfrak{R}$ is termed as **intuitionistic fuzzy maximal** if the following condition holds: For any $a \in \mathfrak{R}$ where $a \notin \mathrm{I}_{\mathrm{I}(0)}$, there exists $h \in \mathfrak{R}$, $\mathrm{I}(1-ha) = \mathrm{I}(0)$.*

Theorem 1. *Let I denote an IFI. Under this assumption, we can establish the following two properties:*

1. *I is an intuitionistic fuzzy maximal ideal if and only if I_* is maximal.*
2. *When I is an intuitionistic fuzzy maximal ideal, the value of $|\mathrm{Im}(I)|$ is equal to 2 .*

Proof. Let $(k, l) = I(0)$ and let $a \in \mathfrak{R} \setminus \mathrm{I}_{(k,l)}$. Then $\gamma_I(a) < k$ and $\zeta_I(a) > l$, it follows that $\mathrm{I}(b - ha) = (k, l)$ for some b and $h \in \mathfrak{R}$.

Hence $b - ha \in I_{(k,l)}$, which implies that $b \in I_{(k,l)} + \langle a\rangle$, this demonstrates that $I_{(k,l)} + \langle a\rangle = \mathfrak{R}$. Accordingly, $I_{(k,l)}$ is maximal.

Now, let $(\alpha, \beta) \in \mathrm{Im}(I)$, $\alpha < k$ and $\beta > l$. Then $I_{(k,l)} \subset I_{(\alpha,\beta)}$ so $I_{(\alpha,\beta)} = \mathfrak{R}$. Hence $|\mathrm{Im}(I)| = 2$.

Conversely, we have $|\mathrm{Im}(I)| = 2$, then I is non-constant; let $a \notin I_*$, then a in invertible. Thus, $h \in \mathfrak{R}$ exists such that $ah = 1$. Then $I(1 - ah) = I(0)$. Hence I is maximal. □

Definition 7. *The intuitionistic fuzzy Jacobson radical of $\mathfrak{R}$, denoted as $IFJR(\mathfrak{R})$, is defined as the intersection of all intuitionistic fuzzy maximal ideals of $\mathfrak{R}$, expressed as:*

$$IFJR(\mathfrak{R}) = \cap\{\theta \mid \theta \text{ is an intuitionistic fuzzy maximal ideal of } \mathfrak{R}\}.$$

Theorem 2. *Let $J = (\gamma_J, \zeta_J)$ denote a non-constant intuitionistic fuzzy ideal of the ring $\mathfrak{R}$ such that $J \subset IFJR(\mathfrak{R})$. Then, for any $a \in J_*$, we have $1 - a \in \mathfrak{R}^\times$, where $\mathfrak{R}^\times$ represents the set of units in the ring $\mathfrak{R}$.*

Proof. Let $J(c) = (k, l) = J(0)$.

Let us suppose that $1 - c$ is not a unit. So, $1 - c$ is contained within a maximal ideal n of $\mathfrak{R}$. Consider $\imath := (\gamma_\imath, \zeta_\imath)$ intuitionistic fuzzy ideal of $\mathfrak{R}$ defined by

$$\imath(z) = \begin{cases} (1, 0), & \text{if } z \in n \\ (t, s), & \text{if } z \in \mathfrak{R} \setminus n, \text{ where } t < k, \text{ and } l < s. \end{cases}$$

Then $\imath$ is maximal.

By hypothesis $k = \gamma_J(c),\ \ l = \zeta_J(c)$, it follows that $\gamma_\imath(c) > k$ and $\zeta_\imath(c) < l$. Consequently $c \in m$, This implies $1 = (1 - c) + c \in n$, which contradicts the maximality of n.

Therefore, $1 - c$ must be a unit. □

4 Intuitionistic fuzzy Nakayama's Lemma

Nakayama's Lemma 1. [5,9] *Let N be a finitely generated $\mathfrak{R}-$module and let A be an ideal of $\mathfrak{R}$ such that $\mathrm{A} \subseteq J(\mathfrak{R})$. If $\mathrm{A}N = N$, then $N = 0$.*

In the intuitionistic fuzzy set theory, the aforementioned Nakayama's Lemma is developed and proven in this section.

Proposition 3. *Let M be a finitely generated module and A be an intuitionistic fuzzy submodule of M, consequently there exists a set of generators $e_1, e_2, \ldots, e_n$ of M such that*

$$A(e_1) = A(e_2) = \cdots = A(e_n) = \Big(\wedge \{\gamma_A(z) \mid z \in \mathrm{M}\},\ \vee\{\zeta_A(z) \mid z \in \mathrm{M}\}\Big).$$

Proof. Let $e_1, \ldots, e_n$ be a set of generators.

For any element a in the module M, there exists $h_i \in \mathfrak{R}$, $a = \sum_{i=0}^{i=n} h_i e_i$.

Then

$$\gamma_A(a) = \gamma_A(\sum_{i=0}^{i=n} h_i e_i) \geq \min_{0 \leq i \leq n} \gamma_A(h_i e_i)$$
$$\geq \min_{0 \leq i \leq n} \gamma_A(e_i).$$

Without loss of generality, assume that $\min\{\gamma_A(e_i)/i = 1, 2, \ldots, n\} = \gamma_A(e_1)$.
Denote $m_1 = e_1$. If $\gamma_A(e_2) = \gamma_A(e_1)$, we put $m_2 = e_2$.
If $\gamma_A(e_2) > \gamma_A(e_1)$, in this case we take $a = (h_l - h_2)e_1 + h_2(e_1 + e_2) + h_3 e_3 + \cdots + h_n e_n$.
Hence $e_1, e_1 + e_2, e_3, \ldots, e_n$, is a set of generators of M.
Additionally we have

$$\gamma_A(e_1) = \gamma_A[(e_1 + e_2) + (-e_2)] \geq \min\{\gamma_A(e_1 + e_2), \gamma_A(-e_2)\}$$
$$\geq \min\{\gamma_A(e_1 + e_2), \gamma_A(e_2)\}$$

and $\gamma_A(e_1) < \gamma_A(e_2)$. It follows that $\gamma_A(e_1) = \gamma_A(e_1 + e_2)$.
Such being the case, we choose

$$m_2 = e_1 + e_2 \text{ with } \gamma_A(m_1) = \gamma_A(m_2) = \min\{\gamma_A(a) \mid a \in \mathrm{M}\}.$$

Similarly we can finally obtain the set of generators $\{m_1, m_2, \ldots, m_n\}$ satisfying the following relation:

$$\gamma_A(m_1) = \gamma_A(m_2) = \cdots = \gamma_A(m_n) = \min\{\gamma_A(a) \mid a \in \mathrm{M}\}.$$

Now in the same way as before, without loss of generality, we assume that $\zeta_A(m_1) = \max_i\{\zeta_A(m_i)\}$ and denote $c_1 = m_1$.
If $\zeta_A(m_2) = \zeta_A(m_1)$, then we take $c_2 = \zeta_A(m_2)$, else we have

$$a = (h_l - h_2)m_1 + h_2(m_1 + m_2) + h_3 m_3 + \cdots + h_n m_n,$$

so $m_1, m_1 + m_2, m_3, \ldots, m_n$, is a set of generators of M, and we can prove that $\gamma_A(c_1) = \gamma_A(m_1 + m_2)$ and $\zeta_A(c_1) = \zeta_A(m_1 + m_2)$.
Then we choose $c_2 = m_1 + m_2$.
By means of this method we can finally obtain the set of generators $\{c_1, c_2, \ldots, c_k\}$ satisfying the following relation:

$$\gamma_A(c_1) = \gamma_A(c_2) = \cdots = \gamma_A(c_k) = \min\{\gamma_A(a) \mid a \in \mathrm{M}\}$$

and

$$\zeta_A(c_1) = \zeta_A(c_2) = \cdots = \zeta_A(c_k) = \max\{\zeta_A(a) \mid a \in \mathrm{M}\}.$$

This completes the proof. □

Now we will define the product between an intuitionistic fuzzy ideal and an intuitionistic fuzzy submodule.

Definition 8. *Let I be an IFI of $\mathfrak{R}$, and let A be an intuitionistic fuzzy submodule of M. We define the product of $I(\gamma_I;\zeta_I)$ and $\mathrm{A}(\gamma_\mathrm{A};\zeta_\mathrm{A})$, denoted by $I\mathrm{A}$, as follows:*

$$(\gamma_I\gamma_\mathrm{A})(x) = \bigvee_{x=\sum_{i<\infty} r_i x_i} \Big(\min_i \left(\min\left(\gamma_I\left(r_i\right), \gamma_\mathrm{A}\left(x_i\right)\right)\right)\Big), x \in \mathrm{M}.$$

$$(\zeta_I\zeta_A)(x) = \bigwedge_{x=\sum_{i<\infty} r_i x_i} \Big(\max_i \left(\max\left(\zeta_I\left(r_i\right), \zeta_A\left(x_i\right)\right)\right)\Big), x \in \mathrm{M}.$$

Proposition 4. *Let $I = (\gamma_I, \zeta_I)$ be an IFI of $\mathfrak{R}$ and let $A = (\gamma_A, \zeta_A)$ be an intuitionistic fuzzy submodule of M. Then $(IA)_{(k,l)} = I_{(k,l)}A_{(k,l)}$, for all $(k,l) \in [0,1]^2$.*

Proof. Let $\mathrm{a} \in (IA)_{(k,l)}$, then we have $k \leq (\gamma_I\gamma_A)(\mathrm{a})$ and $(\zeta_I\zeta_A)(\mathrm{a}) \leq l$.
Let $\eta > 0$, so that $k - \eta < (\gamma_I\gamma_A)(\mathrm{a})$ and $(\zeta_I\zeta_A)(\mathrm{a}) < l + \eta$.

Therefore, there exists a representation $\mathrm{a} = r_1x_1 + \cdots + r_nx_n$, such that

$$k - \eta < \min_i(\min(\gamma_I(r_i), \gamma_A(x_i))),$$
$$\max_i(\max(\zeta_I(r_i), \zeta_A(x_i))) < l + \eta.$$

Then,

$$k - \eta < \min(\gamma_I(r_i), \gamma_A(x_i)),$$
$$\max(\zeta_I(r_i), \zeta_A(x_i)) < l + \eta,$$

for all i. This implies that $k - \eta < \gamma_I(r_i)$, $k - \eta < \gamma_A(x_i)$, $\zeta_I(r_i) < l + \eta$ and $\zeta_A(x_i) < l + \eta$. Hence

$$r_i \in I_{(k-\eta,l+\eta)} \text{ and } x_i \in A_{(k-\eta,l+\eta)}$$

for all i. Therefore

$$\sum_{i<\infty} r_i x_i \in I_{(k-\eta,l+\eta)} A_{(k-\eta,l+\eta)}$$

for all $\eta > 0$. So that

$$\mathrm{a} \in I_{(k,l)} A_{(k,l)}.$$

Consequently

$$(IA)_{(k,l)} \subseteq I_{(k,l)} A_{(k,l)}$$

Conversely, let $\mathrm{a} \in I_{(k,l)} A_{(k,l)}$, such that $\mathrm{a} = \sum_{i<\infty} r_i x_i$ for some $r_i \in I_{(k,l)}$, and $x_i \in A(k,l)$, for $i = 1, \ldots, n$. Now

$$k \leq \min_i(\min(\gamma_I(r_i), \gamma_A(x_i))) \leq (IA)(\mathrm{a}) \text{ and } (IA)(\mathrm{a}) \leq \max_i(\max(\zeta_I(r_i), \zeta_A(x_i))) \leq l$$

Hence, $\mathrm{a} \in (IA)_{(k,l)}$. Finally

$$I_{(k,l)} A_{(k,l)} \subseteq (IA)_{(k,l)}.$$ □

Theorem 5. *Let I be an intuitionistic fuzzy ideal of $\mathfrak{R}$ and let* A *be an intuitionistic fuzzy submodule of* M*. Then the product I*A *is an intuitionistic fuzzy submodule of* M*.*

Proof. Given $x, y \in \mathrm{M}$, let $\eta > 0$ and assume that $\alpha = \min((\gamma_{IA})(x), \gamma_{IA}(y))$ and $\beta = \max((\zeta_{IA})(x), \zeta_{IA}(y))$. Then

$$\alpha - \eta < (\gamma_I \gamma_A)(x) = \bigvee_{x=\sum_{i<\infty} r_i x_i} \Big(\min_i \left(\min\left(\gamma_A\left(r_i\right), \gamma_I\left(x_i\right)\right)\right)\Big),$$
$$\alpha - \eta < (\gamma_I \gamma_A)(y) = \bigvee_{y=\sum_{i<\infty} s_i y_i} \Big(\min_i \left(\min\left(\gamma_I\left(s_i\right), \gamma_A\left(y_i\right)\right)\right)\Big)$$

and

$$(\zeta_I \zeta_A)(x) = \bigwedge_{x=\sum_{i<\infty} r_i x_i} \Big(\max_i \left(\max\left(\zeta_A\left(r_i\right), \zeta_I\left(x_i\right)\right)\right)\Big) < \beta + \eta,$$
$$(\zeta_I \zeta_A)(y) = \bigwedge_{y=\sum_{i<\infty} s_i y_i} \Big(\max_i \left(\max\left(\zeta_A\left(s_i\right), \zeta_I\left(y_i\right)\right)\right)\Big) < \beta + \eta.$$

So, there exists $r_i, s_i \in \mathfrak{R}$ and $x_i, y_i \in \mathrm{M}$ such that $x = r_1x_1 + \cdots + r_nx_n, y = s_1y_1 + \cdots + s_ny_n$ and

$$\alpha - \eta < \min_i \left(\min \left(\gamma_I \left(r_i\right), \gamma_A \left(x_i\right)\right)\right),$$
$$\alpha - \eta < \min_i \left(\min \left(\gamma_I \left(s_i\right), \gamma_A \left(y_i\right)\right)\right),$$
$$\max_i \left(\max \left(\zeta_I \left(r_i\right), \zeta_A \left(x_i\right)\right)\right) < \beta + \eta,$$
$$\max_i \left(\max \left(\zeta_I \left(s_i\right), \zeta_A \left(y_i\right)\right)\right) < \beta + \eta.$$

This impies that for all i

$$\alpha - \eta < \min \left(\gamma_I \left(r_i\right), \gamma_A \left(x_i\right)\right),$$
$$\alpha - \eta < \min \left(\gamma_I \left(s_i\right), \gamma_A \left(y_i\right)\right),$$

and

$$\min \left(\zeta_I \left(r_i\right), \zeta_A \left(x_i\right)\right) < \beta + \eta,$$
$$\min \left(\zeta_I \left(s_i\right), \zeta_A \left(y_i\right)\right) < \beta + \eta.$$

Then $\alpha - \eta < \gamma_I \left(r_i\right)$, $\alpha - \eta < \gamma_A \left(x_i\right), \alpha - \eta < \gamma_I \left(s_i\right), \alpha - \eta < \gamma_A \left(y_i\right)$ and $\zeta_I \left(r_i\right) < \beta + \eta$, $\zeta_A \left(x_i\right) < \beta + \eta, \zeta_I \left(s_i\right) < \beta + \eta, \zeta_A \left(y_i\right) < \beta + \eta$.
Therefore, for all i,

$$\alpha - \eta < \min \left(\gamma_I \left(r_i\right), \gamma_I \left(s_i\right)\right) \leq \gamma_I \left(r_i + s_i\right),$$

$$\zeta_I \left(r_i + s_i\right) \leq \max \left(\zeta_I \left(r_i\right), \zeta_I \left(s_i\right)\right) < \beta + \eta,$$

and

$$\alpha - \eta < \min \left(\gamma_A \left(x_i\right), \gamma_A \left(y_i\right)\right) \leq \gamma_A \left(x_i + y_i\right),$$
$$\zeta_I \left(x_i + y_i\right) \leq \max \left(\zeta_I \left(x_i\right), \zeta_I \left(y_i\right)\right) < \beta + \eta$$

for all $i \in \{1, \ldots, n\}$.

Hence, there exists a representation $x + y = \sum_{i<\infty} (r_ix_i + s_iy_i)$, where $x_i, y_i \in \mathrm{M}, r_i, s_i \in \mathfrak{R}$, for all $i \in \{1, \ldots, n\}$, such that

$$\alpha - \eta < \min \left(\gamma_I \left(r_i + s_i\right), \gamma_A \left(x_i + y_i\right)\right),$$

That is

$$\alpha - \eta < \min_i \left(\min \left(\gamma_I \left(r_i + s_i\right), \gamma_A \left(x_i + y_i\right)\right)\right),$$
$$\max_i \left(\max \left(\zeta_I \left(r_i + s_i\right), \zeta_A \left(x_i + y_i\right)\right)\right) < \beta + \eta.$$

Then

$$\alpha - \eta < \bigvee_{x+y=\sum_{i<\infty}(r_ix_i+s_iy_i)} (\min_i \min(\gamma_I(r_i + s_i), \gamma_A(x_i + y_i))) = (\gamma_I\gamma_A)(x + y).$$
$$(\zeta_I\zeta_A)(x + y) = \bigwedge_{x+y=\sum_{i<\infty}(r_ix_i+s_iy_i)} (\max_i \max(\zeta_I(r_i + s_i), \zeta_A(x_i + y_i))) < \beta + \eta.$$

Since $\eta > 0$ can be any value, it follows that for all $(x, y) \in \mathrm{M}^2$,

$$\begin{aligned} \min((\gamma_I\gamma_A)(x), (\gamma_I\gamma_A)(y)) = \alpha \leq (\gamma_I\gamma_A)(x+y), \\ (\zeta_I\zeta_A)(x+y) \leq \max((\zeta_I\zeta_A)(x), (\zeta_I\zeta_A)(y)) = \beta. \end{aligned} \tag{1}$$

Now similarly, let us show that $\gamma_I\gamma_A(x) \leq \gamma_I\gamma_A(rx)$, and $\zeta_I\zeta_A(rx) \leq \zeta_I\zeta_A(x)$ for all $x \in \mathrm{M}$, and for all $r \in \mathfrak{R}$. Consider $IA(x) = (\alpha, \beta)$. Let $\epsilon > 0$, so

$$\alpha = (\gamma_I\gamma_A)(x) = \bigvee_{x=\sum_{i<\infty} r_i x_i} \left(\min_i \left(\min\left(\gamma_I\left(r_i\right), \gamma_A\left(x_i\right)\right)\right)\right),$$

$$\zeta_I\zeta_A(x) = \bigwedge_{x=\sum_{i<\infty} r_i x_i} \left(\max_i \left(\max\left(\zeta_I\left(r_i\right), \zeta_A\left(x_i\right)\right)\right)\right) = \beta.$$

Therefore, there exists a representation $x = \sum_{i=1}^{i=n} r_i x_i$, where $(r_i, x_i) \in \mathfrak{R} \times M$ such that

$$\alpha - \epsilon < \min_i(\min(\gamma_I(r_i), \gamma_A(x_i)))),$$

$$\max_i(\max(\zeta_I(r_i), \zeta_A(x_i)))) < \beta + \epsilon.$$

Then

$$\alpha - \epsilon < \min(\gamma_I(r_i), \gamma_A(x_i)),$$

$$\max(\zeta_I(r_i), \zeta_A(x_i)) < \beta + \epsilon$$

for all i, so

$$\alpha - \epsilon < \min(\gamma_I(rr_i), \gamma_A(x_i)),$$

$$\max(\zeta_I(rr_i), \zeta_A(x_i)) < \beta + \epsilon$$

for all i and for all $r \in \mathfrak{R}$.
Hence,

$$\alpha - \epsilon < \min_i \min(\gamma_I(rr_i), \gamma_A(x_i)) \leq \bigvee_{rx=\sum_{i<\infty} rr_i x_i} \left(\min_i \left(\min\left(\gamma_I\left(rr_i\right), \gamma_A\left(x_i\right)\right)\right)\right),$$

$$(\zeta_I\zeta_A)(rx) = \bigwedge_{rx=\sum_{i<\infty}(rr_i x_i)} (\max_i \max(\zeta_I(rr_i), \zeta_A(x_i))) \leq \max_i \max(\zeta_I(rr_i), \zeta_A(x_i)) < \beta + \eta.$$

As ϵ is arbitrary, we get

$$\begin{aligned} \gamma_I\gamma_A(x) = \alpha \leq \gamma_I\gamma_A(rx), \\ \zeta_I\zeta_A(rx) \leq \beta = \zeta_I\zeta_A(x) \end{aligned} \tag{2}$$

for all $x \in \mathrm{M}, r \in \mathfrak{R}$. Consequently from (1) and (2), IA is an intuitionistic fuzzy submodule of M. □

Proposition 6. *Consider* M *generated by* $e_1, e_2, \ldots, e_n$ *and let* $IA = (0, 1)$. *Then either* $I = (0, 1)$ *or* $A(e_i) = (0, 1)$, $\forall i$.

Proof. We have

$$0 = (\gamma_I A)(0) = \bigvee_{0=\sum_{i<\infty} r_i e_i} \left(\min_i \left(\min \left(\gamma_I \left(r_i \right), \gamma_A \left(e_i \right) \right) \right) \right),$$

then $0 = \min_i \left(\min \left(\gamma_I \left(r_i \right), \gamma_A \left(e_i \right) \right) \right)$ for all i, such that $0 = \sum_{i<\infty} r_i e_i$, $x_i \in \mathrm{M}, r_i \in R$, and we have $0 = 0.e_i$, so $\min \left(\gamma_I(0), \gamma_A \left(e_i \right) \right) = 0$, then $\gamma_I(0) = 0$ or $\gamma_A \left(e_i \right) = 0$.

Therefore, either $\gamma_I(r) \leq \gamma_I(0) = 0$ for all $r \in R$ or $\gamma_A \left(e_i \right) = 0$ for all i.

For this reason, $\gamma_I = (0)$ or $\gamma_A \left(e_i \right) = 0$, for any i, and

$$1 = (\zeta_{IA})(0) = \bigwedge_{0=\sum_{i<\infty} r_i e_i} \left(\max_i \left(\max \left(\zeta_I \left(r_i \right), \zeta_A \left(e_i \right) \right) \right) \right),$$

it follows that $1 = \max_i \left(\max \left(\zeta_I \left(r_i \right), \zeta_A \left(e_i \right) \right) \right)$ for all i, such that $0 = \sum_{i<\infty} r_i e_i$. Since $0 = 0.e$, so $\max \left(\zeta_I(0), \zeta_A \left(e_i \right) \right) = 1$, that is either $\zeta_I(0) = 1$ or $\zeta_I \left(e_i \right) = 1$.

Then, either $\zeta_I(r) \geq \zeta_I(0) = 1$ for all $r \in R$ or $\zeta_A \left(e_i \right) = 1$ for all i. $\leftarrow$

Therefore, $\zeta_I(r) = 1, \forall r \in R$ or $\zeta_A \left(e_i \right) = 0 \ \forall i \in \{1, \ldots, n\}$. □

Theorem 7. *Consider* M *a finitely generated module and* A *an intuitionistic fuzzy submodule of* M. *If I is an IFI of $\mathfrak{R}$ such that $I\mathrm{A} = \mathrm{A}$, then there exist $k, l \in [0, 1]$ such that $I_{(k,l)} M = M$.*

Proof. Consider $(k, l) = (\bigwedge\{\gamma_A(y) \mid y \in \mathrm{M}\}; \bigvee\{\zeta_A(y) \mid y \in \mathrm{M}\})$. By Proposition 3, there exist generators $e_1, e_2, \ldots, e_n$ of M such that $A(e_i) = (k, l)$ for all i.

Let us consider $y \in \mathrm{M}$ and let $\eta > 0$. Then

$$k - \eta < \gamma_{IA}(y) = \bigvee_{y=\sum_{i<\infty} h_i e_i} \left(\min_i \left(\min \left(\gamma_I \left(h_i \right), \gamma_A \left(e_i \right) \right) \right) \right),$$

$$\zeta_{IA}(y) = \bigwedge_{y=\sum_{i<\infty} h_i e_i} \left(\max_i \left(\max \left(\zeta_I \left(h_i \right), \zeta_A \left(e_i \right) \right) \right) \right) < l + \eta.$$

Therefore, $y = \sum_{i=1}^{i=n} h_i e_i$, where $h_i \in \mathfrak{R}, e_i \in \mathrm{M}$, such that $k - \eta < \min \left(\gamma_I \left(h_i \right), \gamma_A \left(e_i \right) \right)$ and $\max \left(\zeta_I \left(h_i \right), \zeta_A \left(e_i \right) \right) < l + \eta$, for all $i \in \{1, \ldots, n\}$. This implies that $k - \eta < \gamma_I \left(h_i \right)$, $k - \eta < \gamma_A \left(e_i \right)$, $\zeta_I(h_i) < l + \eta$ and $\zeta_A(e_i) < l + \eta$ for all i, and so that $h_i \in I_{(k-\eta, l+\eta)}$ and $e_i \in A_{(k-\eta, l+\eta)}$ for all i, whence:

(i) $y \in I_{(k,l)} A_{(k,l)}$, so that $M \subseteq I_{(k,l)} A_{(k,l)}$.
Therefore $M = I_{(k,l)} A_{(k,l)}$.

(ii) $e_i \in A_{(k,l)}$ for all i, so that $M = A_{(k,l)}$.

From (i) and (ii) we get $I_{(k,l)} M = M$. □

Proposition 8. *Let A be an intuitionistic fuzzy submodule of a finitely generated module* M*, and I be an intuitionistic fuzzy ideal. If $IA = A$ and if $(1, 0) \in \operatorname{Im} A$, then $(1, 0) \in \operatorname{Im} I$ and $A(a) = (1, 0)$ for all $a \in \mathrm{M}$.*

Proof. Clearly, $(IA)(0) = A(0) = (1, 0)$, since $IA = A$ and $(1, 0) \in \operatorname{Im} A$. Then

$$\bigvee_{0=\sum_{i<n} r_i b_i} \Big(\min_i \left(\min \left(\gamma_I \left(r_i \right), \gamma_A \left(b_i \right) \right) \right) \Big) = 1,$$

$$\bigwedge_{0=\sum_{i<n} r_i b_i} \Big(\max_i \left(\max \left(\zeta_I \left(r_i \right), \zeta_A \left(b_i \right) \right) \right) \Big) = 0,$$

where $b_1, \ldots, b_n$ are generators of M. Therefore, a representation $0 = r_1 b_1 + \cdots + r_n b_n$ exists, with $\min \left(\gamma_I \left(r_i \right), \gamma_A \left(b_i \right) \right) = 1$, and $\max \left(\zeta_I \left(r_i \right), \zeta_A \left(b_i \right) \right) = 0$, for any i.

Then, $\gamma_I \left(r_i \right) = 1$, $\gamma_A \left(b_i \right) = 1$, $\zeta_I \left(r_i \right) = 0$ and $\zeta_A \left(b_i \right) = 0$ for all i.

Hence $I(0) = (1, 0)$ and $A(a) = (1, 0)$, for all $a \in \mathrm{M}$.

Consequently, $(1, 0) \in \operatorname{Im} I$ and $A(a) = (1, 0)$, for all $a \in \mathrm{M}$. □

Theorem 9 (Intuitionistic fuzzy Nakayama's Lemma). *Let M be generated by $e_1, e_2, \ldots, e_n$ and let* A *be an intuitionistic fuzzy submodule of M such that* $\mathrm{A} \left(e_i \right) \neq (0, 1)$*,* $\forall i \in \{1, \ldots, n\}$*. If* $I\mathrm{A} = \mathrm{A}$*, where I is an intuitionistic fuzzy ideal of* $\mathfrak{R}$*, and if* $I \subseteq \mathrm{IFJR}(\mathfrak{R})$*, then* $M = 0$*.*

Proof. Let $(k, l) = (\wedge\{\gamma_A(y) \mid y \in \mathrm{M}\}; \vee\{\zeta_A(y) \mid y \in \mathrm{M}\})$.

Since $\mathrm{A} \left(e_i \right) \neq (0, 1)$, Proposition 3 implies that (k, l) cannot be equal to $(0, 1)$.

Now, we will prove that the set $I_{(k,l)}$ is a subset of $J(\mathfrak{R})$.

Given that I is a subset of $\mathrm{IFJR}(\mathfrak{R})$, it could easily be assumed that I is included in every intuitionistic fuzzy maximal ideal of $\mathfrak{R}$.

Let V be any maximal ideal of $\mathfrak{R}$, we define the intuitionistic fuzzy subset ϕ of $\mathfrak{R}$ by

$$\phi(h) = \begin{cases} (1, 0), & \text{if } h \in V \\ (\alpha, \beta), & \text{if } h \notin V \text{ with } (\alpha, \beta) \in [0, k[\times]l, 1]. \end{cases}$$

Then, by Theorem 1, ϕ is an intuitionistic fuzzy maximal ideal of $\mathfrak{R}$. Thus, $I \subseteq \phi$, then $\gamma_I(h) \leq \gamma_\phi(h)$ and $\zeta_I(h) \geq \zeta_\phi(h)$ for all $h \in \mathfrak{R}$.

Now, let us consider $y \in I_{(k,l)}$. Then $\gamma_I(y) > k > \alpha$ and $\zeta_I(y) < l < \beta$, so that $\gamma_{\phi(y)} > \alpha$ and $\zeta_{\phi(y)} < \beta$, that is $\phi(y) = (1, 0)$ and so $y \in V$.

Hence $I_{(k,l)} \subseteq V$, for all maximal ideals V of $\mathfrak{R}$, then $I_{(k,l)} \subseteq J(\mathfrak{R})$.

Therefore, by using Nakayama's Lemma and Proposition 4, it can be seen that $M = 0$ and hence A is constant. □

5 Addition of two intuitionistic fuzzy submodules

First in this section, we will introduce the addition of two intuitionistic fuzzy submodules. And then, using this addition, we will give another intuitionistic fuzzy version of the Nakayama's Lemma.

Definition 9. [4] *Let* A *and* B *be two intuitionistic fuzzy submodules of* M*. The sum of* A *and* B*, written as* $\mathrm{A} + \mathrm{B}$*, is defined by*

$$(A + B)(z) = (\bigvee_{z=x+y} (\min(\gamma_{\mathrm{A}}(x), \gamma_{\mathrm{B}}(y))); \bigwedge_{z=x+y} (\max(\zeta_{\mathrm{A}}(x), \zeta_{\mathrm{B}}(y)))).$$

Obviously, $\mathrm{A}+\mathrm{B}$ is an intuitionistic fuzzy subset of M.

Theorem 10. *Consider* A *and* B *as intuitionistic fuzzy submodules of* M. *Therefore, the sum of* A *and* B, *denoted as* $\mathrm{A}+\mathrm{B}$, *is an intuitionistic fuzzy submodule of* M.

Proof. Consider $w, z \in \mathrm{M}$ and let $\alpha = \min(\gamma_{\mathrm{A+B}}(w), \gamma_{\mathrm{A+B}}(z))$ and $\beta = \max(\zeta_{\mathrm{A+B}}(w), \zeta_{\mathrm{A+B}}(z))$. Let $\eta > 0$ be given.
Then

$$\alpha - \eta < \gamma_{\mathrm{A+B}}(w) = \bigvee_{w=u+v} (\min(\zeta_{\mathrm{A}}(u), \zeta_{\mathrm{B}}(v))),$$
$$\zeta_{\mathrm{A+B}}(w) = \bigwedge_{w=u+v} (\max(\mathrm{A}(u), \mathrm{B}(v))) < \beta + \eta,$$

where $u, v \in \mathrm{M}$, and Therefore,

$$\alpha - \eta < \min(\gamma_{\mathrm{A}}(u), \gamma_{\mathrm{B}}(v)), \quad \max(\zeta_{\mathrm{A}}(u), \zeta_{\mathrm{B}}(v)) < \beta + \eta$$

for some $u, v \in \mathrm{M}$ such that $w = u + v$ and

$$\alpha - \eta < \min(\gamma_A(t), \gamma_B(s)), \quad \max(\zeta_{\mathrm{A}}(t), \zeta_{\mathrm{B}}(s)) < \beta + \eta$$

for some $t, s \in \mathrm{M}$ such that $z = t + s$.
So $\alpha - \eta < \gamma_A(u), \alpha - \eta < \gamma_B(v), \alpha - \eta < \gamma_A(t), \alpha - \eta < \gamma_B(s)$, and $\zeta_A(u) \leq \beta + \eta, \zeta_B(v) \leq \beta + \eta, \zeta_A(t) \leq \beta + \eta, \zeta_B(s) \leq \beta + \eta$, where $w + z = u + v + t + s$, which implies that

$$\alpha - \eta < \min(\gamma_{\mathrm{A}}(u), \gamma_A(t)) \leq \gamma_{\mathrm{A}}(u + t);$$
$$\max(\zeta_{\mathrm{A}}(u), \zeta_{\mathrm{A}}(t)) \leq \zeta_{\mathrm{A}}(u + t) < \beta + \eta,$$

and

$$\alpha - \eta < \min(\gamma_{\mathrm{B}}(v), \gamma_{\mathrm{B}}(s)) \leq \gamma_{\mathrm{B}}(u + t);$$
$$\zeta_{\mathrm{B}}(u + t) \leq \max(\zeta_{\mathrm{B}}(v), \zeta_{\mathrm{B}}(s)) \leq \beta + \eta.$$

Therefore, there is a representation $w + z = u + v + t + s$, where

$$\alpha - \eta < \min(\gamma_{\mathrm{A}}(u + t), \gamma_{\mathrm{B}}(v + s)),$$
$$\min(\zeta_{\mathrm{A}}(u + t), \zeta_{\mathrm{B}}(v + s)) < \beta + \eta,$$

whence

$$\alpha - \eta < \bigvee_{w+z=p+q} (\min(\gamma_{\mathrm{A}}(p), \gamma_{\mathrm{B}}(q))) = \gamma_{\mathrm{A+B}}(z + w)$$
$$\bigwedge_{w+z=p+q} (\max(\zeta_{\mathrm{A}}(p), \zeta_{\mathrm{B}}(q))) = \zeta_{\mathrm{A+B}}(w + z) < \beta + \eta.$$

The arbitrary nature of $\eta > 0$ implies that

$$\begin{aligned} \min(\gamma_{\mathrm{A+B}}(w), \gamma_{\mathrm{A+B}}(z)) = \alpha \leq \gamma_{\mathrm{A+B}}(w + z), \\ \max(\zeta_{\mathrm{A+B}}(w), \zeta_{\mathrm{A+B}}(z)) = \beta \geq \zeta_{\mathrm{A+B}}(w + z) \end{aligned} \tag{3}$$

for all $w, z \in \mathrm{M}$. Next, let $(\alpha_2, \beta_2) = (\mathrm{A}+\mathrm{B})(w)$, then

$$\alpha_2 - \eta < (\gamma_{\mathrm{A}} + \gamma_{\mathrm{B}})(w) = \bigvee_{w=u+v} (\min(\gamma_A(u), \gamma_B(v))),$$
$$(\zeta_{\mathrm{A}} + \zeta_{\mathrm{B}})(w) = \bigwedge_{w=u+v} (\max(\zeta_A(u), \zeta_B(v))) < \beta_2 + \eta.$$

Therefore, $\alpha_2 - \eta < \min(\gamma_{\mathrm{A}}(u), \gamma_{\mathrm{B}}(v))$ and $\min(\zeta_{\mathrm{A}}(u), \zeta_{\mathrm{B}}(v)) < \beta_2 + \eta$ for some $u, v \in \mathrm{M}$ such that $w = u + v$ $\alpha_2 - \eta \leq \min(\gamma_{\mathrm{A}}(ru), \gamma_{\mathrm{B}}(rv))$ and $\max(\zeta_{\mathrm{A}}(ru), \zeta_{\mathrm{B}}(rv)) < \beta_2 + \eta$ for all $r \in \mathfrak{R}$.
Then

$$\alpha_2 - \eta < \bigvee_{rw=p+q} (\min(\gamma_{\mathrm{A}}(p), \gamma_{\mathrm{B}}(q))) = (\gamma_{\mathrm{A+B}})(rw),$$
$$(\zeta_{\mathrm{A+B}})(rw) = \bigwedge_{rw=p+q} (\max(\zeta_{\mathrm{A}}(p), \zeta_{\mathrm{B}}(q))) < \beta_2 + \eta.$$

Hence

$$\begin{aligned}(\gamma_{\mathrm{A+B}})(rw) \geq \alpha_2 = (\gamma_{\mathrm{A+B}})(w),\\ (\zeta_{\mathrm{A+B}})(rw) \leq \beta_2 = (\zeta_{\mathrm{A+B}})(w).\end{aligned} \tag{4}$$

It follows from (3) and (4) that $A + B$ is an intuitionistic fuzzy submodule of M. □

Theorem 11. *Let* A *and* B *be two intuitionistic fuzzy submodules of* M *with* $\mathrm{A} \subseteq \mathrm{B}$. *Let* I *be an intuitionistic fuzzy ideal of* $\mathfrak{R}$. *If* $I\mathrm{A} + \mathrm{B} = \mathrm{A}$, *then there exists* $(k, l) \in [0, 1]^2$ *such that* $\mathrm{A}_{(k,l)}M + \mathrm{B}_{(k,l)} = M$.

Proof. Let $(k, l) = \{(\inf A(w), \sup A(w)) \mid w \in \mathrm{M}\}$. Let $w \in \mathrm{M}$ and $\eta > 0$ be given.
Then

$$\begin{aligned}k - \eta < \gamma_A(w) &= (\gamma_I \gamma_A + \gamma_B)(w)\\ &= \bigvee_{w=u+v} (\min((\gamma_I \gamma_A)(u), \gamma_B(v))),\end{aligned}$$

$$\begin{aligned}\zeta_A(w) &= (\zeta_I \zeta_A + \zeta_B)(w)\\ &= \bigwedge_{w=u+v} (\max((\zeta_I \zeta_A)(u), \zeta_B(v))) < l + \eta,\end{aligned}$$

which implies that $k - \eta < \min((\gamma_I \gamma_A)(u), \gamma_B(v))$ and $\max((\zeta_I \zeta_A)(u), \zeta_B(v)) < l + \eta$, for some $u, v \in \mathrm{M}$ such that $w = u + v$.

Then $k - \eta < (\gamma_I \gamma_A)(u)$, $k - \eta < \gamma_B(v)$, $(\zeta_I \zeta_A)(u) < l + \eta$ and $\zeta_B(v) < l + \eta$, hence $u \in (IA)_{(k-\eta, l+\eta)}$ and $v \in B_{(k-\eta, l+\eta)}$, whence $w = u + v \in (IA)_{(k-\eta, l+\eta)} + B_{(k-\eta, l+\eta)}$, for all $\eta > 0$.

Therefore $M \subseteq (IA)_{(k,l)} + B_{(k,l)} = I_{(k,l)} A_{(k,l)} + B_{(k,l)}$. Thus $M = I_{(k,l)} M + B_{(k,l)}$, since $M = A_{(k,l)}$. □

The following lemma will be instrumental in establishing our next theorem, which presents an alternative version of the intuitionistic fuzzy Nakayama's Lemma.

Lemma 1. [5,9] *Let N be a submodule of a finitely generated module* M *and let A be an ideal of $\mathfrak{R}$ such that $A \subseteq J(\mathfrak{R})$. If $AM + N = M$, then $N = M$.*

Theorem 12. *Let A and B be intuitionistic fuzzy submodules of M such that $A \subseteq B$. Suppose that I is an intuitionistic fuzzy ideal of R and that $IA + B = A$. Additionally, assume that the module M is generated by elements $e_1, \dots, e_n$, that $A(e_i) \neq 0$ for some i and that $I \subseteq$ IFJR(R). Then there exists a pair $(k, l) \in [0, 1]^2$ such that*

$$A_{(k,l)} = B_{(k,l)}.$$

Proof. If $I \subseteq \mathrm{FJR}(\mathfrak{R})$, then $I_{(k,l)} \subseteq J(\mathfrak{R})$ for some $k, l \in [0, 1]$. Hence, by Theorem 11, we obtain that $A_{(k,l)}M + B_{(k,l)} = M$. And by Nakayama's Lemma 1 we get $M = A_{(k,l)}$. Thus

$$A_{(k,l)} = B_{(k,l)}.$$

This completes the proof. □

6 Conclusion

This paper presents a more fruitful approach compared to fuzzy set theory. Its primary objective is to introduce a novel product of an intuitionistic fuzzy ideal and an intuitionistic fuzzy submodule. Additionally, it aims to establish definitions and discussions regarding the summation of two intuitionistic fuzzy submodules. Furthermore, the paper seeks to formulate and analyze the intuitionistic fuzzy version of Nakayama's Lemma. From our standpoint, we aim to extend numerous fundamental results concerning the classification of intuitionistic fuzzy modules, the study of intuitionistic fuzzy module homomorphisms, and the investigation of intuitionistic fuzzy module structures over specific rings.

Acknowledgements

The authors would like to express their sincere gratitude to the editors and anonymous reviewers for their invaluable comments and constructive feedback, which significantly contributed to the enhancement of this paper.

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