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Strong Law of Large Numbers on MV-algebras

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Abstract: The aim is to approve the Strong law of large numbers on MV-algebras by the new approach, using the observable as a distribution function, and not a σ-homomorphism. The main idea is in a local representation of σ-algebras.

The following theorem is proved: Let M be a σ-complete MV-algebra with product, $m : M \longrightarrow [0,1]$ be a σ- additive state, $(x_n)_n$ be a sequence of independent square integrable observables, such that $\sum_{n=1}^{\infty} \frac{\sigma^2(x_n)}{n^2} < \infty$. Then

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i}\left(\frac{1}{n}\sum_{j=1}^{n}(x_j - E(x_j))\right)\left(-\frac{1}{p},\frac{1}{p}\right)\right) = 1$$

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1 MV-algebras

We shall use an excellent characterization of MV-algebras given by D. Mundici ([3]) by the help of l-groups. An l-group is an algebraic system

$$(G, +, \leq)$$

such that

$(G, +)$ is an Abelian group,
$(G, \leq)$ is a partially ordered set being a lattice,
$a \leq b \Longrightarrow a + c \leq b + c$ for and a, b, c in G.

A typical example of an MV-algebra is the unit interval $[0, 1]$ of real numbers with two binary operations $\oplus, \odot$,

$a \oplus b = (a + b) \wedge 1$
$a \odot b = (a + b - 1) \vee 0$

with the partial ordering $\leq$, and two fixed elements $0, 1$ (0 is the least element of $\leq$, 1 is the greatest element of $\leq$).

Generally the situation is analogous.

1.1. Definition. An MV-alebra is an algebraic system

$$(M, \oplus, \odot, \leq, 0, u),$$

where

$$M = [0, u]$$

is an interval in an l-group $G = (G, +, \leq), 0$ is the neutral element of G (i.e. $a + 0 = a$ for any $a \in G$), u is the strong unit of G (i.e. to any $a \in G$ there exists $n \in N$ such that $a \leq u + u + ... + u(n\text{-times})$),

$a \oplus b = (a + b) \wedge u$,

$a \odot b = (a + b - u) \vee 0$.

In two-valued logic Boolean algebras can be considered represented e.g. by charateristic functions $\chi_A : \Omega \to \{0, 1\}$ where,

$\chi_A(v) = 1$, if $v \in A$,

$= 0$, if $v \notin A$.

In multi-valued logic MV-algebras instead of two-valued functions

$$\chi_A : \Omega \to \{0, 1\},$$

multivalued functions

$$\mu_A : \Omega \to [0, 1]$$

are considered. Evidently

$$\chi_A \oplus \chi_B = \chi_{A \cup B}, \chi_A \odot \chi_B = \chi_{A \cap B}.$$

Hence $\mu_A \odot \mu_B$ can be considered as the conjunction, $\mu_A \oplus \mu_B$ as the disjunction, $1 - \mu_A$ as the negation.

1.2. Definition. A state on an MV-algebra M is a mapping $m : M \to [0, 1]$ satisfying the following conditions:

(i) $m(u) = 1, m(0) = 0$;

(ii) $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$;

(iii) $a_n \searrow a \Longrightarrow m(a_n) \searrow m(a)$.

1.3. Definition. Let $\mathcal{J} = \{(-\infty, t); t \in R\}$. An observable on M is any mapping $x : \mathcal{J} \to M$ satisfying the conditions:

(i) $t_n \nearrow \infty \Longrightarrow x((-\infty, t_n)) \nearrow u$;

(ii) $t_n \searrow -\infty \Longrightarrow x\left((-\infty, t_n)\right) \searrow 0$;

(iii) $t_n \nearrow t \Longrightarrow x((-\infty, t_n)) \nearrow x((-\infty, t))$.

1.4. Theorem [3,5]. Let $m : M \to [0, 1]$ be a state, $x : \mathcal{J} \to M$ be an observable. Define $F : R \to [0, 1]$ by the formula

$$F(t) = m(x((-\infty, t))), t \in R$$

Then F has the following properties:

(i) F is non-decreasing;

(ii) $\lim_{t \to \infty} F(t) = 1$;

(iii) $\lim_{t \to -\infty} F(t) = 0$;

(iv) F is left continuous in any point $t \in R$.

Proof is straightforward.

Recall that in the Kolmogorov theory the mean value $E(\xi)$ of an observable $\xi : (\Omega, \mathcal{S}, P) \to R$ is defined as an integral

$$E(\xi) = \int_\Omega \xi dP.$$

Let $g : R \to R$ be a Borel measurable function. The transformation formula states

$$E(g \circ \xi) = \int_R g(t) dF(t),$$

where F is the distribution function of ξ. Therefore

$$E(\xi) = \int_R t dF(t),$$

$$E(\xi^2) = \int_R t^2 dF(t),$$

$$\sigma(\xi)^2 = E(\xi^2) - E(\xi)^2 = \int_R t^2 dF(t) - (\int_R t dF(t))^2.$$

1.5. Definition. An observable $x : \mathcal{J} \to M$ is called to be integrable if there exists

$$E(x) = \int_R t dF(t),$$

where $F : R \to [0, 1]$ is the distribution function of the observable x. The observable x is square integrable, if there exists

$$\int_R t^2 dF(t).$$

2 MV-algebras with Product

Similarly as in [4] and [5] we shall work with a further binary operation called product. Recall that an MV-algebra in an interval is an l-group $(G, +, \leq)$ and denote by - the inverse group operation, i.e. $a - a = 0$ for any $a \in G$.

2.1. Definition. An MV-algebra with product is a pair$(M, .)$, where M is an MV-algebra and . is a commutative and associative binary operation on M satisfying the following conditions:

(i) $u \,.\, a = a$ for any $a \in M$;

(ii) $a \,.((b - c) \vee 0) = (a.b - a.c) \vee 0$ for any $a, b, c \in M$.

The first problem we shall solve is the construction of sums of observables. In the classical case

$$(\xi + \eta) = g \circ T, \quad where\ T = (\xi, \eta)\,;\ \Omega \longrightarrow R^2,\ g : R^2 \longrightarrow R,\ g(u, v) = u + v.$$

Therefore

$$(\xi + \eta)^{-1} = T^{-1} \circ g^{-1}.$$

Of course,

$$g^{-1}\left((-\infty,t)\right)=\{(u,v)\,;u+v<t\}=\Delta t.$$

Hence instead of T^{-1} we may construct a mapping

$$h:\{\Delta t;t\in R\}\longrightarrow M.$$

2.2. Definition. For $t\in R$ put

$$h\left(\Delta t\right)=\bigvee_{n=1}^{\infty}\bigvee_{i=-\infty}^{\infty}x\left(\left\langle\frac{i-1}{2^n},\frac{i}{2^n}\right)\right).\,y\left(\left(-\infty,t-\frac{i}{2^n}\right)\right)$$

2.3. Theorem. Let M be a σ-complete MV-algebra with product, $x,y:\mathcal{J}\longrightarrow M$ be observables, where $a,b\in R$.

$$x\left(\langle a,b)\right)=x\left((-\infty,b)\right)-x\left((-\infty,a)\right)$$

Then the mapping $z:\mathcal{J}\longrightarrow M$ defined by

$$z\left((-\infty,t)\right)=h\left(\Delta t\right),$$

is an observable.

Proof. See [3]

2.4. Definition Let M be a σ-complete MV-algebra with product, $x,y:\mathcal{J}\longrightarrow M$ be observables. Then its sum is defined by the formula

$$\left(x+y\right)\left((-\infty,t)\right)=h\left(\Delta t\right)=h\left(g^{-1}\left((-\infty,t)\right)\right).$$

2.5. Remark. Evidently Theorem 2.3 and Definition 2.4 can be generalized for n summs and $x_1,...,x_n:\mathcal{J}\longrightarrow M$:

$$\Delta_t^n=\{(m_1,...,m_n)\,;m_1+...+m_n<t\}\,,M_n=\{\Delta_t^n;\,t\in R\}\,,$$

$$g_n\left(m_1,...m_n\right)=m_1+...+m_n$$

$$h_n:M_n\longrightarrow M,x_1+,...,+x_n\left((-\infty,t)\right)=h_n\left(\Delta_t^n\right)=h_n\left(g_n^{-1}\left((-\infty,t)\right)\right),$$

hence

$$\left(\sum_{i=1}^{n}x_i\right)\left((-\infty,t)\right)=h_n\left(g_n^{-1}\left((-\infty,t)\right)\right),$$

$$\sum_{i=1}^{n}x_i=h_n\circ g_n^{-1}.$$

The second problem solved in the section is a characterization of independency of observables in MV-algebras.

2.6. Definition. Observables $x_1,...,x_n$ are independent, if for any $t_1,...,t_n\in R$

$$m\left(h_n\left((-\infty,t_1)\times(-\infty,t_2)\times...\times(-\infty,t_n)\right)\right)=$$

$$=m\left(x_1\left((-\infty,t_1)\right)\right).m\left(x_2\left((-\infty,t_2)\right)\right).\,...\,.m\left(x_n\left((-\infty,t_n)\right)\right).$$

2.7. Remark. In the Kolmogorov theory also the following two assertions hold:

1. If $F: R \longrightarrow \langle 0,1\rangle$ is a distribution function, then there exists exactly one probability measure $\lambda_F : \mathcal{B}(R) \longrightarrow \langle 0,1\rangle$ such that

$$\lambda_F(\langle a,b)) = F(b) - F(a)$$

for any $a, b \in R,\ a \leq b$

2. If $F_1, ..., F_n : R \longrightarrow \langle 0,1\rangle$ are distribution functions, then there exists exactly one probability measure $\lambda_F : \mathcal{B}(R^n) \longrightarrow \langle 0,1\rangle$ such that

$$\lambda_F(A_1 \times A_1 \times ... \times A_n) = \lambda_{F_1}(A_1).\lambda_{F_2}(A_2)....\lambda_{F_n}(A_n)$$

for any $A_1, A_2, ..., A_n \in \mathcal{B}(R)$. Notation $\lambda_F = \lambda_{F_1} \times \lambda_{F_2} \times ... \times \lambda_{F_n}$.

2.8. Definition. An observable $x : \mathcal{J} \longrightarrow M$ is called strong, if

$$[a,b) \cap [c,d) = \emptyset \Longrightarrow (x([a,b)).\alpha) \wedge (x([c,d)).\beta) = 0$$

for any $\alpha, \beta \in M$.

2.9. Definition. A state $m : M \longrightarrow \langle 0,1\rangle$ is called σ-additive, if

$$m\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} m(a_n)$$

whenever $a_n \wedge a_m = 0\ (n \neq m)$.

2.10. Theorem. Let M be σ- complete MV-algebra with product, m be a σ-additive state, $x_1, .., x_n$ independent strong observables. Then

$$\lambda_{F_1} \times ... \times \lambda_{F_n}(\Delta_t^n) = m(h(\Delta_t^n))$$

for any $t \in R$.

Proof. We have:

$$\Delta t = \bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} \left\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right) \times \left(-\infty, t - \frac{i}{2^n}\right)$$

$$h(\Delta t) = \bigvee_n \bigvee_i x_1\left(\left\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right)\right).x_2\left(\left(-\infty, t - \frac{i}{2^n}\right)\right)$$

$$\lambda_{F_1}(\langle a,b)) = F_1(b) - F_1(a)$$

$$F_1(b) = m(x_1((-\infty, b)))$$

We shall present it for $n = 2$:

$$\lambda_{F_1} \times \lambda_{F_2}(\Delta_t) = \lambda_{F_1} \times \lambda_{F_2}\left(\bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} \left\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right) \times \left(-\infty, t - \frac{i}{2^n}\right)\right) =$$

$$= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} \lambda_{F_1} \times \lambda_{F_2}\left(\left\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right) \times \left(-\infty, t - \frac{i}{2^n}\right)\right) =$$

$$= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} \lambda_{F_1}\left(\left\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right\rangle\right) . \lambda_{F_2}\left(\left(-\infty, t-\frac{i}{2^n}\right)\right) =$$

$$= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} m\left(x_1\left(\left\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right\rangle\right)\right) . m\left(x_2\left(\left(-\infty, t-\frac{i}{2^n}\right)\right)\right) =$$

$$= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} m\left(x_1\left(\left\langle \frac{i-1}{2^n}, \frac{i}{2^n}\right\rangle\right) . x_2\left(\left(-\infty, t-\frac{i}{2^n}\right)\right)\right) =$$

$$= m\left(h_2\left(\Delta_t^n\right)\right).$$

3 Strong Law of Large Numbers

At first, recall the assertion within the Kolmogorov theory:

3.1. Theorem Let $(\Omega, \mathcal{S}, P)$ be a probability space, $(\xi_n)_n$ be a sequence of independent, square integrable random variables, such that $\sum_{n=1}^{\infty} \frac{\sigma^2(\xi_n)}{n^2} < \infty$. Then

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} \left(\frac{1}{n}\sum_{j=1}^{n} \xi_j - E(\xi_j)\right)^{-1} \left(\left(-\frac{1}{p}, \frac{1}{p}\right)\right)\right) = 1$$

Let $(x_n)_n$ be a sequence of independent observables. Recall that

$$\sum_{i=1}^{n} x_i((-\infty, t)) = h_n \circ g_n^{-1}((-\infty, t)),$$

where

$$g_n(u_1, \ldots, u_n) = u_1 + \cdots + u_n.$$

We have to define

$$\frac{1}{n}\sum_{i=1}^{n} (\xi_i - E(\xi_i))(-\infty, t).$$

Of course,

$$\frac{1}{n}\sum_{i=1}^{n} (\xi_i - E(\xi_i)) < t$$

if and only if

$$\sum_{i=1}^{n} \xi_i < nt + \sum_{i=1}^{n} E(\xi_i).$$

Therefore we define:

3.2 Definition Let M be a σ-complete MV-algebra with product, $x_1, \ldots, x_n$ be independent observables. Then we define

$$\frac{1}{n}\sum_{i=1}^{n} (x_i - E(x_i))(-\infty, t) = (\sum_{i=1}^{n} x_i)(-\infty, nt + \sum_{i=1}^{n} E(x_i)) = h_n(g_n^{-1}(-\infty, nt + \sum_{i=1}^{n} E(x_i)))$$

3.3 Theorem

Let M be a σ-complete MV-algebra with product, $m : M \to [0,1]$ be a σ-additive state, $(x_n)_n$ be an independent sequence of square integrable observables, such that $\sum_{n=1}^{\infty} \frac{\sigma^2(x_n)}{n^2} < \infty$. Then

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i}\left(\frac{1}{n}\sum_{j=1}^{n}(x_j - E(x_j))\left(-\frac{1}{p};\frac{1}{p}\right)\right)\right) = 1$$

Proof. Now we have to construct a Kolmogorov probability space. For any $n \in N$ define

$$\mu_n = \lambda_{F_1} \times \lambda_{F_2} \times ... \times \lambda_{F_n} : \mathcal{B}(R^n) \to [0,1].$$

Then $(\mu_n)_n$ presents a consistent system of probability measures, i.e.

$$\mu_{n+1}|\mathcal{B}(R^n) = \mu_n.$$

Let $\mathcal{C}$ be the family of all cylinders in R^N, i.e. all sets of the form

$$\{(u_n)_{n=1}^{\infty} \in R^N; u_{i_1} \in A_1, u_{i_2} \in A_2, ..., u_{i_k} \in A_k\}.$$

Then, using the Kolmogorov consistency theorem, there exists exactly one probability measure

$$P : \sigma(\mathcal{C}) \to [0,1]$$

such that

$$P(\pi_n^{-1}(A)) = \mu_n(A)$$

for any $n \in N, A \in \mathcal{B}(R^n)$, where $\pi_n : R^N \to R^n$ is the projection

$$\pi_n((u_i)_{i=1}^{\infty}) = (u_1, ..., u_n).$$

Now we have obtained a probability space

$$(R^N, \sigma(\mathcal{C}), P).$$

Next, define

$$\xi_k : R^N \to R$$

by the formula

$$\xi_k((u_i)_{i=1}^{\infty}) = u_k.$$

Compute

$$P_{\xi_k}((-\infty,t)) = P(\{u_n \in R^n; u_k < t\}) =$$
$$= \lambda_{F_1} \times ... \times \lambda_{F_k}(R \times R \times ... \times R \times (-\infty,t))$$
$$= \lambda_{F_k}((-\infty,t)).$$

hence $\xi_k : R^N \to R$ and $x_k : \mathcal{J} \to M$ have the same distribution function. Especially

$$E(\xi_k) = \int_R t dF_k(t) = E(x_k).$$

Moreover, put

$$\eta_n = \frac{1}{n}\sum_{i=1}^{n}(\xi_i - E(\xi_i))$$

and

$$y_n = \frac{1}{n}\sum_{i=1}^{n}(x_i - E(x_i)),$$

i.e.

$$\eta_n^{-1}((-\infty,t)) = \pi_n^{-1}(g_n^{-1}((-\infty, nt + \sum_{i=1}^{n} E(x_i))))$$

and

$$y_n((-\infty,t)) = h_n(g_n^{-1}((-\infty, nt + \sum_{i=1}^{n} E(x_i)))).$$

Then

$$P\left(\bigcap_{n=k}^{k+i}\eta_n^{-1}\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right) \leq m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right)$$

(see [7, Proposition 9.3.3]) and

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i}\left(\frac{1}{n}\sum_{j=1}^{n}(x_j - E(x_j))\left(-\frac{1}{p};\frac{1}{p}\right)\right)\right) = m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right) \geq$$

$$\geq P\left(\bigcap_{n=k}^{k+i}\eta_n^{-1}\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right) = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i}\left(\frac{1}{n}\sum_{j=1}^{n}\xi_j - E(\xi_j)\right)^{-1}\left(\left(-\frac{1}{p},\frac{1}{p}\right)\right)\right) = 1$$

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