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Properties of Fodor's intuitionistic fuzzy implication and negation

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Abstract. Some basic properties of Fodor's intuitionistic fuzzy implication and negation are formulated and checked.
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1 Introduction

The present paper is a continuation of our research [6]. There, we introduced a new operation implication, studied some of its properties and formulate open problems, related to the operations of intuitionistic fuzzy propositional calculus (see [1, 2, 4, 11, 12]. This implication was based on Janos Fodor's fuzzy implication [7], that for $a, c \in [0, 1]$ is defined by

$$a \to c = \begin{cases} 1, & \text{if } a \leq c \\ \max(1 - a, c), & \text{otherwise} \end{cases}$$

In intuitionistic fuzzy propositional calculus, if x is a variable, then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b \rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x. In [5], we called this couple an *intuitionistic fuzzy pair (IFP).*

Below we assume that for the two variables x and y the equalities: $V(x) = \langle a, b \rangle$ and $V(y) = \langle c, d \rangle$ $(a, b, c, d, a + b, c + d \in [0, 1])$ hold.

For the needs of the discussion below, we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1]) by:

$$x \text{ is an IFT if and only if for } V(x) = \langle a, b \rangle \text{ holds: } a \geq b,$$

while x will be a tautology iff $a = 1$ and $b = 0$. As in the case of ordinary logics, x is a tautology, if $V(x) = \langle 1, 0 \rangle$.

2 Preliminary results

The Fodor's intuitionistic fuzzy implication has the form (see [6])

$$V(x \to y) = \langle a, b \rangle \to \langle c, d \rangle = \langle \overline{\text{sg}}(a - c) + \text{sg}(a - c) \max(b, c), \text{sg}(a - c) \min(a, d) \rangle,$$

where we use functions sg and $\overline{\text{sg}}$ defined by,

$$\text{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}, \quad \overline{\text{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}.$$

In [6] we proved that the new implication is defined correctly and it was checked that

$$\langle 0, 0 \rangle \to \langle 0, 0 \rangle = \langle 1, 0 \rangle,$$

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$$\langle 1, 0 \rangle \to \langle 1, 0 \rangle = \langle 1, 0 \rangle,$$

i.e., the new implication has the beheviour of the standard classical logic implication.

The Fodor's intuitionistic fuzzy negation has the form (see [6])

$$\neg \langle a, b \rangle = \langle \overline{\text{sg}}(a) + \text{sg}(a) b, a \rangle.$$

It satisfies the first from the properties below as a tautology (and therefore, as an IFT), but it does not satisfy the other two properties, where

Property P1: $A \to \neg\neg A$ is a tautology (an IFT),
Property P2: $\neg\neg A \to A$ is a tautology (an IFT),
Property P3: $\neg\neg\neg A = \neg A$.

In [6] we proved that the Fodor's intuitionistic fuzzy negation does not satisfy the De Morgan Laws in their standard and modified forms (see [3]):

$$x \wedge y = \neg(\neg x \vee \neg y) \text{ and } x \vee y = \neg(\neg x \wedge \neg y),$$

$$\neg\neg x \wedge \neg\neg y = \neg(\neg x \vee \neg y),$$

$$\neg\neg x \vee \neg\neg y = \neg(\neg x \wedge \neg y).$$

Some variants of fuzzy implications (marked by $I(x, y)$) are described in [8] and the following nine axioms are discussed, where

$$I(x, y) \equiv x \to y.$$

Their validity for the Fodor's intuitionistic fuzzy implication is checked in [6].
Axiom 1 $(\forall x, y)(x \leq y \to (\forall z)(I(x, z) \geq I(y, z)))$.
Axiom 2 $(\forall x, y)(x \leq y \to (\forall z)(I(z, x) \leq I(z, y)))$.
Axiom 3 $(\forall y)(I(0, y) = 1)$.
Axiom 4 $(\forall y)(I(1, y) = y)$.
Axiom 5 $(\forall x)(I(x, x) = 1)$.
Axiom 6 $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.
Axiom 7 $(\forall x, y)(I(x, y) = 1 \text{ iff } x \leq y)$.
Axiom 8 $(\forall x, y)(I(x, y) = I(N(y), N(x)))$, where N is one of the operations for negation.
Axiom 9 I is a continuous function.

3 Main results

First, we give the 17 axioms of the intuitionistic logic (see, e.g. [9]) If A, B and C are arbitrary propositional forms, then:

(IL1) $A \to A$,

(IL2) $A \to (B \to A)$,

(IL3) $A \to (B \to (A \& B))$,

(IL4) $(A \to (B \to C)) \to (B \to (A \to C))$,

(IL5) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

(IL6) $A \to \neg\neg A$,

(IL7) $\neg(A \& \neg A)$,

(IL8) $(\neg A \vee B) \to (A \to B)$,

(IL9) $\neg(A \vee B) \to (\neg A \& \neg B)$,

(IL10) $(\neg A \& \neg B) \to \neg(A \vee B)$, (IL11) $(\neg A \vee \neg B) \to \neg(A \& B)$,

(IL12) $(A \to B) \to (\neg B \to \neg A)$,

(IL13) $(A \to \neg B) \to (B \to \neg A)$,

(IL14) $\neg\neg\neg A \to \neg A$,

(IL15) $\neg A \to \neg\neg\neg A$,

(IL16) $\neg\neg(A \to B) \to (A \to \neg\neg B)$,

(IL17) $(C \to A) \to ((C \to (A \to B)) \to (C \to B))$.

Theorem 1. The Fodor's intuitionistic fuzzy implication satisfies axioms (IL1), (IL2), (IL3), (IL6), (IL8), (IL9), (IL11), (IL14) and (IL15) as tautologies.

Proof. Let us check the validity of (IL3). Let $V(A) = \langle a, b\rangle, V(B) = \langle c, d\rangle$ for $a, b, c, d \in [0, 1]$, so that $a + b \leq 1$ and $c + d \leq 1$.

Then

$$V(A \to (B \to (A \& B)))$$
$$= \langle a, b\rangle \to (\langle c, d\rangle \to (\langle a, b\rangle \& \langle c, d\rangle))$$
$$= \langle a, b\rangle \to (\langle c, d\rangle \to (\langle \min(a, c), \max(b, d)\rangle))$$
$$= \langle a, b\rangle \to \langle \overline{\text{sg}}(c - \min(a, c)) + \text{sg}(c - \min(a, c)) \max(d, \min(a, c)),$$
$$\text{sg}(c - \min(a, c)) \min(c, \max(b, d))\rangle$$

$$= \langle \overline{\text{sg}}(a - \overline{\text{sg}}(c - \min(a, c)) - \text{sg}(c - \min(a, c)) \max(d, \min(a, c)))$$
$$+\text{sg}(a - \overline{\text{sg}}(c - \min(a, c)) - \text{sg}(c - \min(a, c)) \max(d, \min(a, c))) \max(b, \overline{\text{sg}}(c - \min(a, c))$$
$$+\text{sg}(c - \min(a, c)) \max(d, \min(a, c))), \text{sg}(a - \overline{\text{sg}}(c - \min(a, c)) + \text{sg}(c - \min(a, c)) \max(d,$$
$$\min(a, c))) \min(a, \text{sg}(c - \min(a, c)) \min(c, \max(b, d)))\rangle.$$

Let

$$X \equiv \overline{\text{sg}}(a - \overline{\text{sg}}(c - \min(a, c)) - \text{sg}(c - \min(a, c)) \max(d, \min(a, c)))$$
$$+\text{sg}(a - \overline{\text{sg}}(c - \min(a, c)) - \text{sg}(c - \min(a, c)) \max(d, \min(a, c))) \max(b, \overline{\text{sg}}(c - \min(a, c))$$
$$+\text{sg}(c - \min(a, c)) \max(d, \min(a, c))).$$

Let $a < c$. Then

$$X = \overline{\text{sg}}(a - \overline{\text{sg}}(c - a) - \text{sg}(c - a) \max(d, a)) + \text{sg}(a - \overline{\text{sg}}(c - a)$$
$$-\text{sg}(c - a) \max(d, a)) \max(b, \overline{\text{sg}}(c - a) + \text{sg}(c - a) \max(d, a))$$
$$= \overline{\text{sg}}(a - \max(d, a)) + \text{sg}(a - \max(d, a)) \max(b, \max(d, a))$$
$$= 1 + 0 = 1.$$

Let $a \geq c$. Then

$$X = \overline{\text{sg}}(a - \overline{\text{sg}}(c - c) - \text{sg}(c - c)\max(d, c)) + \text{sg}(a - \overline{\text{sg}}(c - c)$$

$$-\text{sg}(c - c)\max(d, c))\max(b, \overline{\text{sg}}(c - c) + \text{sg}(c - c)\max(d, c))$$

$$= \overline{\text{sg}}(a - 1) + \text{sg}(a - 1)\max(b, 1)$$

$$= 1 + 0 = 1.$$

Therefore, $X = 1$.

Let

$$Y \equiv \text{sg}(a - \overline{\text{sg}}(c - \min(a, c)) - \text{sg}(c - \min(a, c))\max(d, \min(a, c)))$$

$$.\min(a, \text{sg}(c - \min(a, c))\min(c, \max(b, d))).$$

Let $a < c$. Then

$$Y = \text{sg}(a - \overline{\text{sg}}(c - a) - \text{sg}(c - a)\max(d, a))\min(a, \text{sg}(c - a)\min(c, \max(b, d)))$$

$$= \text{sg}(a - \max(d, a))\min(a, \min(c, \max(b, d)))$$

$$= 0.$$

Let $a \geq c$. Then

$$Y \equiv \text{sg}(a - \overline{\text{sg}}(c - c) - \text{sg}(c - c)\max(d, c))\min(a, \text{sg}(c - c)\min(c, \max(b, d)))$$

$$= \text{sg}(a - 1)\min(a, 0) = 0.$$

Therefore, $Y = 0$.

Hence, (IL3) is a tautology.

All other assertions are proved analogously. □

Theorem 2. The Fodor's intuitionistic fuzzy implication satisfies axioms (IL1), ..., (IL4), (IL6), ..., (IL15) as IFTs.

Second, we give the list of Kolmogorov's axioms of logic (see, e.g., [10]).

(K1) $A \to (B \to A)$,

(K2) $(A \to (A \to B)) \to (A \to B))$,

(K3) $(A \to (B \to C)) \to (B \to (A \to C))$,

(K4) $(B \to C) \to ((A \to B) \to (A \to C))$,

(K5) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

Theorem 3. The Fodor's intuitionistic fuzzy implication satisfies only axiom (K1) as a tautology.

Theorem 4. The Fodor's intuitionistic fuzzy implication satisfies axioms (K1), ..., (K4) as IFTs.

Third, we give the list of Lukasiewisz-Tarski's axioms of logic (see, e.g., [10])

(LT1) $A \to (B \to A)$,

(LT2) $(A \to B) \to ((B \to C) \to (A \to C))$,

(LT3) $\neg A \to (\neg B \to (B \to A))$,

(LT4) $((A \to \neg A) \to A) \to A$.

Theorem 5. The Fodor's intuitionistic fuzzy implication satisfies axiom (LT1) and (LT3) as tautologies.

Theorem 6. The Fodor's intuitionistic fuzzy implication satisfies all axioms (LT1), ..., (LT4) as IFTs.

4 Conclusion

In next research other new implications will be introduced and studied. All they show that intuitionistic fuzzy sets and logics in the sense, described in [2, 4] correspond to the ideas of Brouwer's intuitionism, for example, related to the omission of the Law for Excluded Middle.

It is note worthy that the search of new implications and negations is important for constructing of rules for multicriteria and intercriteria analyses and their evaluations.

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References

[1] Atanassov, K. T. (1988) Two Variants of Intuitionistic Fuzzy Propositional Calculus, *Mathematical Foundations of Artificial Intelligence Seminar, Sofia, Preprint IM-MFAIS-5-88*. Reprinted: Int. J. Bioautomation, Vol. 19, 2015, Suppl. 2, S146–S155.

[2] Atanassov, K. (1999) *Intuitionistic Fuzzy Sets*. Springer, Heidelberg.

[3] Atanassov, K. (2006) On intuitionistic fuzzy negations and De Morgan Laws. *Proc. of Eleventh International Conf. IPMU* 2006, Paris, July 2–7, 2399–2404.

[4] Atanassov, K. (2012) *On Intuitionistic Fuzzy Sets*, Springer, Berlin.

[5] Atanassov, K., E. Szmidt, & J. Kacprzyk (2013) On intuitionistic fuzzy pairs, *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[6] Atanassov, K., E. Szmidt, & J. Kacprzyk (2015) On Fodors type of intuitionistic fuzzy implication and negation, *Notes on Intuitionistic Fuzzy Sets*, 21(2), 25–34.

[7] Fodor, J., & R. Roubens (1994) *Fuzzy Preference Modelling and Multicriteria Decision Support*, Springer.

[8] Klir, G. & B. Yuan (1995) *Fuzzy Sets and Fuzzy Logic*. Prentice Hall, New Jersey.

[9] Rasiowa H., & R. Sikorski (1963) *The Mathematics of Metamathematics*, Polska Akademia Nauk, Warszawa.

[10] Tabakov, M. (1986) *Logics and Axiomatics*, Nauka i Izkustvo, Sofia (in Bulgarian).

[11] Implications over intuitionistic fuzzy sets. *Ifigenia.net*. Available at: `http://www.ifigenia.org/wiki/IF_implications`

[12] Negations over intuitionistic fuzzy sets. *Ifigenia.net*. Available at: `http://www.ifigenia.org/wiki/IF_negations`