Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2026, Volume 32, Number 3, 280–295
DOI: 10.7546/nifs.2026.32.3.280-295

Intuitionistic fuzzy modal operators in an entropy-weighted TOPSIS framework

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Received: 4 September 2026 **Revised:** 30 September 2026
Accepted: 12 October 2026 **Online First:** 6 November 2026

Abstract: Parametric intuitionistic fuzzy modal operators redistribute the degree of uncertainty in intuitionistic fuzzy evaluations by transforming their membership and non-membership degrees. This paper examines the effect of applying the $E_{\alpha,\beta}$ and $Z^{\omega}_{\alpha,\beta}$ operators to the decision matrix in the entropy-weighted heuristic fuzzy TOPSIS method, where the criterion weights are derived from the Ye, Wang, Wu–Zhang, and Guo entropy measures. Since both operators are affine transformations with a common contraction coefficient of $\alpha\beta$, we show that their direct application to the evaluations in the Euclidean-based TOPSIS method does not alter the proximity coefficients; thus, the effect of these operators manifests solely through the recalculation of entropy weights and the nonlinear weighting step. A numerical example with four alternatives and six criteria, in which all details are defined, shows that in 38 of the 44 operator-entropy combinations considered, the transformations altered the ranking and, in each case, replaced the preferred alternative with a different option; the primary source of these changes is the recalculation of the weights. Analyses examining the parameters individually reveal how the values of α, β, and ω trigger these changes. The results suggest that modal operators should be considered more as a sensitivity analysis tool for the entropy-weighted TOPSIS method rather than as tools that improve the method in a way that preserves the ranking.

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Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy modal operators, Intuitionistic fuzzy entropy, TOPSIS, Multi-criteria decision-making, Sensitivity analysis.
2020 Mathematics Subject Classification: 03E72, 90B50.

1 Introduction

Fuzzy Sets Theory, introduced by Zadeh [24], provides a mathematical framework for representing imprecise and vague information through gradual membership rather than sharp boundaries. It has been widely used in areas such as fuzzy algebra, fuzzy numbers, decision-making and control systems. However, membership degrees alone may be insufficient when positive and negative information are not complementary. To address this issue, Atanassov introduced Intuitionistic Fuzzy Sets (IFS), where membership and non-membership degrees are defined independently under a basic constraint. The remaining part is called the hesitation or indeterminacy degree and represents additional uncertainty [1, 2].

Following the development of IFS theory, Atanassov introduced intuitionistic fuzzy modal operators based on the necessity and possibility concepts of modal logic [2–4]. These operators transform membership and non-membership degrees and therefore affect the hesitation degree. Later, Dencheva [12] and Atanassov [5, 6] developed parameterized forms of these operators, and Çuvalcıoğlu [10, 11] further extended them by introducing more flexible parameterized families. These developments allow intuitionistic fuzzy information to be transformed according to the needs of a decision problem.

Another important development in intuitionistic fuzzy theory is the use of entropy to measure uncertainty. Following Shannon's information-theoretic formulation [16], several entropy measures have been developed for fuzzy and intuitionistic fuzzy sets. Burillo and Bustince [8] studied entropy in intuitionistic fuzzy sets by considering hesitation, while Szmidt and Kacprzyk [17] developed approaches for measuring intuitionistic fuzzy uncertainty. Vlachos and Sergiadis [18] also contributed to the study of intuitionistic fuzzy information and its applications.

Multi-Criteria Decision-Making (MCDM) provides a framework for evaluating and ranking alternatives according to multiple criteria. Among MCDM methods, TOPSIS ranks alternatives according to their distances from positive and negative ideal solutions [14]. Intuitionistic fuzzy TOPSIS extends this approach by considering membership, non-membership, and hesitation information.

The modal operators considered here were introduced in earlier works [10, 11], and their relationship with intuitionistic fuzzy entropy has recently been studied by Bal *et al.* [7], who defined a normal σ-entropy based on the normalized Hamming distance and investigated the equalities it satisfies under $E_{\alpha,\beta}$, $Z^{\omega}_{\alpha,\beta}$ and $Z^{\omega,\theta}_{\alpha,\beta}$. The present paper does not introduce a new operator or a new entropy measure. Its aim is to examine what happens when the operators are inserted into an entropy-weighted intuitionistic fuzzy TOPSIS procedure, that is, how they affect the criterion weights, the closeness coefficients and the resulting rankings.

The contribution of this study is threefold. First, we define the entire procedure, including the position of the modal transformation, and—thanks to the affine structure of the operators—we show that the proximity coefficients in the Euclidean TOPSIS method remain unchanged when these operators are directly applied to the evaluations included in the distance calculation

(Proposition 1). This finding reveals that the recalculation of entropy weights and the nonlinear weighting step are the only channels through which the operators exert their influence. Second, using four different entropy measures (Ye, Wang, Wu–Zhang, and Guo) and eleven different parameter settings, we present a fully reproducible numerical example in which the parameters are examined individually, and we clearly indicate the changes that occur in the rankings. Third, we distinguish the contribution arising from the recalculation of weights from that arising from nonlinear weighting.

The structure of the paper is as follows: Section 2 reviews the necessary definitions, modal operators, entropy measures, and the TOPSIS procedure. Section 3 presents the algorithm, Proposition 1, and numerical results. Section 4 constitutes the conclusion of the paper. The Python implementation is provided in Appendix A.

2 Preliminaries

2.1 Intuitionistic fuzzy sets and modal operators

Definition 1. [2] *An intuitionistic fuzzy set (IFS) on a set X is an object*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A(x)$, $(\mu_A : X \to [0,1])$ *is the degree of membership of x in A,* $\nu_A(x)$, $(\nu_A : X \to [0,1])$ *is the degree of non-membership of x in A and* μ_A *and* ν_A *satisfy the following condition:*

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by IFS(X).
The hesitation degree of x is $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Some fundamental operations on intuitionistic fuzzy sets are given as follow;

Definition 2. [2] *An IFS A is said to be contained in an IFS B* $(A \sqsubseteq B)$ *if and only if, for all*

$$x \in X: \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x).$$
$$A = B \Leftrightarrow A \sqsubseteq B \text{ and } B \sqsubseteq A.$$

Definition 3. [2] *Let* $A \in \text{IFS}$ *and* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *then the complement of A is*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle : x \in X\},$$
$$A \sqcup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}\rangle : x \in X\}.$$

Some special intuitionistic fuzzy sets on *X* are defined as follows:

$$O^* = \{\langle x, 0,1\rangle : x \in X\}$$
$$X^* = \{\langle x, 1,0\rangle : x \in X\}$$

The modal operators introduced by Atanassov transform the hesitation degree of an intuitionistic fuzzy set into either the membership or non-membership component.

Definition 4. [2] *Let* $A \in IFS(X)$. *The necessity modal operator* $\square$ *is defined by*

$$\square A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}.$$

Thus, the membership degree is preserved and the entire hesitation degree is transferred to the non-membership degree.

Definition 5. [2] *Let* $A \in IFS(X)$. *The possibility modal operator* $\Diamond$*is defined by*

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}.$$

Thus, the non-membership degree is preserved and the entire hesitation degree is transferred to the membership degree [2].

Definition 6. [10] *Let* $A \in IFS(X)$ *and* $\alpha, \beta \in [0,1]$. *The parametric intuitionistic fuzzy modal operator* $E_{\alpha,\beta}$ *is defined as*

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta)\rangle : x \in X\}.$$

The parameters α and β provide independent control over the transformation of membership and non-membership information.

Definition 7. [11] *Let* $A \in IFS(X)$ *and* $\alpha, \beta, \omega \in [0,1]$. *The extended modal operator* $Z^{\omega}_{\alpha,\beta}$ *is defined by*

$$Z^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega\alpha), \alpha(\beta\nu_A(x) + \omega - \omega\beta)\rangle : x \in X\}.$$

The additional parameter ω extends the flexibility of the parametric transformation and allows the operator to be adjusted according to the characteristics of the decision problem.

2.2 Intuitionistic fuzzy entropy measures

In this study, four intuitionistic fuzzy entropy measures are employed to determine the criterion weights (Table 1). The measures introduced by Ye [23], Wang [19] and Wu [22] satisfy the axioms of Szmidt and Kacprzyk [17]; in particular, they attain the maximum value 1 if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$. The Wang measure coincides with the Szmidt–Kacprzyk entropy, as shown by Wei *et al*. [20, Theorem 3.1], and the Wu measure is the score-function-based entropy of Wu and Zhang [22, Eq. (17)]. We also note that the cosine-based simplification

$$\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{2}\cos\frac{\mu_A(x_i)-\nu_A(x_i)}{4}\pi - 1\right]\frac{1}{\sqrt{2}-1},$$

which appears in the literature as a separate measure, is identical to the Ye entropy [15, 21], and it is therefore not considered separately. The Guo criterion [13] is based on a different perspective that relies on the amount of information conveyed by an intuitive fuzzy set; in this approach, the degree of hesitation is also treated as a source of uncertainty. This measure reaches its maximum value of 1 only when $\pi_A(x_i) = 1$ for all $x_i \in X$ [13, Definition 5].

Moreover, E_G can be written as the Wu measure weighted by the factor $\frac{1+\pi_A(x_i)}{2}$, which makes the role of the hesitancy degree explicit. Using criteria from both perspectives allows for an examination of the effect of modal operators under different interpretations of intuitionistic fuzzy uncertainty.

Table 1. Entropy formulations

Entropy measure	Mathematical formulation
Ye [23]	$E_Y(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\sin\frac{\mu_A(x_i)+1-\nu_A(x_i)}{4}\pi + \sin\frac{\nu_A(x_i)+1-\mu_A(x_i)}{4}\pi - 1\right)\frac{1}{\sqrt{2}-1}\right]$
Wang [19]	$E_W(A) = \frac{1}{n}\sum_{i=1}^{n}\frac{\min\{\mu_A(x_i),\nu_A(x_i)\}+\pi_A(x_i)}{\max\{\mu_A(x_i),\nu_A(x_i)\}+\pi_A(x_i)}$
Wu [22]	$E_{Wu}(A) = \frac{1}{n}\sum_{i=1}^{n}(1-\lvert\mu_A(x_i)-\nu_A(x_i)\rvert)$
Guo [13]	$E_G(A) = \frac{1}{n}\sum_{i=1}^{n}(1-\lvert\mu_A(x_i)-\nu_A(x_i)\rvert)\frac{1+\pi_A(x_i)}{2}$

2.3 Intuitionistic fuzzy TOPSIS

Multi-Criteria Decision Making (MCDM) provides a systematic framework for evaluating alternatives based on multiple criteria. TOPSIS, one of the MCDM methods, is based on the principle that the preferred alternative must be closest to the Intuitionistic Fuzzy Positive Ideal Solution (IFPIS) and farthest from the Intuitionistic Fuzzy Negative Ideal Solution (IFNIS). In an intuitionistic fuzzy context, this principle allows for the simultaneous consideration of membership, non-membership, and uncertainty information.

3 Numerical investigation

3.1 Data and algorithm

We use the intuitionistic fuzzy decision matrix from Chen and Li's [9] numerical example, which includes four alternatives $A_1, \dots, A_4$ and six general criteria $C_1, \dots, C_6$ (Table 2). Since this example concerns only the determination of weights, for the sake of clarity, we assume that criteria C_1, C_2, C_4, C_6 are benefit criteria, while criteria C_3, C_5 are cost criteria. The entire procedure followed is presented below; a Python implementation that reproduces all numerical results is provided in Appendix A.

Step 0 *(Modal transformation).* A parametric operator is applied to each entry of the decision matrix. Both operators are encapsulated in a single expression

$$Z^{\omega}_{\alpha,\beta}(\mu,\nu) = \big(\alpha\beta\mu + \omega\beta(1-\alpha),\ \alpha\beta\nu + \omega\alpha(1-\beta)\big),$$

since $E_{\alpha,\beta} = Z^{1}_{\alpha,\beta}$, and $\alpha = \beta = 1$ gives the identity (the classical procedure).

Step 1 *(Entropy weights).* For the selected entropy measure E (Table 1), the criterion entropy is $E_j = \frac{1}{m}\sum_{i=1}^{m} E\left(x_{ij}\right)$ and the weights are

$$w_j = \frac{1 - E_j}{n - \sum_{k=1}^{n} E_k}, \qquad j = 1, \ldots, n.$$

The weights are computed from the *transformed* matrix.

Step 2 *(Weighted matrix).* $r_{ij} = w_j \otimes x_{ij} = \left\langle 1 - \left(1 - \mu_{ij}\right)^{w_j},\ \nu_{ij}^{w_j} \right\rangle$, with $\pi_{ij} = \left(1 - \mu_{ij}\right)^{w_j} - \nu_{ij}^{w_j}$.

Step 3 *(Ideal solutions).* For benefit criteria $r_j^+ = \langle \max_i \mu_{ij}, \min_i \nu_{ij} \rangle$ and $r_j^- = \langle \min_i \mu_{ij}, \max_i \nu_{ij} \rangle$, the roles are reversed for cost criteria.

Step 4 *(Distances).* $S_i^{\pm} = \left(\frac{1}{2n} \sum_{j=1}^{n} \left[\left(\mu_{ij} - \mu_j^{\pm}\right)^2 + \left(\nu_{ij} - \nu_j^{\pm}\right)^2 + \left(\pi_{ij} - \pi_j^{\pm}\right)^2\right]\right)^{1/2}$.

Step 5 *(Ranking).* $CC_i = S_i^- / (S_i^+ + S_i^-)$; alternatives are ranked by decreasing CC_i.

3.2 Effect of the operators on the closeness coefficients

Proposition 1. *Let* $\alpha\beta > 0$ *and let* $Z_{\alpha,\beta}^{\omega}$ *be applied directly to the weighted values* r_{ij} *that enter Steps 3–5, the ideal solutions being determined from the transformed values. Then every closeness coefficient* CC_i *is unchanged, provided* $S_i^+ + S_i^- \neq 0$.

Proof. The operator performs a transformation on each component of the form $t \mapsto \alpha\beta t + c$ (where $\alpha\beta > 0$); this preserves the order of membership and non-membership degrees. Therefore, the ideal solutions for the transformed values are the transformed versions of the original ideal solutions. For any two evaluations, the differences between the membership and non-membership degrees are multiplied by $\alpha\beta$; due to the relationship $\pi = 1 - \mu - \nu$, the difference between the degrees of uncertainty is also multiplied by $\alpha\beta$ in the same way. As a result, both the S_i^+ and S_i^- values are multiplied by $\alpha\beta$ and the CC_i value—which is the ratio of these two—remains unchanged. □

Consequently, the effect of modal operators in the current procedure stems from only two sources: (i) the recalculation of entropy weights from the transformed matrix (Step 1) and (ii) the nonlinearity of the weighting of the form $r_{ij} = w_j \otimes x_{ij}$ (Step 2). Table 7 distinguishes between these two effects: while the preferred alternative changes in 14 out of 44 cases when the classical weights are preserved, it changes in 38 out of 44 cases when the recalculated weights are used. Therefore, the recalculation of weights is the dominant mechanism.

Remark 1. *If* $\alpha = \beta$, *then* $\mu' - \nu' = \alpha\beta(\mu - \nu)$ *for both operators, so the Wu entropy satisfies* $1 - E'_{Wu} = \alpha\beta(1 - E_{Wu})$ *elementwise and the Wu weights are invariant. Under* $E_{\alpha,\alpha}$ *the Wu results therefore differ from the classical ones only through the nonlinear weighting (Table 7, rows* $\alpha = \beta$*).*

Remark 2. *As* α *and* β *approach 0, both operators converge to the constant evaluation at the point* $(0,0)$; *when* $\alpha = 0$ *or* $\beta = 0$, *these operators become independent of the input, and thus their proximity coefficients become undefined. For this reason, small values of the parameters do not reproduce the classical procedure; the identity condition occurs when* $\alpha = \beta = 1$.

3.3 Results

Table 2. Intuitionistic fuzzy decision matrix $\langle \mu_{ij}, \nu_{ij} \rangle$ (C1, C2, C4, C6: benefit; C3, C5: cost)

	C1 (B)	**C2 (B)**	**C3 (C)**	**C4 (B)**	**C5 (C)**	**C6 (B)**
A1	⟨0.39, 0.33⟩	⟨0.31, 0.46⟩	⟨0.11, 0.35⟩	⟨0.24, 0.69⟩	⟨0.36, 0.16⟩	⟨0.01, 0.53⟩
A2	⟨0.13, 0.58⟩	⟨0.27, 0.49⟩	⟨0.31, 0.68⟩	⟨0.91, 0.08⟩	⟨0.01, 0.14⟩	⟨0.04, 0.72⟩
A3	⟨0.39, 0.49⟩	⟨0.70, 0.10⟩	⟨0.47, 0.02⟩	⟨0.21, 0.62⟩	⟨0.01, 0.04⟩	⟨0.26, 0.11⟩
A4	⟨0.54, 0.34⟩	⟨0.95, 0.00⟩	⟨0.08, 0.02⟩	⟨0.22, 0.27⟩	⟨0.19, 0.02⟩	⟨0.08, 0.46⟩

Table 3. Criterion entropies E_j and entropy weights w_j for the original decision matrix

Measure		**C1**	**C2**	**C3**	**C4**	**C5**	**C6**
Ye	E_j	0.9332	0.6616	0.8954	0.7277	0.9772	0.7673
	w_j	0.0644	0.3261	0.1008	0.2624	0.0220	0.2243
Wang	E_j	0.7316	0.4661	0.6684	0.4807	0.8567	0.5462
	w_j	0.1193	0.2372	0.1473	0.2308	0.0637	0.2017
Wu	E_j	0.7975	0.5200	0.7200	0.5650	0.8675	0.5675
	w_j	0.1032	0.2446	0.1427	0.2217	0.0675	0.2204
Guo	E_j	0.4771	0.3181	0.5529	0.3606	0.7713	0.4235
	w_j	0.1689	0.2202	0.1444	0.2065	0.0738	0.1862

Table 4. Closeness coefficients and rankings of the classical (untransformed) procedure

Parameters	**Measure**	CC_1	CC_2	CC_3	CC_4	**Ranking**
Classical	Ye	0.2114	0.3909	0.3954	0.5934	A4 > A3 > A2 > A1
	Wang	0.2623	0.4015	0.3364	0.5825	A4 > A2 > A3 > A1
	Wu	0.2614	0.3951	0.3512	0.5835	A4 > A2 > A3 > A1
	Guo	0.2674	0.3902	0.3249	0.5934	A4 > A2 > A3 > A1

Table 5. Closeness coefficients and rankings under $E_{\alpha,\beta}$

Parameters	**Measure**	CC_1	CC_2	CC_3	CC_4	**Ranking**
$\alpha = 0.1$, $\beta = 0.9$	Ye	0.3803	0.5279	0.4232	0.4500	A2 > A4 > A3 > A1
	Wang	0.3801	0.5269	0.4257	0.4480	A2 > A4 > A3 > A1
	Wu	0.3808	0.5274	0.4249	0.4478	A2 > A4 > A3 > A1
	Guo	0.3809	0.5272	0.4258	0.4469	A2 > A4 > A3 > A1

Contd.

$\alpha = 0.4$, $\beta = 0.9$	Ye	0.3472	0.5157	0.4001	0.4928	A2 > A4 > A3 > A1
	Wang	0.3589	0.5213	0.4104	0.4682	A2 > A4 > A3 > A1
	Wu	0.3620	0.5224	0.4077	0.4692	A2 > A4 > A3 > A1
	Guo	0.3621	0.5220	0.4163	0.4588	A2 > A4 > A3 > A1
$\alpha = 0.4$, $\beta = 0.1$	Ye	0.3404	0.5195	0.4732	0.4648	A2 > A3 > A4 > A1
	Wang	0.3382	0.5225	0.4682	0.4693	A2 > A4 > A3 > A1
	Wu	0.3389	0.5222	0.4690	0.4679	A2 > A3 > A4 > A1
	Guo	0.3372	0.5231	0.4669	0.4710	A2 > A4 > A3 > A1
$\alpha = 0.9$, $\beta = 0.1$	Ye	0.3245	0.5094	0.4695	0.4824	A2 > A4 > A3 > A1
	Wang	0.3202	0.5095	0.4649	0.4885	A2 > A4 > A3 > A1
	Wu	0.3224	0.5100	0.4669	0.4857	A2 > A4 > A3 > A1
	Guo	0.3208	0.5100	0.4655	0.4879	A2 > A4 > A3 > A1
$\alpha = 0.1$, $\beta = 0.1$	Ye	0.2434	0.4764	0.5219	0.5679	A4 > A3 > A2 > A1
	Wang	0.2874	0.4969	0.4966	0.5178	A4 > A2 > A3 > A1
	Wu	0.2873	0.4969	0.4967	0.5178	A4 > A2 > A3 > A1
	Guo	0.3380	0.5252	0.4590	0.4785	A2 > A4 > A3 > A1
$\alpha = 0.9$, $\beta = 0.9$	Ye	0.1920	0.4401	0.4626	0.5417	A4 > A3 > A2 > A1
	Wang	0.2649	0.4841	0.4323	0.4920	A4 > A2 > A3 > A1
	Wu	0.2615	0.4708	0.4471	0.4966	A4 > A2 > A3 > A1
	Guo	0.2933	0.4881	0.4259	0.4879	A2 > A4 > A3 > A1

Table 6. Closeness coefficients and rankings under $Z_{\alpha,\beta}^{\omega}$

Parameters	**Measure**	CC_1	CC_2	CC_3	CC_4	**Ranking**
$\alpha = 0.1$, $\beta = 0.1$, $\omega = 0.1$	Ye	0.2891	0.5048	0.4822	0.4996	A2 > A4 > A3 > A1
	Wang	0.3182	0.5156	0.4627	0.4859	A2 > A4 > A3 > A1
	Wu	0.3181	0.5156	0.4627	0.4859	A2 > A4 > A3 > A1
	Guo	0.3372	0.5203	0.4522	0.4812	A2 > A4 > A3 > A1
$\alpha = 0.1$, $\beta = 0.9$, $\omega = 0.1$	Ye	0.4077	0.5367	0.3375	0.4847	A2 > A4 > A1 > A3
	Wang	0.3995	0.5130	0.3558	0.4912	A2 > A4 > A1 > A3
	Wu	0.4000	0.5135	0.3552	0.4908	A2 > A4 > A1 > A3
	Guo	0.3930	0.5052	0.3689	0.4920	A2 > A4 > A1 > A3
$\alpha = 0.1$, $\beta = 0.9$, $\omega = 0.9$	Ye	0.3812	0.5268	0.4190	0.4544	A2 > A4 > A3 > A1
	Wang	0.3810	0.5259	0.4217	0.4521	A2 > A4 > A3 > A1
	Wu	0.3817	0.5264	0.4209	0.4519	A2 > A4 > A3 > A1
	Guo	0.3816	0.5261	0.4218	0.4510	A2 > A4 > A3 > A1

Contd.

Parameters	Measure	CC_1	CC_2	CC_3	CC_4	Ranking
$\alpha = 0.9$, $\beta = 0.1$, $\omega = 0.9$	Ye	0.3278	0.5112	0.4714	0.4780	A2 > A4 > A3 > A1
	Wang	0.3235	0.5120	0.4662	0.4844	A2 > A4 > A3 > A1
	Wu	0.3256	0.5123	0.4682	0.4816	A2 > A4 > A3 > A1
	Guo	0.3238	0.5124	0.4665	0.4840	A2 > A4 > A3 > A1
$\alpha = 0.1$, $\beta = 0.5$, $\omega = 0.9$	Ye	0.3457	0.5295	0.4451	0.4713	A2 > A4 > A3 > A1
	Wang	0.3465	0.5276	0.4484	0.4684	A2 > A4 > A3 > A1
	Wu	0.3468	0.5280	0.4479	0.4681	A2 > A4 > A3 > A1
	Guo	0.3466	0.5272	0.4493	0.4677	A2 > A4 > A3 > A1

Table 7. Source of the ranking changes: rankings with weights recomputed from the transformed matrix versus weights kept at their classical values

Operator, parameters	Measure	Recomputed weights	Classical weights kept
$E, \alpha = 0.1, \beta = 0.9$	Ye	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
	Wang	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Wu	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Guo	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
$E, \alpha = 0.4, \beta = 0.9$	Ye	A2 > A4 > A3 > A1	A4 > A3 > A2 > A1
	Wang	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Wu	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
	Guo	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
$E, \alpha = 0.4, \beta = 0.1$	Ye	A2 > A3 > A4 > A1	A4 > A3 > A2 > A1
	Wang	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
	Wu	A2 > A3 > A4 > A1	A4 > A3 > A2 > A1
	Guo	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
$E, \alpha = 0.9, \beta = 0.1$	Ye	A2 > A4 > A3 > A1	A4 > A3 > A2 > A1
	Wang	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
	Wu	A2 > A4 > A3 > A1	A4 > A3 > A2 > A1
	Guo	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
$E, \alpha = 0.1, \beta = 0.1$	Ye	A4 > A3 > A2 > A1	A4 > A3 > A2 > A1
	Wang	A4 > A2 > A3 > A1	A4 > A2 > A3 > A1
	Wu	A4 > A2 > A3 > A1	A4 > A2 > A3 > A1
	Guo	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1

Contd.

Operator, parameters	Measure	Recomputed weights	Classical weights kept
$E, \alpha = 0.9, \beta = 0.9$	Ye	A4 > A3 > A2 > A1	A4 > A3 > A2 > A1
	Wang	A4 > A2 > A3 > A1	A4 > A2 > A3 > A1
	Wu	A4 > A2 > A3 > A1	A4 > A2 > A3 > A1
	Guo	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
$Z, \alpha = 0.1, \beta = 0.1, \omega = 0.1$	Ye	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Wang	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Wu	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Guo	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
$Z, \alpha = 0.1, \beta = 0.9, \omega = 0.1$	Ye	A2 > A4 > A1 > A3	A4 > A2 > A3 > A1
	Wang	A2 > A4 > A1 > A3	A4 > A2 > A3 > A1
	Wu	A2 > A4 > A1 > A3	A4 > A2 > A3 > A1
	Guo	A2 > A4 > A1 > A3	A4 > A2 > A3 > A1
$Z, \alpha = 0.1, \beta = 0.9, \omega = 0.9$	Ye	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
	Wang	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Wu	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Guo	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
$Z, \alpha = 0.9, \beta = 0.1, \omega = 0.9$	Ye	A2 > A4 > A3 > A1	A4 > A3 > A2 > A1
	Wang	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
	Wu	A2 > A4 > A3 > A1	A4 > A3 > A2 > A1
	Guo	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
$Z, \alpha = 0.1, \beta = 0.5, \omega = 0.9$	Ye	A2 > A4 > A3 > A1	A4 > A3 > A2 > A1
	Wang	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1
	Wu	A2 > A4 > A3 > A1	A4 > A2 > A3 > A1
	Guo	A2 > A4 > A3 > A1	A2 > A4 > A3 > A1

3.4 Analysis of ranking results

Classical procedure (Table 4). Alternative A_4 is preferred across all four entropy metrics, while A_1 ranks last. Ye ranks A_3 above A_2, whereas Wang, Wu, and Guo rank A_2 above A_3.

Ranking changes. Modal transformations alter the ranking in 38 out of 44 operator-entropy combinations (Tables 5–6), and in each of these cases, the preferred alternative shifts from $A_4$to A_2. The ranking is preserved only for the Ye, Wang, and Wu criteria and for $E_{\alpha,\alpha}$ ($\alpha = \beta = 0.1$ and $\alpha = \beta = 0.9$). A_1 remains in last place in all scenarios except for the $Z^{0.1}_{0.1,0.9}$ case; in the $Z^{0.1}_{0.1,0.9}$ case, A_1 and A_3 swap the last two positions.

Role of the parameters (Figures 1–3). Keeping one parameter constant shows that as the α value increases while $\beta = 0.9$, the leader shifts from A_2 to A_4 (within the range of approximately $\alpha = 0.5 - 0.95$, depending on the criterion; in the Wang and Guo criteria, it returns to A_2 when $\alpha = 1$). During the variation of the β parameter (β −scan), the Ye criterion yields six different rankings. During the variation of the ω parameter (ω −scan), A2 occupies the first rank for all values where $\omega \geq 0.1$. The sharp changes near the value $\beta = 1$ stem from inputs where $v_{ij} = 0$, and the v^w value is highly sensitive to these inputs.

Dependence on the entropy measure. Under the transformations in question, all four measures yield very similar proximity coefficients and identical rankings in most cases; the greatest differences arise in the case of $E_{\alpha,\alpha}$, where the classical procedure and the Ye measure diverge from the others.

These observations are specific to the example under consideration and should not be interpreted as general properties of the operators.

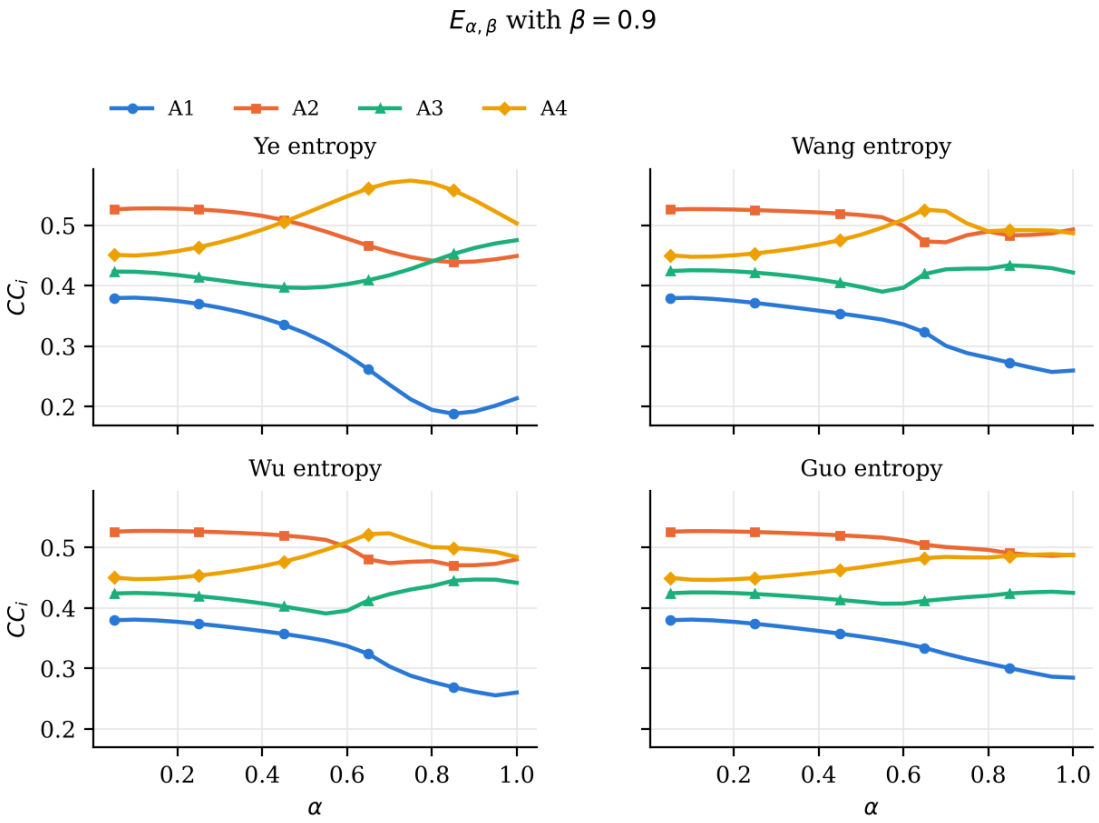


Figure 1. Closeness coefficients under $E_{\alpha,\beta}$ as α varies, with $\beta = 0.9$ fixed.

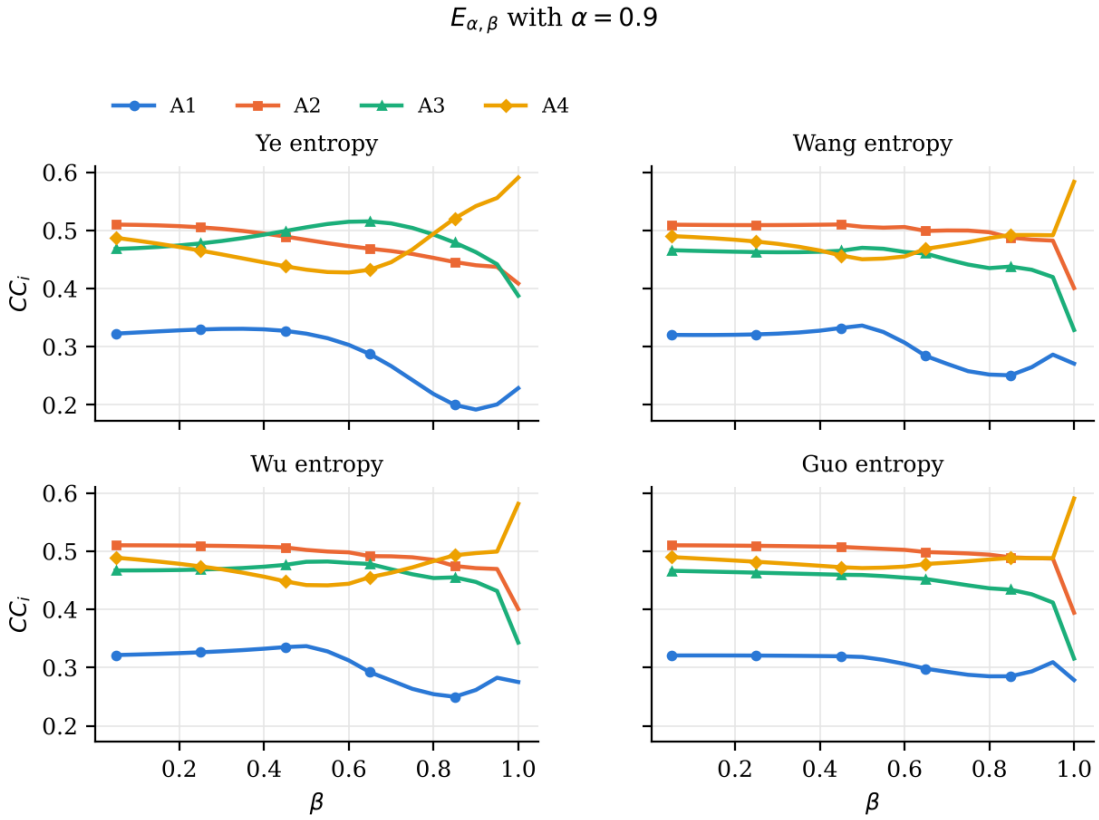


Figure 2. Closeness coefficients under $E_{\alpha,\beta}$ as β varies, with $\alpha = 0.9$ fixed.

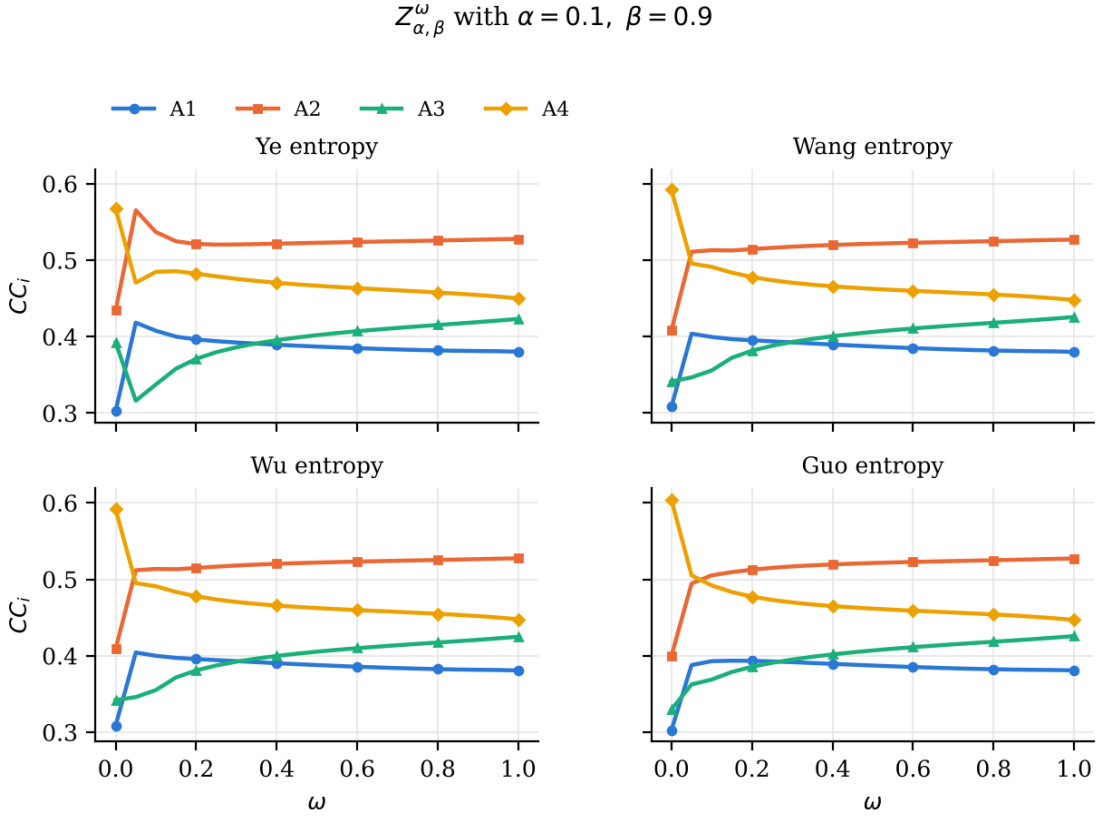


Figure 3. Closeness coefficients under $Z^{\omega}_{\alpha,\beta}$ as ω varies, with $\alpha = 0.1$ and $\beta = 0.9$ fixed ($\omega = 0$ is a pure scaling).

4 Conclusion

This paper examines the integration of the parametric intuitionistic fuzzy modal operators $E_{\alpha,\beta}$ and $Z^{\omega}_{\alpha,\beta}$ into the entropy-weighted intuitionistic fuzzy TOPSIS method. Both operators act on each criterion as an affine transformation with contraction coefficients $\alpha\beta$; therefore, the direct application of these operators to the criteria used in the Euclidean TOPSIS method does not alter the proximity coefficients (Proposition 1). In the method discussed in this study, these operators influence the decision process solely through the recalculation of entropy weights and the nonlinear $w_j \otimes x_{ij}$ weighting.

In the numerical example, the classical procedure prefers alternative A_4 across all four entropy metrics. The transformations alter the ranking in 38 of the 44 operator-entropy combinations, and in each of these cases, the preferred alternative becomes A_2; the ranking is preserved only for the $E_{\alpha,\alpha}$ operator in conjunction with the Ye, Wang, and Wu–Zhang criteria. When the classical weights are held constant, the preferred alternative changes in only 14 of the 44 combinations; this indicates that the recalculation of weights is the dominant mechanism. For the $\alpha = \beta$ case, the Wu–Zhang weights remain unchanged under both operators. Individual analyses reveal that the parameters can shift the decision outcome across different rankings, and that inputs with a non-membership value of zero make the results particularly sensitive around the value $\beta = 1$.

These findings are limited to the sample under consideration and do not reveal the general ranking characteristics of the operators. However, they demonstrate that these modal operators provide a structured method for examining the sensitivity of entropy-weighted TOPSIS decisions to systematic changes in input evaluations. Future studies may analytically identify the parameter regions where rank reversal occurs, evaluate other distance metrics and aggregation steps, extend the analysis to the $Z^{\omega,\theta}_{\alpha,\beta}$ framework, and apply this approach to real-world decision problems involving data based on expert opinions.

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Appendix A Python implementation

The following script implements Steps 0–5 from Section 3.1 using Python 3 and NumPy. When the script is run, the criterion weights from Table 3, the proximity coefficients and rankings from Tables 4–6, as well as the rankings obtained by keeping the classical weights constant (Table 7) are printed to the screen. The results obtained have been verified by comparing them with an independent spreadsheet application in which the formulas are active; the difference between the two methods is less than 10^{-12}.

```
"""Entropy-weighted IF-TOPSIS with the modal operators E and Z.
Implements Steps 0-5 of Section 3.1. Running the file prints the
criterion weights (Table 3), the closeness coefficients and rankings
(Tables 4-6) and the fixed-weight comparison (Table 7).
"""
import numpy as np


# Decision matrix (Table 2): rows A1-A4, columns C1-C6
MU = np.array([[0.39, 0.31, 0.11, 0.24, 0.36, 0.01],
               [0.13, 0.27, 0.31, 0.91, 0.01, 0.04],
               [0.39, 0.70, 0.47, 0.21, 0.01, 0.26],
               [0.54, 0.95, 0.08, 0.22, 0.19, 0.08]])
NU = np.array([[0.33, 0.46, 0.35, 0.69, 0.16, 0.53],
               [0.58, 0.49, 0.68, 0.08, 0.14, 0.72],
               [0.49, 0.10, 0.02, 0.62, 0.04, 0.11],
               [0.34, 0.00, 0.02, 0.27, 0.02, 0.46]])
BENEFIT = np.array([True, True, False, True, False, True])
ALT = ["A1", "A2", "A3", "A4"]
ENTROPIES = ["Ye", "Wang", "Wu", "Guo"]

def entropy(name, mu, nu):
    """Entropy of single IF values (Table 1), elementwise."""
    pi, d = 1 - mu - nu, mu - nu
    if name == "Ye":
        return (np.sin(np.pi * (1 + d) / 4)
                + np.sin(np.pi * (1 - d) / 4) - 1) / (np.sqrt(2) - 1)
    if name == "Wang":
        return (np.minimum(mu, nu) + pi) / (np.maximum(mu, nu) + pi)
    if name == "Wu":
        return 1 - np.abs(d)
    if name == "Guo":
        return (1 - np.abs(d)) * (1 + pi) / 2
    raise ValueError(name)

def Z_op(mu, nu, a, b, w=1.0):
    """Step 0: Z^w_{a,b}; w = 1 gives E_{a,b}, a = b = 1 the identity."""
    return b * (a * mu + w - w * a), a * (b * nu + w - w * b)

def weights(name, mu, nu):
    """Step 1: E_j = mean_i E(x_ij), w_j = (1 - E_j)/(n - sum_k E_k)."""
    Ej = entropy(name, mu, nu).mean(axis=0)
    return (1 - Ej) / (len(Ej) - Ej.sum())

def closeness(mu, nu, w):
    """Steps 2-5: weighting, ideal solutions, distances, CC_i."""
    rm, rn = 1 - (1 - mu) ** w, nu ** w                          # Step 2
```

```
    pm = np.where(BENEFIT, rm.max(0), rm.min(0))                # Step 3
    pn = np.where(BENEFIT, rn.min(0), rn.max(0))
    qm = np.where(BENEFIT, rm.min(0), rm.max(0))
    qn = np.where(BENEFIT, rn.max(0), rn.min(0))
    def dist(am, an):                                           # Step 4
        dp = (1 - rm - rn) - (1 - am - an)
        s = (rm - am) ** 2 + (rn - an) ** 2 + dp ** 2
        return np.sqrt(s.sum(axis=1) / (2 * mu.shape[1]))
    Sp, Sm = dist(pm, pn), dist(qm, qn)
    return Sm / (Sp + Sm)                                       # Step 5

def run(name, a=1.0, b=1.0, w=1.0, fixed_w=None):
    mu, nu = Z_op(MU, NU, a, b, w)
    wts = weights(name, mu, nu) if fixed_w is None else fixed_w
    return closeness(mu, nu, wts)

def ranking(cc):
    return " > ".join(ALT[i] for i in np.argsort(-cc))

if __name__ == "__main__":
    E_SET = [(0.1, 0.9), (0.4, 0.9), (0.4, 0.1),
             (0.9, 0.1), (0.1, 0.1), (0.9, 0.9)]
    Z_SET = [(0.1, 0.1, 0.1), (0.1, 0.9, 0.1), (0.1, 0.9, 0.9),
             (0.9, 0.1, 0.9), (0.1, 0.5, 0.9)]
    cases = ([("classical", (1, 1, 1))]
             + [("E", (a, b, 1)) for a, b in E_SET]
             + [("Z", p) for p in Z_SET])
    for name in ENTROPIES:
        print(name, "weights:", np.round(weights(name, MU, NU), 4))
    for op, p in cases:
        for name in ENTROPIES:
            cc = run(name, *p)
            w0 = weights(name, MU, NU)
            cc_fix = run(name, *p, fixed_w=w0)
            print(op, p, name, np.round(cc, 4), ranking(cc),
                  "| fixed weights:", ranking(cc_fix))
```