Notes on Intuitionistic Fuzzy Sets
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A study on four extended intuitionistic fuzzy modal operators

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Abstract: In this study, extended modal operators $L_B^D(A)$ and $K_B^D(A)$ are presented. The relationships between the newly defined operators and the previously introduced $T_B(A)$ and $S_B(A)$ operators with the Necessity and Possibility operators have been examined, and moreover, their algebraic properties regarding inclusion and complement have also been studied.
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1 Introduction

Fuzzy set theory was first introduced by Zadeh as an extension of crisp set theory [12]. In the subsequent years, numerous extensions of fuzzy sets were proposed. Among these, the concept of intuitionistic fuzzy sets—introduced by Atanassov in 1983 [1]—constitutes a significant enhancement by extending the truth-value domain to the set $[0,1] \times [0,1]$.

Intuitionistic fuzzy modal operators were initially introduced by Atanassov [1], followed by several notable extensions developed by various researchers [2, 3, 6–9, 11]. Atanassov also provided extended forms of these modal operators in subsequent studies [4, 5]. The algebraic properties and the interrelationships of these operators have been widely examined in the literature. More recently, an extended form of the highest extension of intuitionistic fuzzy modal operators, $X_{a,b,c,d,e,f}$, has been introduced [5].

Intuitionistic fuzzy extended modal operators $L_B^{\omega}(A), K_B^{\omega}(A), T_B(A)$ and $S_B(A)$ were defined in [10]. In this study, the new extended definitions of $L_B^{\omega}(A)$ and $K_B^{\omega}(A)$ are presented and also, algebraic properties of them are studied. The relations with other well-established intuitionistic fuzzy modal operators, $\Box(A)$ and $\Diamond(A)$, are demonstrated for new operators and for $T_B(A)$ and $S_B(A)$ modal operators.

2 Preliminaries

In this section, we present the definition of intuitionistic fuzzy sets and the definitions of algebraic operations. In addition, the definitions of the considered modal operators are also provided.

Definition 1. [1] *An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the degree of membership of x in A, $\nu_A(x), (\nu_A : X \to [0,1])$ is called the "degree of nonmembership of x in A" and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 2. [1] *An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 3. [1] *Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, then the above set is called the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}.$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\},$$
$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}.$$

The concept of intuitionistic fuzzy modal operator was first defined by Atanassov in 1983 and its basic properties were investigated.

Definition 4. [1] *Let X be the universe set and $A \in IFS(X)$, then*

1. $\Box(A) = \{\langle x, \mu_A(x),\ 1 - \mu_A(x)\rangle \mid x \in X\}$;
2. $\Diamond(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

The one type modal operators $L^{\omega}_{\alpha,\beta}$ and $K^{\omega}_{\alpha,\beta}$ were studied by Yılmaz and Bal in 2014 [11]. After then, the second type modal operators $T_{\alpha,\beta}$ and $S_{\alpha,\beta}$ were defined by Çuvalcıoğlu and Yılmaz [8].

Definition 5. [11] *Let X be the universe set and $A \in IFS(X), \alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$*

1. $L^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\rangle \mid x \in X\}$;
2. $K^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha(1-\beta)\mu_A(x) + \alpha\beta(1-\omega), \alpha\nu_A(x) + \omega(1-\alpha)\rangle \mid x \in X\}$.

Definition 6. [8] *Let X be the universe set and $A \in IFS(X), \alpha, \beta, \alpha + \beta \in [0,1]$.*

1. $T_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x) + \alpha), \alpha(\nu_A(x) + (1-\beta)\mu_A(x))\rangle \mid x \in X\}$ *where $\alpha + \beta \in [0,1]$;*
2. $S_{\alpha,\beta}(A) = \{\langle x, \alpha(\mu_A(x) + (1-\beta)\nu_A(x)), \beta(\nu_A(x) + (1-\alpha)\mu_A(x) + \alpha)\rangle \mid x \in X\}$ *where $\alpha + \beta \in [0,1]$.*

The extended modal operators were introduced in 2017 by Tarsuslu (Yılmaz) *et al* [10].

Definition 7. [10] *Let X be the universe set and $A, B \in IFS(X)$ and $\omega \in [0,1]$. The generalized modal operators are defined as follows:*

1. $$L^{\omega}_{B}(A) = \{\langle x, \mu_B(x)\mu_A(x) + \omega\big(1 - \mu_B(x)\big), \mu_B(x)\big(1 - \nu_B(x)\big)\nu_A(x) + \mu_B(x)\nu_B(x)\big(1-\omega\big)\rangle \mid x \in X\};$$
2. $$K^{\omega}_{B}(A) = \{\langle x, \mu_B(x)\big(1 - \nu_B(x)\big)\mu_A(x) + \mu_B(x)\nu_B(x)\big(1-\omega\big), \mu_B(x)\mu_A(x) + \omega\big(1 - \mu_B(x)\big)\rangle \mid x \in X\};$$
3. $$T_B(A) = \{\langle x, \nu_B(x)\big(\mu_A(x) + \big(1 - \mu_B(x)\big)\nu_A(x) + \mu_B(x)\big), \mu_B(x)\big(\nu_A(x) + \big(1 - \nu_B(x)\big)\mu_A(x)\big)\rangle \mid x \in X\};$$
4. $$S_B(A) = \{\langle x, \mu_B(x)\big(\mu_A(x) + \big(1 - \nu_B(x)\big)\nu_A(x)\big), \nu_B(x)\big(\nu_A(x) + \big(1 - \mu_B(x)\big)\mu_A(x) + \mu_B(x)\big)\rangle \mid x \in X\}.$$

3 Main results

In this section, the newly extended modal operators $L^D_B(A)$ and $K^D_B(A)$ are defined. In addition to presenting some of their algebraic properties, their relationships with the Necessity and Possibility operators are proved. Furthermore, new relationships of the previously defined operators $T_B(A)$ and $S_B(A)$ with the Necessity and Possibility operators are examined, along with some of their characteristic properties.

Definition 8. *Let X be the universe set and $A, B, D \in IFS(X)$. The extended modal operators are defined as follows:*

1. $L_B^D(A) = \{\langle x, \mu_B(x)\mu_A(x) + \mu_D(x)\big(1 - \mu_B(x)\big),$
$\mu_B(x)\big(1 - \nu_B(x)\big)\nu_A(x) + \mu_B(x)\nu_B(x)\nu_D(x)\rangle \mid x \in X\}$
2. $K_B^D(A) = \{\langle x, \mu_B(x)\big(1 - \nu_B(x)\big)\mu_A(x) + \mu_B(x)\nu_B(x)\nu_D(x),$
$\mu_B(x)\nu_A(x) + \mu_D(x)\big(1 - \mu_B(x)\big)\rangle \mid x \in X\}$

Let us prove that these definitions are correct, for $L_B^D(A)$:

$$\begin{aligned}
&\mu_B(x)\mu_A(x) + \mu_D(x)(1-\mu_B(x)) + \mu_B(x)(1-\nu_B(x))\nu_A(x) + \mu_B(x)\nu_B(x)\nu_D(x)\\
&\le \mu_B(x)\mu_A(x) + \mu_D(x)(1-\mu_B(x)) + \mu_B(x)(1-\nu_B(x))\nu_A(x) + \mu_B(x)\nu_B(x)(1-\mu_D(x))\\
&= \mu_B(x)(\mu_A(x) + \nu_A(x)) + \mu_B(x)\nu_B(x) - \mu_B(x)\nu_B(x)\nu_A(x)\\
&\quad + \mu_D(x)(1 - \mu_B(x) - \mu_B(x)\nu_B(x))\\
&\le \mu_B(x) + \mu_B(x)\nu_B(x) + 1 - \mu_B(x) - \mu_B(x)\nu_B(x)\\
&= 1.
\end{aligned}$$

$K_B^D(A)$, can also be proven in the same way.

Proposition 1. *Let X be the universe set and $A, B, D \in IFS(X)$, then $\left(L_B^D(A)\right)^c = K_B^D(A^c)$ and $L_B^D(A^c) = \left(K_B^D(A)\right)^c$.*

Proof. It is clear from the definition. □

Theorem 1. *Let X be the universe set and $A, B, D_1 D_2, \in IFS(X)$. If $D_1 \sqsubseteq D_2$, then*

1. $L_B^{D_1}(A) \sqsubseteq L_B^{D_2}(A)$
2. $K_B^{D_2}(A) \sqsubseteq K_B^{D_1}(A)$

Proof. Let $\mu_{D_1}(x) \le \mu_{D_2}(x)$ and $\nu_{D_1}(x) \ge \nu_{D_2}(x)$, then

$$\begin{aligned}
\mu_{D_1}(x)(1 - \mu_B(x)) &\le \mu_{D_2}(x)(1 - \mu_B(x)) \quad \Rightarrow\\
\mu_B(x)\mu_A(x) + \mu_{D_1}(x)(1 - \mu_B(x)) &\le \mu_B(x)\mu_A(x) + \mu_{D_2}(x)(1 - \mu_B(x))
\end{aligned}$$

and

$$\begin{aligned}
\mu_B(x)\nu_B(x)\nu_{D_1}(x) &\ge \mu_B(x)\nu_B(x)\nu_{D_2}(x) \quad \Rightarrow\\
&\quad \mu_B(x)(1 - \nu_B(x))\nu_A(x) + \mu_B(x)\nu_B(x)\nu_{D_1}(x)\\
&\ge \mu_B(x)(1 - \nu_B(x))\nu_A(x) + \mu_B(x)\nu_B(x)\nu_{D_2}(x) \quad \Rightarrow\\
L_B^{D_1}(A) &\sqsubseteq L_B^{D_2}(A).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mu_B(x)\nu_B(x)\nu_{D_1}(x) &\ge \mu_B(x)\nu_B(x)\nu_{D_2}(x) \quad \Rightarrow\\
&\quad \mu_B(x)(1 - \nu_B(x))\mu_A(x) + \mu_B(x)\nu_B(x)\nu_{D_1}(x)\\
&\ge \mu_B(x)(1 - \nu_B(x))\mu_A(x) + \mu_B(x)\nu_B(x)\nu_{D_2}(x)
\end{aligned}$$

and

$$
\begin{aligned}
\mu_{D_1}\left(1-\mu_B\left(x\right)\right) &\leq \mu_{D_2}\left(1-\mu_B\left(x\right)\right) \quad \Rightarrow \\
\mu_B\left(x\right)\nu_A\left(x\right)+\mu_{D_1}\left(1-\mu_B\left(x\right)\right) &\leq \mu_B\left(x\right)\nu_A\left(x\right)+\mu_{D_2}\left(1-\mu_B\left(x\right)\right) \quad \Rightarrow \\
K_B^{D_2}\left(A\right) &\sqsubseteq K_B^{D_1}\left(A\right).
\end{aligned}
$$

□

Proposition 2. *Let X be the universe set and $A, B, D \in IFS(X)$, then*

1. $L_B^D\left(\Box A\right) \sqsubseteq L_B^D\left(\Diamond A\right)$
2. $K_B^D\left(\Box A\right) \sqsubseteq K_B^D\left(\Diamond A\right)$

Proof. Since $\mu_A\left(x\right) \leq 1-\nu_A\left(x\right)$, then

$$
\mu_B\left(x\right)\mu_A\left(x\right)+\mu_D\left(x\right)\left(1-\mu_B\left(x\right)\right) \leq \mu_B\left(x\right)\left(1-\nu_A\left(x\right)\right)+\mu_D\left(x\right)\left(1-\mu_B\left(x\right)\right)
$$

and

$$
\begin{aligned}
&\mu_B\left(x\right)\left(1-\nu_B\left(x\right)\right)\left(1-\mu_A\left(x\right)\right)+\mu_B\left(x\right)\nu_B\left(x\right)\nu_D\left(x\right) \\
\geq\ &\mu_B\left(x\right)\left(1-\nu_B\left(x\right)\right)\nu_A\left(x\right)+\mu_B\left(x\right)\nu_B\left(x\right)\nu_D\left(x\right).
\end{aligned}
$$

So, $L_B^D\left(\Box A\right) \sqsubseteq L_B^D\left(\Diamond A\right)$. It can be proved by analogy for K_B^D. □

Proposition 3. *Let X be the universe set and $A, B, D \in IFS(X)$, then*

1. $K_B^{\Diamond D}\left(A^c\right) \sqsubseteq \left(L_B^{\Box D}\left(A\right)\right)^c$
2. $\left(K_B^{\Box D}\left(A\right)\right)^c \sqsubseteq L_B^{\Diamond D}\left(A^c\right)$

Proof. 1. With the inequality $\nu_D\left(x\right) \leq 1-\mu_D\left(x\right)$,

$$
\begin{aligned}
\mu_B\left(x\right)\nu_B\left(x\right)\nu_D\left(x\right) &\leq \mu_B\left(x\right)\nu_B\left(x\right)\left(1-\mu_D\left(x\right)\right) \quad \Rightarrow \\
&\quad\ \mu_B\left(x\right)\left(1-\nu_B\left(x\right)\right)\nu_A\left(x\right)+\mu_B\left(x\right)\nu_B\left(x\right)\nu_D\left(x\right) \\
&\leq \mu_B\left(x\right)\left(1-\nu_B\left(x\right)\right)\nu_A\left(x\right)+\mu_B\left(x\right)\nu_B\left(x\right)\left(1-\mu_D\left(x\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(1-\nu_D\left(x\right)\right)\left(1-\mu_B\left(x\right)\right) &\geq \mu_D\left(x\right)\left(1-\mu_B\left(x\right)\right) \quad \Rightarrow \\
\mu_B\left(x\right)\mu_A\left(x\right)+\left(1-\nu_D\left(x\right)\right)\left(1-\mu_B\left(x\right)\right) &\geq \mu_B\left(x\right)\mu_A\left(x\right)+\mu_D\left(x\right)\left(1-\mu_B\left(x\right)\right).
\end{aligned}
$$

$K_B^{\Diamond D}\left(A^c\right) \sqsubseteq \left(L_B^{\Box D}\left(A\right)\right)^c$ inclusion provided.

2. Since $\mu_D\left(x\right) \leq 1-\nu_D\left(x\right)$, then

$$
\mu_B\left(x\right)\nu_A\left(x\right)+\mu_D\left(x\right)\left(1-\mu_B\left(x\right)\right) \leq \mu_B\left(x\right)\nu_A\left(x\right)+\left(1-\nu_D\left(x\right)\right)\left(1-\mu_B\left(x\right)\right)
$$

and

$$
\begin{aligned}
&\mu_B\left(x\right)\mu_A\left(x\right)\left(1-\nu_B\left(x\right)\right)+\mu_B\left(x\right)\nu_B\left(x\right)\left(1-\mu_D\left(x\right)\right) \\
\geq\ &\mu_B\left(x\right)\left(1-\nu_B\left(x\right)\right)\mu_A\left(x\right)+\mu_B\left(x\right)\nu_B\left(x\right)\nu_D\left(x\right).
\end{aligned}
$$

So, $\left(K_B^{\Box D}\left(A\right)\right)^c \sqsubseteq L_B^{\Diamond D}\left(A^c\right)$. □

The extended forms of the modal operators T_B and S_B were defined by Tarsuslu (Yılmaz) *et al* [10]. Now, we will investigate some new results.

Theorem 2. *Let X be the universe set and $A, B \in IFS(X)$. If $A \sqsubseteq B$, then*

1. $T_B(A) \sqsubseteq T_A(B)$
2. $S_A(B) \sqsubseteq S_B(A)$

Proof. Since $A \sqsubseteq B$, then $\nu_A(x) - \nu_B(x) \geq 0$ and $\mu_B(x) - \mu_A(x) \geq 0$. Now,

$$\begin{aligned} 0 &\leq (\mu_A(x) + \mu_B(x))(\nu_A(x) - \nu_B(x)) + \nu_A(x)\nu_B(x)(\mu_B(x) - \mu_A(x)) \quad \Rightarrow \\ &\quad \nu_B(x)\mu_A(x) - \nu_A(x)\nu_B(x)\mu_B(x) + \mu_B(x)\nu_B(x) \\ &\leq \nu_A(x)\mu_B(x) - \nu_A(x)\nu_B(x)\mu_A(x) + \mu_A(x)\nu_A(x) \end{aligned}$$

and

$$\begin{aligned} \mu_A(x)\mu_B(x)(\nu_A(x) - \nu_B(x)) &\geq 0 \text{ and } \mu_B(x)\nu_A(x) \geq \mu_A(x)\nu_B(x) \quad \Rightarrow \\ \mu_B(x)\nu_A(x) - \mu_A(x)\mu_B(x)\nu_B(x) &\geq \nu_B(x)\mu_A(x) - \mu_A(x)\mu_B(x)\nu_A(x) \end{aligned}$$

hold. This completes the proof. The other inclusion can be similarly shown. □

Theorem 3. *Let X be the universe set and $A, B \in IFS(X)$. Then the following inclusions hold:*

1. $\left(T_{\Box(B)}(A)\right)^c \sqsubseteq T_{\Diamond(B^c)}(A^c)$
2. $S_{\Diamond(B^c)}(A^c) \sqsubseteq \left(S_{\Box(B)}(A)\right)^c$
3. $\Diamond(T_{B^c}(A^c)) \sqsubseteq (\Box(T_B(A)))^c$
4. $\Diamond(S_{B^c}(A^c)) \sqsubseteq (\Box(S_B(A)))^c$

Proof. We will examine inclusions 1. and 4., and the rest are proven by analogy.

1. Since $\mu_B(x)(1 - \mu_B(x)) \geq 0$, then

$$\mu_B(x)(\nu_A(x) + \mu_A(x)\mu_B(x)) \leq \mu_B(x)(\nu_A(x) + \mu_A(x)\mu_B(x)) + \mu_B(x)(1 - \mu_B(x))$$

and

$$\begin{aligned} &\mu_A(x)(1 - \mu_B(x)) + (1 - \mu_B(x))^2 \nu_A(x) \\ \leq\ &\mu_A(x)(1 - \mu_B(x)) + (1 - \mu_B(x))^2 \nu_A(x) + \mu_B(x)(1 - \mu_B(x)). \end{aligned}$$

4. It is clear that

$$\begin{aligned} &1 - (\mu_B(x)(\mu_A(x) + (1 - \nu_B(x))\nu_A(x) + \nu_B(x))) \\ \leq\ &1 - (\mu_B(x)(\mu_A(x) + (1 - \nu_B(x))\nu_A(x))) \end{aligned}$$

and

$$\mu_B(x)(\mu_A(x) + (1 - \nu_B(x))\nu_A(x) + \nu_B(x)) \geq \mu_B(x)(\mu_A(x) + (1 - \nu_B(x))\nu_A(x)).$$

Thus the proof is completed. □

4 Conclusion

A generalization of the operators $L^{\omega}_{\alpha,\beta}(A)$ and $K^{\omega}_{\alpha,\beta}(A)$ was introduced in 2017 [10]. The novelty of the present work is to define an extension of these operators based on the D-intuitionistic fuzzy set. In addition, the fundamental algebraic properties of these operators and their relationships with the operators $\Box A$ and $\Diamond A$ are investigated. Similarly, some properties of the operators $T_B(A)$ and $S_B(A)$, which were also presented in 2017, are examined.

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