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Properties of the intuitionistic fuzzy implication $\rightarrow_{189}$

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Abstract: In [7], the new intuitionistic fuzzy implication $\rightarrow_{189}$ was defined and some of its properties were studied. Here, new properties of this implication will be discussed.
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1 Introduction

The present paper is a continuation of author's research over the new intuitionistic fuzzy implication $\rightarrow_{189}$, defined in [7]. The text is influenced by results in [2, 4, 5], related to two other intuitionistic fuzzy implications ($\rightarrow_{187}$ and $\rightarrow_{188}$), introduced by K. Atanassov, E. Szmidt and J. Kacprzyk. In their paper [6], five new intuitionistic fuzzy operations were defined and their properties were studied. On their basis, implication $\rightarrow_{189}$ was defined.

All notations and definitions in the text are used from [1, 7].

Here, following [1], we define only the classical operations "negation", "disjunction" and "conjunction". If the truth-values of variables p are q are $\langle a, b\rangle$ and $\langle c, d\rangle$, respectively, then these operations are the follows:

$$V(\neg_1 p) = \langle b, a\rangle,$$

$$V(p \vee q) = \langle \max(a, c), \min(b, d)\rangle,$$

$$V(p \wedge q) = \langle \min(a, c), \max(b, d)\rangle.$$

In [3], K. Atanassov, E. Szmidt and J. Kacprzyk called the object $\langle a, b\rangle$ an Intuitionistic Fuzzy Pair (IFP), where $a, b, a + b \in [0, 1]$.

In [7], the intuitionistic fuzzy implication $\rightarrow_{189}$ was defined by:

$$x \rightarrow_{189} y = \langle bc, ad\rangle.$$

and it was shown that

$$\neg\langle a, b\rangle = \langle a, b\rangle \rightarrow_{189} \langle 0, 1\rangle = \langle 0, a\rangle,$$

therefore, we obtain a new, 54-th intuitionistic fuzzy negation

$$\neg_{54}\langle a, b\rangle = \langle 0, a\rangle.$$

2 Main results

For brevity, below we will write $\rightarrow$ instead of $\rightarrow_{189}$.

In [7], it was checked the validity of G. F. Rose's formula [9, 11] that has the form:

$$((\neg\neg x \rightarrow x) \rightarrow (\neg\neg x \vee \neg x)) \rightarrow (\neg\neg x \vee \neg x),$$

which in the present case has the form:

$$((\neg_1\neg_1 x \rightarrow_{189} x) \rightarrow_{189} (\neg_1\neg_1 x \vee \neg_1 x)) \rightarrow_{189} (\neg_1\neg_1 x \vee \neg_1 x)$$

when negation is the classical negation $\neg_1$. Now, we prove

Theorem 1. *Rose's formula is an IFT for* $\neg_{54}$ *and* $\rightarrow_{189}$.

Proof. Sequentially, we obtain:

$$\begin{aligned}
&((\neg_{54}\neg_{54} x \rightarrow x) \rightarrow (\neg_{54}\neg_{54} x \vee \neg_{54} x)) \rightarrow (\neg_{54}\neg_{54} x \vee \neg_{54} x)\\
&= ((\neg_{54}\neg_{54}\langle a, b\rangle \rightarrow \langle a, b\rangle) \rightarrow (\neg_{54}\neg_{54}\langle a, b\rangle \vee \neg_{54}\langle a, b\rangle))\\
&\rightarrow (\neg_{54}\neg_{54}\langle a, b\rangle \vee \neg_{54}\langle a, b\rangle)\\
&= ((\neg_{54}\langle 0, a\rangle \rightarrow \langle a, b\rangle) \rightarrow (\neg_{54}\langle 0, a\rangle \vee \neg_{54}\langle a, b\rangle))\\
&\rightarrow (\neg_{54}\langle 0, a\rangle \vee \neg_{54}\langle a, b\rangle)\\
&= (((\langle 0, 0\rangle \rightarrow \langle a, b\rangle) \rightarrow (\langle 0, 0\rangle \vee \langle 0, a\rangle)) \rightarrow (\langle 0, 0\rangle \vee \langle 0, a\rangle)\\
&= (\langle 0, 0\rangle \rightarrow (\langle 0, 0\rangle)) \rightarrow \langle 0, 0\rangle\\
&= \langle 0, 0\rangle \rightarrow \langle 0, 0\rangle) = \langle 0, 0\rangle,
\end{aligned}$$

which is an IFT. □

Second, we check C. A. Meredith's axiom (see, e.g., [8]).

Theorem 2. *For every five formulas A, B, C, D and E, Meredith's axiom*

$$((((A \to B) \to (\neg C \to \neg D)) \to C) \to E) \to ((E \to A) \to (D \to A))$$

is an IFT for $\to_{189}$ and for $\neg_{54}$.

Proof. Let $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$, $V(C) = \langle e, f\rangle$, $V(D) = \langle g, h\rangle$, $V(E) = \langle i, j\rangle$, where $a, b, \dots, j \in [0, 1]$ and $a + b \leq 1$, $c + d \leq 1$, $e + f \leq 1$, $g + h \leq 1$ and $i + j \leq 1$. Then

$$V(((((A \to B) \to (\neg_{54} C \to \neg_{54} D)) \to C) \to E)$$

$$\to ((E \to A) \to (D \to A)))$$

$$= (((((\langle a, b\rangle \to \langle c, d\rangle) \to (\langle 0, e\rangle \to \langle 0, g\rangle)) \to \langle e, f\rangle) \to \langle i, j\rangle)$$

$$\to (((\langle i, j\rangle \to \langle a, b\rangle) \to (\langle g, h\rangle \to \langle a, b\rangle)))$$

$$= ((((\langle bc, ad\rangle \to \langle 0, 0\rangle) \to \langle e, f\rangle) \to \langle i, j\rangle) \to (\langle aj, bi\rangle \to \langle ah, bg\rangle)$$

$$= (((\langle 0, 0\rangle \to \langle e, f\rangle) \to \langle i, j\rangle) \to \langle abhi, abgj\rangle$$

$$= (\langle 0, 0\rangle \to \langle i, j\rangle) \to \langle \min(bi, a, h), \min(a, j)bg\rangle$$

$$= (\langle 0, 0\rangle \to \langle i, j\rangle) \to \langle \min(bi, a, h), \min(a, j)bg\rangle$$

$$= \langle 0, 0\rangle.$$

□

The next assertions are proved in the same manner so we will omit their proofs.

The axioms of the intuitionistic logic (see, e.g., [10]) are the following.

(IL1) $A \to A$,

(IL2) $A \to (B \to A)$,

(IL3) $A \to (B \to (A \wedge B))$,

(IL4) $(A \to (B \to C)) \to (B \to (A \to C))$,

(IL5) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

(IL6) $A \to \neg\neg A$,

(IL7) $\neg(A \wedge \neg A)$,

(IL8) $(\neg A \vee B) \to (A \to B)$,

(IL9) $\neg(A \vee B) \to (\neg A \wedge \neg B)$,

(IL10) $(\neg A \wedge \neg B) \to \neg(A \vee B)$,

(IL11) $(\neg A \vee \neg B) \to \neg(A \wedge B)$,

(IL12) $(A \to B) \to (\neg B \to \neg A)$, (IL13) $(A \to \neg B) \to (B \to \neg A)$,

(IL14) $\neg\neg\neg A \to \neg A$,

(IL15) $\neg A \to \neg\neg\neg A$,

(IL16) $\neg\neg(A \to B) \to (A \to \neg\neg B)$,

(IL17) $(C \to A) \to ((C \to (A \to B)) \to (C \to B))$.

Theorem 3. *Axioms IL1, IL4, IL6, IL7, IL9, ..., IL16 are IFTs, but none of them is a tautology.*

The fact that the rest axioms are not IFTs is checked with finding counterexamples. For example, for IL2 we have:

$$V(A \to (B \to A)) = \langle a, b\rangle \to \langle ab, ab\rangle = \langle ab^2, a^2b\rangle.$$

Now, for $a = \frac{1}{2}, b = \frac{1}{3}$ we obtain that

$$\langle ab^2, a^2b\rangle = \langle \frac{1}{18}, \frac{1}{12}\rangle$$

which, obviously, is not an IFT.

The axioms of A. Kolmogorov (see, e.g., [12]) are the following.

(K1) $A \to (B \to A)$,

(K2) $(A \to (A \to B)) \to (A \to B))$,

(K3) $(A \to (B \to C)) \to (B \to (A \to C))$,

(K4) $(B \to C) \to ((A \to B) \to (A \to C))$,

(K5) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

Theorem 4. *Only axioms K3, K4, and K5 of Kolmogorov are IFTs.*

The axioms of J. Łukasiewicz and A. Tarski (see, e.g., [12]) are the following.

(LT1) $A \to (B \to A)$,

(LT2) $(A \to B) \to ((B \to C) \to (A \to C))$,

(LT3) $\neg A \to (\neg B \to (B \to A))$,

(LT4) $((A \to \neg A) \to A) \to A$.

Theorem 5. *Only axioms LT2 and LT3 of Łukasiewicz and Tarski are IFTs.*

3 Conclusion

In a next research, we will study the forms and properties of the conjunctions and disjunctions that can be constructed on the basis of the implication $\to_{189}$ and negation $\neg_{54}$.

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