Interval Valued Fuzzy Sets, Mass Assignment and Possibility Assignment

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Background

- There are many variants of Fuzzy Sets.
  - Fuzzy Sets
  - Interval Valued Fuzzy Sets
  - Mass Assignment
- There are relationships between these represenatations
  - This paper explores Interval Valued Fuzzy Sets and Mass Assignment
- Introduces Possibility Mass Assignment

Definitions

Definition 1 (Fuzzy sets) *Let* $\mathbb{U}$ *denote a universe of discourse. Then a fuzzy set* A *in* $\mathbb{U}$ *is defined as a set of ordered pairs*

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in \mathbb{U}\}$$

where $\mu_A : \mathbb{U} \longrightarrow [0, 1]$ *is a function of* A *that delivers the grade of membership of* x *in* A*.*

Definitions

Definition 2 (Interval Valued Fuzzy Sets) *Let* $D[0,1]$ *be the set of all closed subintervals of the interval [0,1], and* $\mathbb{U}$ *denote a universe of discourse.*

$$A^{+} = \{\langle x, \mu_A(x)\rangle\} \mid x \in \mathbb{U}\}$$

where $\mu_A : \mathbb{U} \longrightarrow D[0,1]$ *is a function of* A *that delivers the grade of membership of* x *in* A*.*

Definitions

Alternatively

Definition 3 (Interval Valued Fuzzy Sets) *Let $\mathbb{U}$ denote a universe of discourse. Then an interval valued fuzzy set A in $\mathbb{U}$ is defined as a set of ordered pairs*

$$A = \{\left\langle x, [\underline{\mu}_A(x), \overline{\mu}_A(x)]\right\rangle \mid x \in \mathbb{U}\}$$

where $\overline{\mu}_A : \mathbb{U} \longrightarrow [0,1]$ is a function of A that delivers the upper grade of membership of x in A and $\underline{\mu}_A : \mathbb{U} \longrightarrow [0,1]$ is a function of A that delivers the lower grade of membership of x in A.

Mass Assignment

- Mass Assignment is a way of linking fuzzy sets with probability
- Events are assigned to subsets of the Domain of discourse $\mathbb{U} = \{a, b, c\}$
  - $\mathrm{m}_A(\mathbb{U}) = 1.0$
    - Means all events belong to the Universe
    - $\mathrm{Pl}_A(\mathbb{U}) = 1.0$
    - $\mathrm{Bel}_A(\mathbb{U}) = 1.0$

Mass Assignment

- $\mathrm{Pl}_A(X) \underline{\triangle}$ the possibility of the set X in A
- $\mathrm{Bel}_A(X) \underline{\triangle}$ the belief in the set X in A

Mass Assignment

- $\mathrm{m}_A(\{a,b\}) = 0.8, \mathrm{m}_A(\{a\}) = 0.2$
  - $\mathrm{Pl}_A(\mathbb{U}) = 1.0$
  - $\mathrm{Bel}_A(\mathbb{U}) = 1.0$
  - $\mathrm{Pl}_A(\{a,b\}) = 1.0$
  - $\mathrm{Bel}_A(\{a,b\}) = 1.0$
  - $\mathrm{Pl}_A(\{a\}) = 1.0$
  - $\mathrm{Bel}_A(\{a\}) = 0.2$
  - $\mathrm{Pl}_A(\{b\}) = 0.8$
  - $\mathrm{Bel}_A(\{b\}) = 0.0$

Mass Assignment

- $\mathrm{Pl}^{+}{}_{A}(X)\underline{\triangle}$ the possibility of the set X in A including $\emptyset$
- $\mathrm{Bel}^{+}{}_{A}(X)\underline{\triangle}$ the belief in the set X in A including $\emptyset$
- $\mathrm{Pl}^{-}{}_{A}(X)\underline{\triangle}$ the possibility of the set X in A not including $\emptyset$
- $\mathrm{Bel}^{-}{}_{A}(X)\underline{\triangle}$ the belief in the set X in A not including $\emptyset$

Mass Assignment

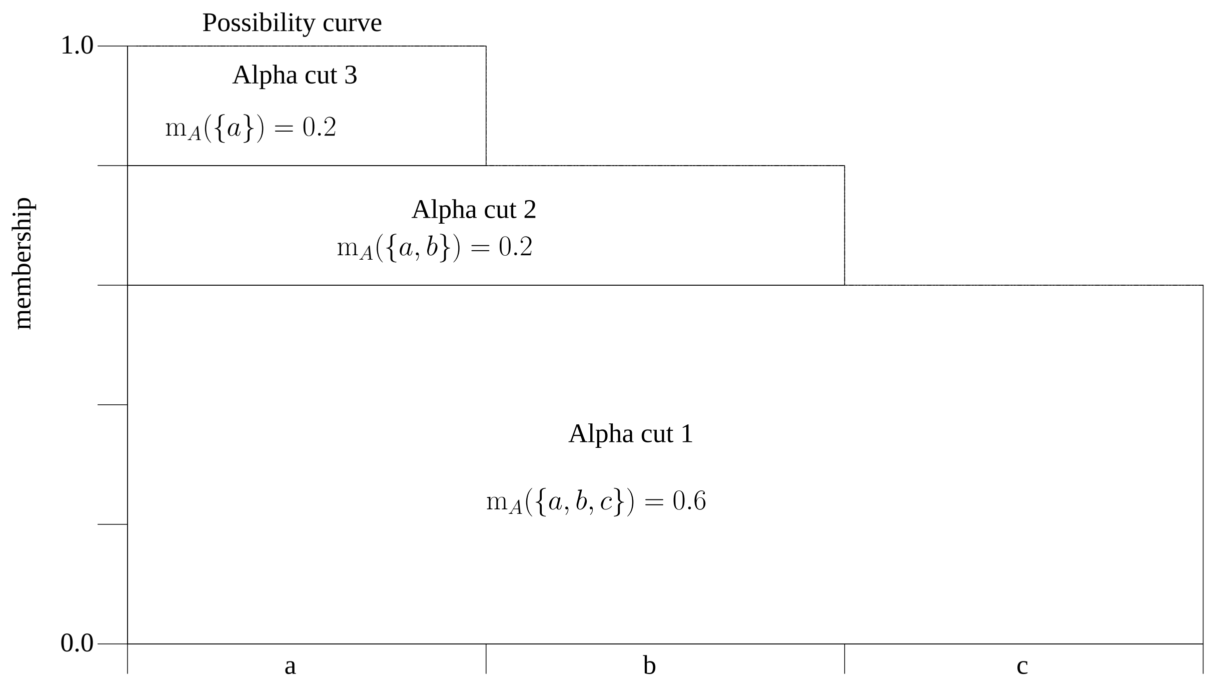
- $\mathrm{m}_A(\{a,b\}) = 0.6, \mathrm{m}_A(\{a\}) = 0.2, \mathrm{m}_A(\{\}) = 0.2$
  - $\mathrm{Pl}^+(\mathbb{U}) = 1.0$
  - $\mathrm{Pl}^-(\mathbb{U}) = 0.8$
  - $\mathrm{Bel}^+(\mathbb{U}) = 1.0$
  - $\mathrm{Bel}^-(\mathbb{U}) = 0.8$
  - $\mathrm{Pl}^-(\{a,b\}) = 0.8$
  - $\mathrm{Bel}^-(\{a,b\}) = 0.8$
  - $\mathrm{Pl}^-(\{a\}) = 0.8$
  - $\mathrm{Bel}^-(\{a\}) = 0.2$
  - $\mathrm{Pl}^-(\{b\}) = 0.6$
  - $\mathrm{Bel}^-(\{b\}) = 0.0$
- Fuzzy set is now not normalised, and inconsistent

Mass Assignment and Intervals

- Intervals may be obtained from Mass Assignments
- Can Mass Assignments be obtained from Intervals?

Mass Assignment and Fuzzy Sets

$$A = \{\langle a, 1.0\rangle, \langle b, 0.8\rangle, \langle c, 0.6\rangle\}$$



Showing the possibility curve, the alpha cuts and the induced mass assignments

Mass Assignment and Fuzzy Sets

- Fuzzy Set
  - $$A = \{\langle a, 1.0\rangle, \langle b, 0.8\rangle, \langle c, 0.6\rangle\}$$
- Possibilities
  - $$\underset{A}{\mathrm{Pl}}^{-}(\{a\}) = \underset{A}{\mathrm{Pl}}^{+}(\{a\}) = 1.0, \qquad \underset{A}{\mathrm{Pl}}^{-}(\{b\}) = \underset{A}{\mathrm{Pl}}^{+}(\{b\}) = 0.8$$
    $$\underset{A}{\mathrm{Pl}}^{-}(\{c\}) = \underset{A}{\mathrm{Pl}}^{+}(\{c\}) = 0.6$$
- Masses
  - $$\underset{A}{\mathrm{m}}(\{a\}) = 0.2, \qquad \underset{A}{\mathrm{m}}(\{a, b\}) = 0.2$$
    $$\underset{A}{\mathrm{m}}(\{a, b, c\}) = 0.6$$

Mass Assignment and Fuzzy Sets

- Beliefs
  -

$$\underset{A}{\mathrm{Bel}}^{-}(\{a\}) = 0.2, \qquad \underset{A}{\mathrm{Bel}}^{+}(\{a\}) = 0.2$$
$$\underset{A}{\mathrm{Bel}}^{-}(\{b\}) = 0.0, \qquad \underset{A}{\mathrm{Bel}}^{+}(\{b\}) = 0.0$$
$$\underset{A}{\mathrm{Bel}}^{-}(\{c\}) = 0.0, \qquad \underset{A}{\mathrm{Bel}}^{+}(\{c\}) = 0.0$$

- Possibilities
  -

$$\underset{A}{\mathrm{Pl}}^{-}(\{a\}) = \underset{A}{\mathrm{Pl}}^{+}(\{a\}) = 1.0, \qquad \underset{A}{\mathrm{Pl}}^{-}(\{b\}) = \underset{A}{\mathrm{Pl}}^{+}(\{b\}) = 0.8$$
$$\underset{A}{\mathrm{Pl}}^{-}(\{c\}) = \underset{A}{\mathrm{Pl}}^{+}(\{c\}) = 0.6$$

- Intervals
  -

$$A = \{\langle a, [0.2, 1.0]\rangle, \langle b, [0.0, 0.8]\rangle, \langle c, [0.0, 0.6]\rangle\}$$

Mass Assignment and Intervals

- Intervals may be obtained from Mass Assignments
- Can Mass Assignments be obtained from Intervals?

Mass Assignment and Intervals

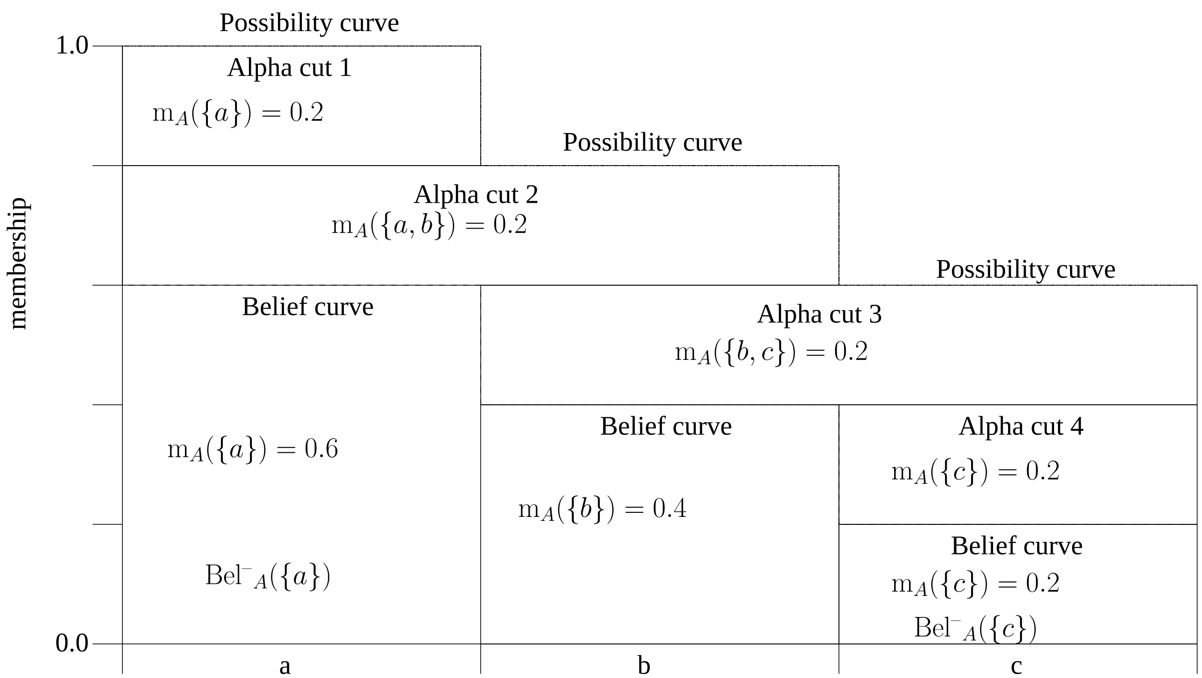
- Intervals may be obtained from Mass Assignments
- Can Mass Assignments be obtained from Intervals?
  - Are there problems?

Mass Assignment and Intervals

- Kyborg proves that not all intervals can be represented as a Mass Assignment
- Given a normalised Interval Valued Fuzzy Set then
  - no mass can be assigned to any singletons apart from the supremum possibility
- Given an unnormalised Interval Valued Fuzzy Set then
  - only mass up to the degree of unnormalisation
    - can be assigned to any singletons apart from the supremum possibility set
      - often a singleton set

Mass Assignment and Intervals

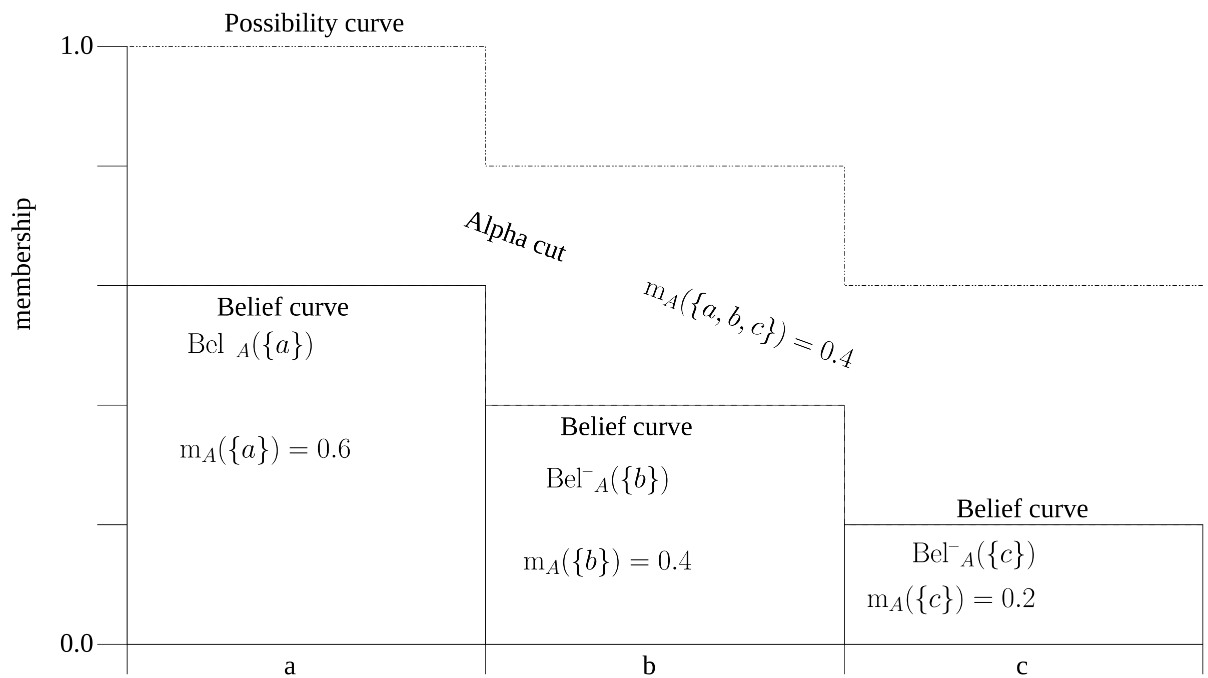
$$A = \{\langle a, [0.6, 1.0]\rangle, \langle b, [0.4, 0.8] =\rangle, \langle c, [0.2, 0.6]\rangle\}$$



Showing the possibility curve, the horizontal alpha cuts and the induced mass assignments

Mass Assignment and Intervals

$$A = \{\langle a, [0.6, 1.0]\rangle\ , \langle b, [0.4, 0.8] =\rangle\ , \langle c, [0.2,]0.6\rangle\}$$



Showing the possibility curve, the horizontal alpha cuts and the induced mass assignments
Clear that the total exceeds 1.0, by quite a margin

Mass Assignment and Intervals

- Given these restrictions can we propose an algorithm for translating intervals to Mass Assignments?
- **Lemma 1** *Let* A *be an interval valued fuzzy set.*
  Let $s, x \in \mathbb{U} \mid \overline{\mu}_A(s) = \sup \overline{\mu}_A(x)$
  $\mathrm{m}_A(s) = \underline{\mu}_A(s)$
  $\sum_{\forall x, x \neq s} \mathrm{Bel^+}_A(x) \leq 1.0 - \overline{\mu}_A(s)$
- this restates that we will exceed a total mass of 1.0 very easily

Mass Assignment and Intervals

- Given these restrictions can we propose an algorithm for translating intervals to Assignments?

Mass Assignment and Intervals

- Given these restrictions can we propose an algorithm for translating intervals Assignments?
  - Yes provided we abandon the idea of probability mass

Mass Assignment and Intervals

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  - Yes provided we abandon the idea of probability mass

Possibility Assignment and Intervals

- Algorithm described in detail in paper

Possibility Assignment and Intervals

- Propose a measure on sets that is a Possibility Mass $\mathcal{P}_A(X) X \subseteq \mathbb{U}$
- This may exceed a total mass of 1.0

Possibility Assignment and Intervals

- How does it work

Example

$$
\begin{aligned}
A &= \{\langle a, 0.2, 0.8\rangle, \langle b, 0.1, 1.0\rangle, \langle c, 0.0, 0.6\rangle\} \\
B &= \{\langle a, 0.0, 0.8\rangle, \langle b, 0.2, 1.0\rangle, \langle c, 0.0, 0.6\rangle\}
\end{aligned}
$$

$$
\begin{aligned}
\underset{A}{\mathcal{P}}(\{a\}) &= 0.2, & \underset{A}{\mathcal{P}}(\{b\}) &= 0.1 \\
\underset{A}{\mathcal{P}}(\{a, b\}) &= 0.3, & \underset{A}{\mathcal{P}}(\{b, c\}) &= 0.3 \\
\underset{A}{\mathcal{P}}(\{a, b, c\}) &= 0.3 \\
\underset{B}{\mathcal{P}}(\{b\}) &= 0.2, & \underset{B}{\mathcal{P}}(\{a, b\}) &= 0.2 \\
\underset{B}{\mathcal{P}}(\{a, b, c\}) &= 0.6
\end{aligned}
$$

Example

Set up a tableau

$A \cap B$	B: $\{b\} : 0.2$	$\{a, b\} : 0.2$	$\{a, b, c\} : 0.6$
A: $\{a\} : 0.2$	$\{\}$	$\{a\}$	$\{a\}$
$\{b\} : 0.1$	$\{b\}$	$\{b\}$	$\{b\}$
$\{a, b\} : 0.3$	$\{b\}$	$\{a, b\}$	$\{a, b\}$
$\{b, c\} : 0.3$	$\{b\}$	$\{b\}$	$\{b, c\}$
$\{a, b, c\} : 0.3$	$\{b\}$	$\{a, b\}$	$\{a, b, c\}$

Showing the tableau for the intersection of the Possibility Assignments

Example

$A \cap B$		B: $\{b\} : 0.2$	$\{a,b\} : 0.2$	$\{a,b,c\} : 0.6$
A	$\{a\} : 0.2$	$\{\} : 0.04$	$\{a\} : 0.04$	$\{a\} : 0.12$
	$\{b\} : 0.1$	$\{b\} : 0.02$	$\{b\} : 0.02$	$\{b\} : 0.06$
	$\{a,b\} : 0.3$	$\{b\} : 0.06$	$\{a,b\} : 0.06$	$\{a,b\} : 0.18$
	$\{b,c\} : 0.3$	$\{b\} : 0.06$	$\{b\} : 0.06$	$\{b,c\} : 0.18$
	$\{a,b,c\} : 0.3$	$\{b\} : 0.06$	$\{a,b\} : 0.06$	$\{a,b,c\} : 0.18$

Showing the result of multiplicative intersection of the Possibility Assignments

Example

Extracting the Possibility Masses

$$\underset{A\cap B}{\mathcal{P}}(\{a\}) = 0.16, \qquad \underset{A\cap B}{\mathcal{P}}(\{b\}) = 0.34$$
$$\underset{A\cap B}{\mathcal{P}}(\{a,b\}) = 0.30, \qquad \underset{A\cap B}{\mathcal{P}}(\{b,c\}) = 0.18$$
$$\underset{A\cap B}{\mathcal{P}}(\{a,b,c\}) = 0.18, \qquad \underset{A\cap B}{\mathcal{P}}(\{\}) = 0.04$$

This results in the intervals

$$A \cap B = \{\langle a, 0.16, 0.64\rangle, \langle b, 0.34, 0.8\rangle, \langle c, 0.0, 0.36\rangle\}$$

A bit restrictive

Example

Multiplicative Possibility Assignment

$$A \cap B = \{\langle a, 0.16, 0.64\rangle, \langle b, 0.34, 0.8\rangle, \langle c, 0.0, 0.36\rangle\}$$

Intersection of the Intervals

$$A \cap B = \{\langle a, 0.2, 0.8\rangle, \langle b, 0.2, 1.0\rangle, \langle c, 0.0, 0.6\rangle\}$$

Example

Constructing the maximal restriction of the Possibility tableau
Extracting the Possibility Masses

$$\underset{A\cap B}{\mathcal{P}}(\{a\}) = 0.2, \qquad \underset{A\cap B}{\mathcal{P}}(\{b\}) = 0.24$$
$$\underset{A\cap B}{\mathcal{P}}(\{a,b\}) = 0.16, \qquad \underset{A\cap B}{\mathcal{P}}(\{b,c\}) = 0.1$$
$$\underset{A\cap B}{\mathcal{P}}(\{a,b,c\}) = 0.5$$

This results in the intervals

$$A \cap B = \{\langle a, 0.2, 0.86\rangle, \langle b, 0.24, 1.0\rangle, \langle c, 0.0, 0.6\rangle\}$$

Compared to

$$A \cap B = \{\langle a, 0.2, 0.8\rangle, \langle b, 0.2, 1.0\rangle, \langle c, 0.0, 0.6\rangle\}$$

Comments

- Possibility Assignment allows us to map Interval Valued sets across to a Variant of Mass Assignment
- Operators such as conditionalisation become available
- As do many others