Notes on Intuitionistic Fuzzy Sets
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$\mathbb{Z}_2$-graded intuitionistic L-fuzzy q-deformed quantum subspaces of A_q

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Abstract: In this paper, assuming that $\langle L, \leq \rangle$ is a lattice set with a few specific conditions, intuitionistic L-fuzzy subalgebras, intuitionistic L-fuzzy subcoalgebras and intuitionistic L-fuzzy left (right) coideals are defined and the properties of intuitionistic L-fuzzy subcoalgebras under homomorphisms of coalgebras are investigated. Duality of intuitionistic L-fuzzy subalgebras and duality of intuitionistic L-fuzzy subcoalgebras are also discussed. Intuitionistic L-fuzzy subbialgebras as well as intuitionistic L-fuzzy Hoph subalgebras are studied. Intuitionistic L-fuzzy quantum subsets of $k_q[x, y]$ are established and also $\mathbb{Z}_2$-graded intuitionistic L-fuzzy q-deformed quantum subspaces of A_q are introduced.
Keywords: Intuitionistic L-fuzzy subcoalgebras, Intuitionistic L-fuzzy Hoph subalgebras, $\mathbb{Z}_2$-graded intuitionistic L-fuzzy q-deformed quantum subspaces.
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1 Introduction

In the works of Faddeev et al. [18] and Drinfeld [15], quantum groups were introduced and a new class of Hopf algebras was constructed. Vallin [32] developed Hopf C^*-algebra theory. Recently, different approaches are contained in the papers of Vaksman and Soibelman [31], Kruszynski and Woronowicz [24], and Brown and Goodearl [11]. A compact matrix pseudo group is defined in [34] as a non-commutative compact space endowed with a group structure.

Manin [25] defined quantum plane as a particular start point to construct quantum groups and introduced multiparametric quantum deformation of the general linear supergroup in [26] and later Wachter [33] discussed analysis on q-deformed quantum spaces. Recently the author

[28] defined quantum partial derivatives of Q-analytic functions on quantum superspace A_q and investigated relations between them and some automorphisms of O_q. For more details on Hoph algebras, we refer the readers to [1] and [13].

In 1983, Atanassov [3], introduced the concept of intuitionistic fuzzy sets as a generalization of the concept of fuzzy sets defined by Zadeh [35], to overcome the difficulties in dealing with uncertainties. In intuitionistic fuzzy sets, computational complexity is more as two types of uncertainties are used. But, for obtaining better results, where uncertainty present is more, especially in diagnosis based on medical images, accurate result is very much important, compromising the computational complexity. Later, with Stoeva, Atanassov [8] further generalized that concept to an intuitionistic L-fuzzy set, where L stands for some lattice coupled with a special negation. He also published some results himself in [5, 6, 19–21]. Many other mathematicians have been generalized the concept of intuitionistic fuzzy sets, such as Biswas [10], El-Badawy Yehia [17], Hur et al. [22, 23], Banerjee and Basnet [9], Davvaz and Dudek [14] and Akram [2]. Atanassov provided a comprehensive, complete coverage of virtually all results obtain up to 2012 in the area of the theory and applications of intuitionistic fuzzy sets in the book [5].

So far algebraic approches to fuzzy quantum spaces and compatibility [16], fuzzy quantum spaces [29], quantum structures and fuzzy set theory [30] and fuzzy subcoalgebras and duality [12] have been established and discussed but no intuitionistic L-fuzzification of q-deformed quantum space has been done. The goal of this paper is to introduce the concept of $\mathbb{Z}_2$-graded intuitionistic L-fuzzy q-deformed quantum subspaces of A_q. In order to discuss the L-fuzzification of the concepts of geometry, we estabilshed the concept of C^∞ L-fuzzy manifold with L-gradation of openness in [27].

We proceed as follows. In Section 2, we recall the definition of intuitionistic L-fuzzy vector subspaces and dual of them. In Section 3, we discuss intuitionistic L-fuzzy subcoalgebras as well as intuitionistic L-fuzzy Hoph subalgebras. In Section 4, intuitionistic L-fuzzy quantum subsets of $k_q[x, y]$ and in Section 5, $\mathbb{Z}_2$-graded intuitionistic L-fuzzy q-deformed quantum subspaces of A_q are introduced and some interesting examples are given.

2 Preliminaries

We assume that $L = \langle L, \leq \rangle$ is a complete lattice set (or a complete chain, or a complete ordered semi-ring, etc.) with an (unary) involutive order reversing operation $N : L \to L$.

Definition 2.1 (see [5]). Let M be a nonempty set. An intuitionistic L-fuzzy subset B of M is defined as an object having the form $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in M\}$ or $B = (\mu_B, \nu_B)$, where the functions $\mu_B : M \to L$ and $\nu_B : M \to L$ denote the degree of membership and the degree of non-membership of each element $x \in M$ to the set B, respectively, and $\mu_B(x) \leq N(\nu_B(x))$ for each $x \in M$.

Definition 2.2 (see [12]). Let V be a k-vector space where $k = \mathbb{R}$, $\mathbb{C}$ or any field with characteristic ≥ 2. An intuitionistic L-fuzzy subset $B = (\mu_B, \nu_B)$ of V is called an intuitionistic L-fuzzy vector subspace of V if

$$\mu_B(\alpha x+\beta y)\geq \mu_B(x)\wedge\mu_B(y),\quad \nu_B(\alpha x+\beta y)\leq \nu_B(x)\vee\nu_B(y)$$

for any $x,y\in V$ and $\alpha\in k$.

Definition 2.3 (see [7]). The intuitionistic L-fuzzy subsets $\tilde{0}$ and $\tilde{1}$ are defned by $\mu_{\tilde{0}}(x)=0$, $\nu_{\tilde{0}}(x)=1$ if $x\neq 0$ and $\mu_{\tilde{0}}(0)=1,\ \nu_{\tilde{0}}(0)=0$, and $\mu_{\tilde{1}}(x)=1,\ \nu_{\tilde{1}}(x)=0,\ \forall x\in M$. Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be intuitionistic L-fuzzy subsets of M. We say $A\leq B$ if for all $x\in M$, we have $\mu_A(x)\leq\mu_B(x),\ \nu_A(x)\geq\nu_B(x)$. We define intuitionistic L-fuzzy subsets $A\cap B,\ A\cup B$ by

$$\mu_{A\cap B}(x)=\mu_A(x)\wedge\mu_B(x),\quad \nu_{A\cap B}(x)=\nu_A(x)\vee\nu_B(x)$$

$$\mu_{A\cup B}(x)=\mu_A(x)\vee\mu_B(x),\quad \nu_{A\cup B}(x)=\nu_A(x)\wedge\nu_B(x).$$

Definition 2.4 (see [5]). Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be intuitionistic L-fuzzy vector subspaces of a k-vector space V. The intuitionistic L-fuzzy subsets $A+B$ and $\alpha.A$ of V for each $\alpha\in k,\ x\in X$, are defined by

$$\mu_{A+B}(x)=\begin{cases}\sup\limits_{x=a+b}\{\mu_A(a)\wedge\mu_B(b)\}, & \text{if } x=a+b\\ 0, & \text{otherwise}\end{cases}$$

$$\nu_{A+B}(x)=\begin{cases}\inf\limits_{x=a+b}\{\nu_A(a)\vee\nu_B(b)\}, & \text{if } x=a+b,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\mu_{\alpha.A}(x)=\begin{cases}\mu_A(\alpha x), & \text{if }\ \alpha\neq 0\\ 1, & \text{if }\ \alpha=0,\ x=0\\ 0, & \text{if }\ \alpha=0,\ x\neq 0\end{cases}$$

$$\nu_{\alpha.A}(x)=\begin{cases}\nu_A(\alpha x), & \text{if }\ \alpha\neq 0\\ 0, & \text{if }\ \alpha=0,\ x=0\\ 1, & \text{if }\ \alpha=0,\ x\neq 0.\end{cases}$$

Further if $A\cap B=\tilde{0}$, then $A+B$ is said to be the direct sum and denoted by $A\oplus B$.

Lemma 2.5. *Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be intuitionistic L-fuzzy vector subspaces of an L-fuzzy vector space V. Then $A+B=(\mu_{A+B},\nu_{A+B}),\ A\cap B=(\mu_{A\cap B},\nu_{A\cap B})$ and $\alpha.A=(\mu_{\alpha.A},\nu_{\alpha.A})$ for each $\alpha\in k$, are also intuitionistic L-fuzzy vector subspaces of V.*

Definition 2.6. Let f be a mapping from a k-vector space V to a k-vector space V' . If $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ are intuitionistic L-fuzzy vector subspaces of V and V', respectively, then the preimage of $B=(\mu_B,\nu_B)$ under f is defined to be an intuitionistic L-fuzzy set $f^{-1}[B]=(\mu_{f^{-1}[B]},\nu_{f^{-1}[B]})$ where $\mu_{f^{-1}[B]}(x)=\mu_B(f(x)),\ \ \nu_{f^{-1}[B]}(x)=\nu_B(f(x))$ for any $x\in V$ and the image of $A=(\mu_A,\nu_A)$ under f is defined to be an intuitionistic L-fuzzy set $f[A]=(\mu_{f[A]},\nu_{f[A]})$ where

$$\mu_{f[A]}(x) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \{\mu_A(x)\}, & \text{if } y \in f(V) \\ 0, & \text{if } y \notin f(V), \end{cases}$$

$$\nu_{f[A]}(x) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \{\nu_A(x)\}, & \text{if } y \in f(V) \\ 0, & \text{if } y \notin f(V). \end{cases}$$

Lemma 2.7. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic L-fuzzy vector subspaces of V and V', respectively, and $f : V \to V'$ be a mapping. Then $f^{-1}[B] = (\mu_{f^{-1}[B]}, \nu_{f^{-1}[B]})$ and $f[A] = (\mu_{f[A]}, \nu_{f[A]})$ are intuitionistic L-fuzzy vector subspaces of V and V', respectively.*

Notation remark 2.8. *From now on we assume that $L = \langle L, \leq, \bigwedge, \bigvee, ' \rangle$ is a complete distributive lattice set with at least 2 elements and $\langle L, + \rangle$ is also an additive group. To obtain the best generalization of notions of intuitionistic fuzzy sets* [3]*, fuzzy subcoalgebras* [12] *and fuzzy quantum spaces* [15]*, we consider the condition $0 \leq \mu_B(x) + \nu_B(x) \leq 1$ instead of the condition $\mu_B(x) \leq N(\nu_B(x))$, for each $x \in M$ in Definition* 2.1.
We denote all intuitionistic L-fuzzy subsets of M by IL^M.

3 Intuitionistic L-fuzzy Hoph subalgebras

Proposition 3.1. *Let V be a k-vector space and $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy vector subspace of V. We define $\mu_{B^*}, \nu_{B^*} : V^* \to L$ by*

$$\mu_{B^*}(f) = \begin{cases} \frac{1}{2} - \frac{1}{2}\sup\{\mu_B(x) \mid x \in V,\ f(x) \neq 0\}, & \text{if } f \neq 0 \\ \frac{1}{2} - \frac{1}{2}\inf\{\mu_B(x) \mid x \in V\}, & \text{if } f = 0 \end{cases}$$

$$\nu_{B^*}(f) = \begin{cases} \frac{1}{2} - \frac{1}{2}\inf\{\nu_B(x) \mid x \in V,\ f(x) \neq 0\}, & \text{if } f \neq 0 \\ \frac{1}{2} - \frac{1}{2}\sup\{\nu_B(x) \mid x \in V\}, & \text{if } f = 0 \end{cases}$$

Then $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy vector subspace of V^*.

Proof. First we show that $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy subset of V^*.
By Notation remark 2.8, it should be shown that $0 \leq \mu_{B^*}(f) + (\nu_{B^*}(f)) \leq 1$ for each $f \in V^*$.
If $f \neq 0$, then

$$\mu_{B^*}(f) + \nu_{B^*}(f) = 1 - \frac{1}{2}\Big(\inf\{\mu_B(x)|x \in V,\ f(x) \neq 0\} + \inf\{\nu_B(x)|x \in V,\ f(x) \neq 0\}\Big) \leq 1$$

If $f = 0$, then

$$\mu_{B^*}(f) + \nu_{B^*}(f) = 1 - \frac{1}{2}\Big(\inf\{\mu_B(x)|x \in M\} + \inf\{\nu_B(x)|x \in V\}\Big) \leq 1$$

Hence $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy subset of V^*.

Now it must be shown that

$$\nu_{B^*}(\alpha f+\beta g)\leq \nu_{B^*}(f)\vee \nu_{B^*}(g),\qquad \forall f,g\in V^*,\ \ 0\neq \alpha,\beta\in k.$$

We consider three cases:

- If $f=0$ and $g=0$, then

$$\nu_{B^*}(\alpha f+\beta g)=\nu_{B^*}(0)=\frac{1}{2}-\frac{1}{2}\sup\{\nu_{B}^*(x)|x\in V\}=\nu_{B^*}(f)=\nu_{B^*}(g)$$

- If $f=0$ and $g\neq 0$, then

$$\begin{aligned}\nu_{B^*}(\alpha f+\beta g)&=\frac{1}{2}-\frac{1}{2}\inf\{\nu_{B^*}(x)|x\in V,\beta g(x)\neq 0\}\\&\leq \big(\frac{1}{2}-\frac{1}{2}\sup\{\mu_{B^*}(x)|x\in V\}\big)\bigwedge\big(\frac{1}{2}-\frac{1}{2}\inf\{\mu_{B^*}(x)|x\in V,\ g(x)\neq 0\}\big)\\&=\nu_{B^*}(f)\vee\nu_{B^*}(g)\end{aligned}$$

- If $f\neq 0$ and $g\neq 0$, then

$$\begin{aligned}\nu_{B^*}(\alpha f+\beta g)&=\frac{1}{2}-\frac{1}{2}\inf\{\nu_{B^*}(x)|x\in V,\ (\alpha f+\beta g)(x)\neq 0\}\\&\leq\ \ \big(\big(\frac{1}{2}-\frac{1}{2}\inf\{\mu_{B}^*(x)|x\in V,\ f(x)\neq 0\}\big)\\&\quad\bigwedge\big(\frac{1}{2}-\frac{1}{2}\inf\{\mu_{B}^*(x)|x\in V,\ g(x)\neq 0\}\big)\\&=\nu_{B^*}(f)\vee\nu_{B^*}(g)\end{aligned}$$

We can prove similarly that

$$\mu_{B^*}(\alpha f+\beta g)\geq \mu_{B^*}(f)\wedge \mu_{B^*}(g),\qquad \forall f,g\in V^*,\ \ 0\neq \alpha,\beta\in k.$$

Hence $B^*=(\mu_{B^*},\nu_{B^*})$ is an intuitionistic L-fuzzy vector subspace of V^*. □

Definition 3.2. Let (A,M,U) be a k-algebra. An intuitionistic L-fuzzy subset $B=(\mu_B,\nu_B)$ of A is called an intuitionistic L-fuzzy subalgebra of A if it satisfies the following conditions:

1) $B=(\mu_B,\nu_B)$ is an intuitionistic L-fuzzy vector subspace of A,

2) $\mu_B(xy)\geq\mu_B(x)\wedge\mu_B(y),\quad \nu_B(xy)\leq\nu_B(x)\vee\nu_B(y),$ for any $x,y\in A$.

Definition 3.3. If we set the condition

$$\mu_B(xy)\geq\mu_B(y),\quad \nu_B(xy)\leq\nu_B(y),\qquad \Big(\mu_B(xy)\geq\mu_B(x),\quad \nu_B(xy)\leq\nu_B(x)\Big),\qquad \forall x,y\in A,$$

instead of condition 2) in Definition 3.2, then $B=(\mu_B,\nu_B)$ is called an intuitionistic L-fuzzy left (right) ideal of A.

Definition 3.4. Let (C, Δ, ε) be a coalgebra and for any $x \in C$,

$$\Delta(x) = \sum_{i=1}^{n} x_{i1} \otimes x_{i2}.$$

An intuitionistic L-fuzzy subset $B = (\mu_B, \nu_B)$ of C is called an intuitionistic L-fuzzy subcoalgebra of C if it satisfies the following conditions:

1) $B = (\mu_B, \nu_B)$ is an intuitionistic L-fuzzy vector subspace of C,

2) $\mu_B(x) \leq \mu_B(x_{i1}) \bigwedge \mu_B(x_{i2}), \quad \nu_B(x) \geq \nu_B(x_{i1}) \bigvee \nu_B(x_{i2}), \quad \forall x \in C,\ \forall i.$

Definition 3.5. If we set the condition

$$\mu_B(x) \leq \mu_B(x_{i2}), \quad \nu_B(x) \geq \nu_B(x_{i2}), \quad \Big(\mu_B(x) \leq \mu_B(x_{i1}), \quad \nu_B(x) \geq \nu_B(x_{i1})\Big),\ \forall x \in C,\ \forall i,$$

instead of the condition 2) in Definition 3.4, then $B = (\mu_B, \nu_B)$ is called an intuitionistic L-fuzzy left (right) coideal of C.

Remark 3.6. i) An intuitionistic L-fuzzy subcoalgebra is an intuitionistic L-fuzzy left (right) coideal.

ii) If $B = (\mu_B, \nu_B)$ is an intuitionistic L-fuzzy left and right coideal, by Definition 3.5, we have

$$\mu_B(x) \leq \mu_B(x_{i2}),\ \ \mu_B(x) \leq \mu_B(x_{i1}) \quad \Rightarrow \quad \mu_B(x) \leq \mu_B(x_{i1}) \bigwedge \mu_B(x_{i2})$$

$$\nu_B(x) \geq \nu_B(x_{i2}),\ \ \nu_B(x) \geq \nu_B(x_{i1}) \quad \Rightarrow \quad \nu_B(x) \geq \nu_B(x_{i1}) \bigvee \nu_B(x_{i2}),$$

$\forall x \in C,\ \forall i$. Hence B is an intuitionistic L-fuzzy subcoalgebra.

Theorem 3.7. *An intuitionistic L-fuzzy subset $B = (\mu_B, \nu_B)$ of the coalgebra C is an intuitionistic L-fuzzy left (right) coideal if and only if the level sets*

$$B_{r,s} = \big\{x \in C : \mu_B(x) \geq r,\ \nu_B(x) \leq s\big\}$$

are left (right) coideals of **C** *where*

$$r + s \in L, \quad 0 \leq r \leq \mu_B(0), \quad \nu_B(0) \leq s \leq 1.$$

Proof. $(\Longrightarrow)$ Let $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy left coideal of C and $r, s \in L$, such that $r + s \in L,\ \ 0 \leq r \leq \mu_B(0),\ \ \nu_B(0) \leq s \leq 1$. Since $0 \in B_{r,s}$, then $B_{r,s} \neq \varnothing$. Let $x, y \in B_{r,s}$ and $\alpha, \beta \in A$. Using (1) of Definition 3.2 we have

$$\mu_B(\alpha x + \beta y) \geq \mu_B(x) \wedge \mu_B(y) \geq r, \qquad \nu_B(\alpha x + \beta y) \leq \nu_B(x) \vee \nu_B(y) \leq s.$$

So $(\alpha x + \beta y) \in B_{r,s}$. Let $x \in B_{r,s} \subset C$. By Definition 3.4 we have

$$r \leq \mu_B(x) \leq \mu_B(x_{i2}), \qquad s \geq \nu_B(x) \geq \nu_B(x_{i2}).$$

Then $x_{i2} \in B_{r,s}$. Hence we have $\Delta(B_{r,s}) \subset C \otimes B_{r,s}$. Therefore $B_{r,s}$ is a left coideals of C.

($\Longleftarrow$) Let $x, y \in C$. We assume that $\mu_B(x) = r,\ \nu_B(x) = s$ and $\mu_B(y) = r',\ \nu_B(y) = s'$. Hence $r, s, r', s' \in L,\ s.t.\ r + s \in L,\quad r' + s' \in L,\quad 0 \leq r, r' \leq \mu_B(0),\quad \mu_B(0) \leq s, s' \leq 1$. Then $x \in B_{r,s}$ and $y \in B_{r',s'}$. Without loss of generality we assume only two cases:

$i)$ $r \leq r'$ and $s \leq s'$. Hence $\mu_B(x), \mu_B(y) \geq r$ and $\nu_B(x), \nu_B(y) \leq s'$. Therefore $x, y \in B_{r,s'}$. Since $B_{r,s'}$ is a subspace of C, hence $\alpha x + \beta y \in B_{r,s'}$ for each $\alpha, \beta \in A$. Thus

$$\mu_B(\alpha x + \beta y) \geq r = r \wedge r' = \mu_B(x) \wedge \mu_B(y),$$

$$\nu_B(\alpha x + \beta y) \leq s' = s \vee s' = \nu_B(x) \vee \nu_B(y).$$

$ii)$ $r \leq r'$ and $s' \leq s$. Hence $\mu_B(x), \mu_B(y) \geq r$ and $\nu_B(x), \nu_B(y) \leq s$. Therefore $x, y \in B_{r,s}$. Since $B_{r,s}$ a subspace of C, hence $\alpha x + \beta y \in B_{r,s}$ for each $\alpha, \beta \in A$. Thus

$$\mu_B(\alpha x + \beta y) \geq r = r \wedge r' = \mu_B(x) \wedge \mu_B(y),$$

$$\nu_B(\alpha x + \beta y) \leq s = s \vee s' = \nu_B(x) \vee \nu_B(y).$$

Therefore B satisfies the condition $1)$ of Definition 3.2.
To prove condition $2)$, we use the assumption that $B_{r,s}$ is a left coideals of C. Then we have $\Delta(B_{r,s}) \subset C \otimes B_{r,s}$. Hence $x_{i2} \in B_{r,s}$. Thus

$$\mu_B(x_{i2}) \geq r = \mu_B(x), \qquad \nu_B(x_{i2}) \leq s = \nu_B(x).$$

Therefore, B is an intuitionistic L-fuzzy left coideal of C.

The proof of the intuitionistic L-fuzzy right coideal is similar. $\square$

Definition 3.8. Let C and D be two coalgebras. The linear map $f : C \to D$ is a morphism of coalgebras, if

$$\Delta_D \circ f = (f \otimes f) \circ \Delta_D, \quad \varepsilon_D \circ f = \varepsilon_C.$$

$$\sum_{i=1}^{n} f(x)_{i1} \otimes f(x)_{i2} = \sum_{j=1}^{m} f(x_{j1}) \otimes f(x_{j2}), \tag{3.1}$$

Proposition 3.9. *Let $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy subcoalgebra (respectively, left / right coideal) of D and $f : C \to D$ be a morphism of coalgebras. $f^{-1}[B] = (\mu_{f^{-1}[B]}, \nu_{f^{-1}[B]})$ is an IL-fuzzy subcoalgebra (respectively, fuzzy left / right coideal) of C.*

Proof. Let $x, y \in C$ and $\alpha, \beta \in A$. Then

$$\begin{aligned}
\mu_{f^{-1}[B]}(\alpha x + \beta y) &= \mu_B\big(f(\alpha x + \beta y)\big) \\
&= \mu_B\big(\alpha f(x) + \beta f(y)\big) \\
&\geq \mu_B(f(x)) \wedge \mu_B(f(y)) \\
&= \mu_{f^{-1}[B]}(x) \bigwedge \mu_{f^{-1}[B]}(y)
\end{aligned}$$

$$\begin{aligned}\nu_{f^{-1}[B]}(\alpha x+\beta y) &= \nu_B\big(f(\alpha x+\beta y)\big)\\ &= \nu_B\big(\alpha f(x)+\beta f(y)\big)\\ &\leq \nu_B(f(x))\bigvee \nu_B(f(y))\\ &= \nu_{f^{-1}[B]}(x)\bigvee \nu_{f^{-1}[B]}(y)\end{aligned}$$

Let $\Delta_C(x)=\sum_{j=1}^m x_{j1}\otimes x_{j2}$. Since

$$\Delta_D(f(x))=\sum_{i=1}^n f(x)_{i1}\otimes f(x)_{i2}=\sum_{j=1}^m f(x_{j1})\otimes f(x_{j2}), \tag{3.2}$$

$$\begin{aligned}\mu_{f^{-1}[B]}(x) &= \mu_B\big(f(x)\big)\\ &\leq \mu_B(f(x)_{i1})\bigwedge \mu_B(f(x)_{i2})\\ &= \mu_B(f(x_{j1}))\bigwedge \mu_B(f(x_{j2}))\\ &= \mu_{f^{-1}[B]}(x_{j1})\bigwedge \mu_{f^{-1}[B]}(x_{j2})\end{aligned}$$

Similarly we can show that

$$\nu_{f^{-1}[B]}(x)\geq \nu_{f^{-1}[B]}(x_{j1})\bigvee \nu_{f^{-1}[B]}(x_{j2}).$$

So $f^{-1}[B]$ is an intuitionistic L-fuzzy subcoalgebra of C.
The case of intuitionistic L-fuzzy left (right) coideal can be proved by the similar manner. □

Proposition 3.10. *Let $A=(\mu_A,\nu_A)$ be an intuitionistic L-fuzzy subcoalgebra (respectively, left / right coideal) of C and $f:C\to D$ be a morphism of subcoalgebras. Then $f[A]=(\mu_{f[A]},\nu_{f[A]})$ is an intuitionistic L-fuzzy subcoalgebra (respectively, left / right coideal) of D.*

Proof. Let $x,y\in D$. We show that

$$\mu_{f[A]}(x+y)\geq \mu_{f[A]}(x)\bigwedge \mu_{f[A]}(y),\quad \nu_{f[A]}(x+y)\leq \nu_{f[A]}(x)\bigvee \nu_{f[A]}(y). \tag{3.3}$$

If $f^{-1}(x)=\varnothing$ or $f^{-1}(y)=\varnothing,$ then $0=\mu_{f[A]}(x)\bigwedge\mu_{f[A]}(y)$ and $1=\nu_{f[A]}(x)\bigvee\nu_{f[A]}(y)$. So (3.3) holds. Now we assume that there exist $r,s\in C$ such that $f(r)=x,\ f(s)=y$. So by linearity of f, we have $f(r+s)=x+y$. Hence $r+s\in f^{-1}(x+y)$ and we have

$$\begin{aligned}\mu_{f[A]}(x+y) &= \sup\big\{\mu_A(z)\mid z\in f^{-1}(x+y)\big\}\\ &\geq \sup\big\{\mu_A(r+s)\mid (r+s)\in f^{-1}(x+y)\big\}\\ &\geq \sup\big\{(\mu_A(r)\wedge\mu_A(s))\mid (r+s)\in f^{-1}(x+y)\big\}\\ &\geq \sup\big\{\mu_A(r)\mid r\in f^{-1}(x)\big\}\bigwedge\ \sup\big\{\mu_A(s)\mid s\in f^{-1}(y)\big\}\\ &\geq \mu_{f[A]}(x)\bigwedge \mu_{f[A]}(y)\end{aligned}$$

$$
\begin{aligned}
\nu_{f[A]}(x+y) &= \inf\left\{\nu_A(z) \mid z \in f^{-1}(x+y)\right\}\\
&\leq \inf\left\{\nu_A(r+s) \mid (r+s) \in f^{-1}(x+y)\right\}\\
&\leq \inf\left\{(\nu_A(r) \vee \nu_A(s)) \mid (r+s) \in f^{-1}(x+y)\right\}\\
&\leq \inf\left\{\nu_A(r) \mid r \in f^{-1}(x)\right\} \bigvee \sup\left\{\nu_A(s) \mid s \in f^{-1}(y)\right\}\\
&\leq \nu_{f[A]}(x) \bigvee \nu_{f[A]}(y).
\end{aligned}
$$

Let $x \in D$ and $\alpha \in A$. For each $z \in f^{-1}(x)$, we have $f(\alpha z) = \alpha f(z) = \alpha x$. So we get

$$
\begin{aligned}
\mu_{f[A]}(\alpha x) &= \sup\left\{\mu_A(w) \mid w \in f^{-1}(\alpha x)\right\}\\
&\geq \sup\left\{\mu_A(\alpha z) \mid (\alpha z) \in f^{-1}(\alpha x)\right\}\\
&\geq \sup\left\{\mu_A(z) \mid z \in f^{-1}(x)\right\}\\
&= \mu_{f[A]}(x)
\end{aligned}
$$

$$
\begin{aligned}
\nu_{f[A]}(\alpha x) &= \inf\left\{\nu_A(w) \mid w \in f^{-1}(\alpha x)\right\}\\
&\leq \inf\left\{\nu_A(\alpha z) \mid (\alpha z) \in f^{-1}(\alpha x)\right\}\\
&\leq \inf\left\{\nu_A(z) \mid z \in f^{-1}(x)\right\}\\
&= \nu_{f[A]}(x)
\end{aligned}
$$

Let $x \in D$ and $\Delta_D(x) = \sum_{j=1}^{m} x_{j1} \otimes x_{j2}$. We want show that

$$
\mu_{f[A]}(x) \leq \mu_{[A]}(x_{j1}) \bigwedge \mu_{f[A]}(x_{j2}), \qquad \nu_{f[A]}(x) \geq \nu_{f[A]}(x_{j1}) \bigvee \nu_{f[A]}(x_{j2}) \tag{3.4}
$$

If $f^{-1}(x) = \varnothing$, then $0 = \mu_{f[A]}(x)$ and $1 = \nu_{f[A]}(x)$. So (3.4) holds. Now we assume that there exists $z \in C$ such that $f(z) = x$. Since

$$
\begin{aligned}
\sum_{j=1}^{m} x_{j1} \otimes x_{j2} &= \Delta_D(x) = \Delta_D(f(z))\\
&= (f \otimes f) \circ \Delta_C(z)\\
&= \sum_{i=1}^{n} f(z_{i1}) \otimes f(z_{i2})
\end{aligned}
$$

So $z_{i1} \in f^{-1}(x_{j1})$ and $z_{i2} \in f^{-1}(x_{j2})$. Hence

$$
\begin{aligned}
\mu_{f[A]}(x) &= \sup\left\{\mu_A(z) \mid z \in f^{-1}(x)\right\}\\
&\leq \sup\left\{\left(\mu_A(z_{i1}) \bigwedge \mu_A(z_{i2})\right) \mid z_{i1} \in f^{-1}(x_{j1}),\ z_{i2} \in f^{-1}(x_{j2})\right\}\\
&= \sup\left\{\mu_A(z_{i1}) \mid z_{i1} \in f^{-1}(x_{j1})\right\} \bigwedge \sup\left\{\mu_A(z_{i2}) \mid z_{i2} \in f^{-1}(x_{j2})\right\}\\
&= \mu_{f[A]}(x_{j1}) \bigwedge \mu_{f[A]}(x_{j2})
\end{aligned}
$$

We can similarly show that $\nu_{f[A]}(x) \geq \nu_{f[A]}(x_{j1}) \bigvee \nu_{f[A]}(x_{j2})$.
Therefore $f[A] = (\mu_{f[A]}, \nu_{f[A]})$ is an IF-fuzzy subcoalgebra of D.
The case of intuitionistic L-fuzzy left (right) coideal can be proved in a similar manner. $\square$

Let (C, Δ, ε) be a k-coalgebra. We define the mappings $M : C^* \to C^* \otimes C^*,\ M = \Delta^* \circ \rho$, where ρ is defined as $\rho : C^* \otimes C^* \to (C \otimes C)^*$ by $\rho(f \otimes g)(x \otimes y) = f(x)g(y)$ and $U : k \to C^*$ by $U = \varepsilon^* \circ \phi$, where $\phi : k \to k^*$ is the canonical isomorphism. Using [13, Proposition 1.3.6], we have (C^*, M, U) is an algebra. The multiplication is denoted by $M(f \otimes g) = f * g$ and we have

$$(f * g)(x) = \sum_{i=1}^{n} f(x_{i1})\, g(x_{i2}).$$

Proposition 3.11. *Let (C, Δ, ε) be a coalgebra.*

i) Let $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy subcoalgebra of C. Then $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy ideal of C^*.

ii) Let $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy left (right) coideal of C. Then $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy left (right) ideal of C^*.

Proof. $i)$ Using Proposition 3.1, $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy vector subspace of C^*. We should prove that

$$\nu_{B^*}\big(f * g\big) \le \nu_{B^*}(f) \bigwedge \mu_{B^*}(g), \text{ for any } f, g \in C^*. \tag{3.5}$$

Let $f \neq 0$ and $g \neq 0$. We have

$$\begin{aligned}
\nu_{B^*}\big(f * g\big) &= \frac{1}{2} - \frac{1}{2} \inf \big\{\nu_B(x) \mid x \in V,\ (f * g)(x) \neq 0\big\} \\
&= \frac{1}{2} - \frac{1}{2} \inf \Big\{\big(\nu_B(x_{i1}) \bigwedge \nu_B(x_{i2})\big) \mid x_{i1}, x_{i2} \in V,\ \sum_{i=1}^{n} f(x_{i1})\, g(x_{i2}) \neq 0\Big\} \\
&= \frac{1}{2} - \frac{1}{2} \inf \Big\{\big(\nu_B(x_{i1}) \bigwedge \nu_B(x_{i2})\big) \mid x_{i1}, x_{i2} \in V,\ f(x_{i1})\, g(x_{i2}) \neq 0\Big\} \\
&= \frac{1}{2} - \frac{1}{2} \inf \Big\{\big(\nu_B(x_{i1}) \bigwedge \nu_B(x_{i2})\big) \mid x_{i1}, x_{i2} \in V,\ f(x_{i1}) \neq 0,\ g(x_{i2}) \neq 0\Big\} \\
&\le \frac{1}{2} - \frac{1}{2} \inf \Big\{\big(\nu_B(x_{i1}) \mid x_{i1} \in V,\ f(x_{i1}) \neq 0\Big\} \\
&\bigwedge \inf \Big\{\big(\nu_B(x_{i2})\big) \mid x_{i2} \in V,\ g(x_{i2}) \neq 0\Big\} \\
&\le \frac{1}{2} - \frac{1}{2} \inf \Big\{\big(\nu_B(u) \mid u \in V,\ f(u) \neq 0\Big\} \\
&\bigwedge \inf \Big\{\big(\nu_B(v)\big) \mid v \in V,\ g(v) \neq 0\Big\} \\
&\le \nu_{B^*}(f) \bigwedge \mu_{B^*}(g)
\end{aligned}$$

Let $f = 0$ and $g \neq 0$. Then we have $(f * g)(x) = \sum_{i=1}^{n} f(x_{i1})\, g(x_{i2}) = 0$. Thus

$$\begin{aligned}\nu_{B^*}\big(f * g\big) &= \nu_{B^*}(0) = \nu_{B^*}(f)\\ &= \frac{1}{2} - \frac{1}{2}\sup\{\nu_B(x) \mid x \in V\}\\ &\leq \frac{1}{2} - \frac{1}{2}\sup\{\nu_B(x) \mid x \in V,\ g(x) \neq 0\}\\ &\leq \frac{1}{2} - \frac{1}{2}\inf\{\nu_B(x) \mid x \in V,\ g(x) \neq 0\} = \nu_{B^*}(g).\end{aligned}$$

Hence we have $\nu_{B^*}\big(f * g\big) \leq \nu_{B^*}(f) \bigwedge \mu_{B^*}(g)$.
Let $f = 0$ and $g = 0$. Then $\nu_{B^*}\big(f * g\big) = \nu_{B^*}(0) = \nu_{B^*}(f) = \mu_{B^*}(g)$ and the result is obtained. We can prove similarly that

$$\mu_{B^*}\big(f * g\big) \geq \mu_{B^*}(f) \bigvee \mu_{B^*}(g), \text{ for any } f, g \in C^*.$$

Hence $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy ideal of C^*.

$ii)$ This can be proved in a similar manner. □

Let the k-algebra (A, M, U) be a finite dimensional algebra. We define the mappings $\Delta : A^* \to A^* \otimes A^*$ and $\varepsilon : A^* \to k$ by $\Delta = \rho^{-1} \circ M^*$ and $\varepsilon = \psi \circ U^*$ where $\rho : A^* \otimes A^* \to (A \otimes A)^*$ by $\rho(f \otimes g)(x \otimes y) = f(x)g(y)$ and $\psi : k^* \to k$ by $\psi(f) = 1$. Then $(A^*, \Delta, \varepsilon)$ is a coalgebra.

Proposition 3.12. *Let (A, M, U) be a finite dimensional algebra.*

i) Let $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy ideal of A. Then $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy subcoalgebra of A^*.

ii) Let $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy left (right) ideal of A. Then $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy left (right) coideal of A^*.

Proof. $i)$ Using Proposition 3.1, $B^* = (\mu_{B^*}, \nu_{B^*})$ is an intuitionistic L-fuzzy vector subspace of C^*. Let $\Delta(f) = \sum_{j=0}^{n} f_{j1} \otimes f_{j2}$. We show that

$$\nu_{B^*}\big(f\big) \geq \nu_{B^*}\big(f_{j1}\big) \bigvee \nu_{B^*}\big(f_{j2}\big).$$

Let $f \neq 0$, then we have

$$\begin{aligned}\nu_{B^*}(f) &= \frac{1}{2} - \frac{1}{2}\inf\{\nu_B(x) \mid x \in A,\ \ f(x) \neq 0\}\\ &\geq \frac{1}{2} - \frac{1}{2}\inf\{\nu_B(uv) \mid uv \in A,\ \ f(uv) \neq 0\}\\ &\geq \frac{1}{2} - \frac{1}{2}\inf\{\big(\nu_B(u) \bigvee \nu_B(v)\big) \mid uv \in V,\ \ \sum_{j=0}^{n} f_{j1}(u) f_{j2}(v) \neq 0\}\\ &\geq \frac{1}{2} - \frac{1}{2}\inf\{\big(\nu_B(u) \bigvee \nu_B(v)\big) \mid uv \in V,\ \ f_{j1}(u) f_{j2}(v) \neq 0\}\\ &\geq \frac{1}{2} - \frac{1}{2}\big(\inf\{\nu_B(u) \mid u \in V,\ f_{j1}(u) \neq 0\}\big) \bigwedge \inf\{\nu_B(v) \mid v \in V,\ f_{j2}(v) \neq 0\}\end{aligned}$$

If

$$\inf\{\nu_{_B}(u)|\, u\in V,\ f_{j1}(u)\neq 0\}\geq \inf\{\nu_{_B}(v)|\, v\in V,\ f_{j2}(v)\neq 0\},$$

then $\nu_{_{B^*}}\big(f_{j1}\big)\leq \nu_{_{B^*}}\big(f_{j2}\big)$. Hence $\nu_{_{B^*}}\big(f\big)\geq \nu_{_{B^*}}\big(f_{j2}\big)$.

If

$$\inf\{\nu_{_B}(u)|\, u\in V,\ f_{j1}(u)\neq 0\}\leq \inf\{\nu_{_B}(v)|\, v\in V,\ f_{j2}(v)\neq 0\},$$

then $\nu_{_{B^*}}\big(f_{j1}\big)\geq \nu_{_{B^*}}\big(f_{j2}\big)$. Hence $\nu_{_{B^*}}\big(f\big)\geq \nu_{_{B^*}}\big(f_{j1}\big)$. Therefore $\nu_{_{B^*}}\big(f\big)\geq \nu_{_{B^*}}\big(f_{j1}\big)\bigvee \nu_{_{B^*}}\big(f_{j2}\big)$.

Let $f=0$. Then $f_{j1}=f_{j2}=0$. So $\nu_{_{B^*}}\big(f\big)\geq \nu_{_{B^*}}\big(f_{j1}\big)\bigvee \nu_{_{B^*}}\big(f_{j2}\big)$.

We can also show that

$$\mu_{_B}\big(f\big)\leq \mu_{_B}\big(f_{j1})\big)\bigwedge \mu_{_B}\big(f_{j2})\big)$$

and this completes the proof.

$ii)$ This can be proved similarly to $i)$. □

Definition 3.13. Let $(H,M,U,\Delta,\varepsilon)$ be a k-bialgebra. If $B=(\mu_{_B},\nu_{_B})$ is either an intuitionistic L-fuzzy subalgebra or intuitionistic L-fuzzy subcoalgebra of H, then it is called an intuitionistic L-fuzzy subbialgebra of H. Let $S:H\to H$ be the antipode of H, then for each $x\in H$ we have $(S\otimes I)\Delta(x)=(I\otimes S)\Delta(x)=U(\varepsilon(x)$. If

$$\nu_{_B}\big(\sum_{(x)}S(x_{i_1})x_{i_2}\big)=\nu_{_B}\big(\sum_{(x)}x_{i_1}S(x_{i_2})\big)=\nu_{_B}\big(U(\varepsilon(x)\big), \tag{3.6}$$

$$\mu_{_B}\big(\sum_{(x)}S(x_{i_1})x_{i_2}\big)=\mu_{_B}\big(\sum_{(x)}x_{i_1}S(x_{i_2})\big)=\mu_{_B}\big(U(\varepsilon(x)\big), \tag{3.7}$$

then $B=(\mu_{_B},\nu_{_B})$ is called an intuitionistic L-fuzzy Hoph subalgebra of H.

Example 3.14. Let A be the algebra generated by an invertible element a and an element b such that $b^n=0$ and $ab=\lambda ba$, where λ is a primitive $2n$-th root of unity. It can be shown that A is a Hopf algebra with the comultiplication, counit and antipode defined by

$$\Delta(a)=a\otimes a,\quad \Delta(b)=a\otimes b+b\otimes a^{-1},\quad \varepsilon(a)=1,\quad \varepsilon(b)=0$$

$$S(a)=a^{-1},\quad S(b)=-a^{-1}ba.$$

An interesting solution is given by [13, Exercise 5.6.24] as follows:

"Let $C=\langle a\rangle$ be an infinite cyclic group and $a^*(a)=\sqrt{\lambda}$. Clearly $A\simeq H(C,n,a^2,a^*)$. So A has the properties of Ore extension Hopf algebras of the form $H(C,n,c,c^*,a,b)$."

Now we define an intuitionistic I-fuzzy subset $B=(\mu_{_B},\nu_{_B})$ of A by

$$\mu_{_B}(a)=\mu_{_B}(a^{-1})=0.7,\ \ \nu_{_B}(a)=\nu_{_B}(a^{-1})=0.2$$

$$\mu_{_B}(b)=0.4,\ \ \nu_{_B}(b)=0.6,\qquad \mu_{_B}(1)=1,\ \ \nu_{_B}(1)=0,\qquad \mu_{_B}(0)=0.5,\ \ \mu_{_B}(0)=0.5$$

It is easy to prove that $B=(\mu_{_B},\nu_{_B})$ is an intuitionistic I-fuzzy subbialgebra of A.

Furthermore

$$\begin{aligned}
\mu_B\big((S\otimes I)\Delta(a)\big) &= \mu_B\big((S\otimes I)(a\otimes a)\\
&= \mu_B\big(S(a)a\big)\\
&= \mu_B\big(a^{-1}a\big)\\
&= \mu_B(1)=1\\
&= \mu_B\big(U(\varepsilon(a)\big),
\end{aligned}$$

$$\begin{aligned}
\mu_B\big((S\otimes I)\Delta(b)\big) &= \mu_B\big((S\otimes I)(a\otimes b+b\otimes a^{-1})\\
&= \mu_B\big(S(a)\,b+S(b)\,a^{-1}\big)\\
&= \mu_B\big(a^{-1}b+(-a^{-1}ba)\,a^{-1}\big)\\
&= \mu_B(0)=0.5\\
&= \mu_B\big(U(\varepsilon(b)\big).
\end{aligned}$$

Similarly, we can show that $\mu_B\big((I\otimes S)\Delta(x)\big)=\mu_B\big(U(\varepsilon(x)\big),\ \forall x\in A.$

Hence $B=(\mu_B,\nu_B)$ satisfies the condition (3.7) and similarly satisfies (3.6). Therefore $B=(\mu_B,\nu_B)$ is an intuitionistic I-fuzzy Hoph subalgebra of A.

Example 3.15. Let H be a k-vector space with the basis $\{c_i\}_{i=0}^{\infty}$. Define two k-linear mappings $M: H\otimes H\to H$ and $U: k\to H$ by $M(c_i\otimes c_j)=c_ic_j$ for all $i,j\geq 0$ and $U(1)=c_0$. Also define two k-linear mappings $\Delta: H\to H\otimes H$ and $\varepsilon: H\to k$ by

$$\Delta(c_n)=\sum_{i=0}^{n} c_i\otimes c_{n-i},\quad \varepsilon(c_n)=\delta_{n,0}\quad \forall n\geq 0,$$

and define the antipode $S: H\to H,\ S(c_n)=\delta_{n,0}\,c_n$. It is proved in [1] that $(H,M,U,\Delta,\varepsilon,S)$ is a Hoph algebra. Let $I=[0,1]$. We define an intuitionistic I-fuzzy subset $B=(\mu_B,\nu_B)$ of H by

$$\begin{aligned}
\mu_B(c_n) &= \frac{1}{n+1}, & \nu_B(c_n) &= \frac{n}{n+1},\quad \forall n\geq 0,\\
\mu(\alpha x+\beta y) &= \mu(x)\vee\mu(y), & \nu(\alpha x+\beta y) &= \nu(x)\wedge\nu(y),\\
\mu(xy) &= \mu(x)\wedge\mu(y), & \nu(xy) &= \nu(x)\vee\nu(y),
\end{aligned}$$

for all $x,y\in H$ and for all $\alpha,\beta\in k$.

Therefore B is an intuitionistic I-fuzzy subalgebra of H.

Since we have for each $i=0,\dots,n$:

$$\frac{1}{n+1}\leq\frac{1}{i+1}\wedge\frac{1}{n-i+1}\Longrightarrow \mu_B(c_n)\leq\mu_B(c_i)\wedge\mu_B(c_{n-i})\Longrightarrow \mu_B(x)\leq\mu_B(x_{i1})\bigwedge\mu_B(x_{i2})$$

$$\frac{n}{n+1}\geq\frac{i}{i+1}\vee\frac{n}{n-i+1}\Longrightarrow \nu_B(c_n)\geq\nu_B(c_i)\vee\nu_B(c_{n-i})\Longrightarrow \nu_B(x)\geq\nu_B(x_{i1})\bigvee\nu_B(x_{i2}).$$

Hence B is an intuitionistic I-fuzzy subcoalgebra of H. Now

$$\begin{aligned}
\mu_B\big(\sum_{i=0}^{n} S(c_i)\, c_{n-i}\,\big) &= \bigvee_{i=0}^{n} \mu_B(\delta_{i,0}\, c_i) \;\wedge\; \mu_B(c_{n-i}) \\
&= \bigvee_{i=0}^{n} 1 \;\wedge\; \frac{1}{n-i+1} = 1 \\
&= \bigvee_{i=0}^{n} \frac{1}{i+1} \;\wedge\; 1 \\
&= \bigvee_{i=0}^{n} \mu_B(c_i) \;\wedge\; \mu_B(\delta_{n-i,0}\, c_{n-i}) \\
&= \;\mu_B\big(\sum_{i=0}^{n} c_i\, S(c_{n-i})\,\big).
\end{aligned}$$

$$\mu_B\big((U \circ \varepsilon)(c_0)\big) = \mu_B(U(\delta_{0,0}) = \mu_B(1) = 1.$$

$$\mu_B\big((U \circ \varepsilon)(c_n)\big) = \mu_B(U(\delta_{n,0}) = \mu_B(0) = 1.$$

Hence B satisfies (3.7) and we can similarly show that (3.6) holds. Therefore $B = (\mu_B, \nu_B)$ is an intuitionistic I-fuzzy Hoph subalgebra of H.

4 Intuitionistic L-fuzzy quantum subsets of $k_q[x, y]$

Let q be an invertible real (complex) number less than 1, and let I_q be the two-sided ideal of the free algebra $k[x, y]$ generated by the element $yx - qxy$.

The quantum plane is defined as the quotient-algebra

$$k_q[x, y] = k[x, y]/I_q. \tag{4.1}$$

Definition 4.1. Let $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy Hoph subalgebra of $k[x, y]$. If it satisfies the condition

$$\mu_B(yx) = \mu_B(qxy), \quad \nu_B(yx) = \nu_B(qxy)$$

then $B = (\mu_B, \nu_B)$ is called an intuitionistic L-fuzzy quantum subspace of $k_q[x, y]$.

Example 4.2. We can consider $H = k[x, y]$ as a k-vector space with the basis $\{x^m y^n\}_{m,n=0}^{\infty}$ and define

$$M(x^m y^n \otimes x^k y^l) = x^{m+k} y^{n+l}, \quad U(1) = x^0 y^0 = 1.$$

$$\Delta(x^m y^n) = \sum_{i=0}^{m} \sum_{j=0}^{n} x^i y^j \otimes x^{m-i} y^{n-i}, \qquad \varepsilon(x^m y^n) = \delta_{m+n,0},$$

$$S(x^m y^n) = \delta_{m+n,0}\, x^m y^n, \qquad \forall m, n, k, l \geq 0.$$

Then similarly to Example 3.15 we can prove that $k[x, y]$ is a Hoph algebra.
Define an intuitionistic I-fuzzy subset $B = (\mu_B, \nu_B)$ of H by

$$\mu_B(x^m y^n) = \frac{1}{m+n+1}, \qquad \nu_B(x^m y^n) = \frac{m+n}{m+n+1}, \quad \forall m, n \geq 0,$$

Then $B = (\mu_{_B}, \nu_{_B})$ is an intuitionistic I-fuzzy Hoph subalgebra of H. Also we have

$$\mu_{_B}(xy) = \frac{1}{3} = \mu_{_B}(qyx), \quad \nu_{_B}(xy) = \frac{2}{3} = \nu_{_B}(qyx)$$

Hence $B = (\mu_{_B}, \nu_{_B})$ is an intuitionistic I-fuzzy quantum subset of $k_q[x, y]$.

5 $\mathbb{Z}_2$-graded intuitionistic L-fuzzy quantum subspaces of A_q

Let k be a field of characteristic $\neq 2$ ($\mathbb{R}$ or $\mathbb{C}$). Suppose the format $(a_1, \ldots, a_n)$ be an arbitrary sequence, $a_i \in \mathbb{Z}_2$ and the multiparameter

$$q = \{q_{ij} : |q_{ij}| \leq 1, \quad 1 \leq i < j \leq n\}, \tag{5.1}$$

be the family of non-zero elements of k. The quantum superspace A_q or, rather, the polynomial function ring on it is generated by coordinates $x_1, \ldots,\ x_n$ with parity $\hat{x_i} = \hat{\imath}$ and commutation rules:

$$x_i^2 = 0 \quad \text{for} \quad \hat{\imath} = 1 \qquad \text{and} \qquad x_j x_i = (-1)^{\hat{i}\hat{j}} q_{ij}\, x_i\, x_j \quad \text{for} \quad i < j. \tag{5.2}$$

We call x_i an odd (even) coordinate if $\hat{\imath} = 1 (\hat{\imath} = 0)$ and we have $k_i = 0; 1 (k_i \geq 0)$. We denote briefly an polynomial function

$$f(x_1, \ldots, x_n) = \sum_{0 \leq k_1, \ldots, k_n \leq m} \alpha_{k_1, \ldots, k_n}\, x_1^{k_1} \ldots x_n^{k_n},$$

by $f(x) = \sum \alpha_k\, x^k$.

Proposition 5.1. *The multiplication of any two polynomial functions using* (5.2)*, can be briefly written:*

$$f(x).\ g(x) = \sum \alpha_k\, x^k.\ \sum \beta_l\, x^l = \sum \alpha_k\, \beta_l\, \gamma_{kl}\, x^{k+l}. \tag{5.3}$$

where

$$\gamma_{kl} = \prod_{1 \leq i \leq n-1} \prod_{i<j \leq n} (-1)^{l_i k_j \hat{\imath}\, \hat{\jmath}}\, q_{ij}^{l_i k_j}.$$

Proof.

$$\begin{aligned}
x_1^{k_1} \ldots\ x_n^{k_n}\, x_1^{l_1} \ldots\ x_n^{l_n} &= \left(x_1^{k_1} \ldots\ x_n^{k_n}\right)\ \left(x_1^{l_1} \ldots\ x_n^{l_n}\right) \\
&= \left(\prod_{1<j \leq n} (-1)^{l_1 k_j \hat{1}\, \hat{\jmath}}\, q_{1j}^{l_1 k_j}\right) \left(x_1^{k_1+l_1} x_2^{k_2} \ldots\ x_n^{k_n}\right)\ \left(x_2^{l_2} \ldots\ x_n^{l_n}\right) \\
&= \left(\prod_{1 \leq i \leq n-1} (-1)^{l_i k_j \hat{1}\, \hat{\jmath}} \prod_{i<j \leq n} q_{ij}^{l_i k_j}\right) \left(x_1^{k_1+l_1} \ldots\ x_n^{k_n+l_n}\right) \\
&= \gamma_{kl}\, x_1^{(k_1+l_1)} \ldots\ x_n^{(k_n+l_n)}.
\end{aligned}$$

□

Definition 5.2. An intuitionistic L-fuzzy subset $B = (\mu_{_B}, \nu_{_B})$ of A_q is called a $\mathbb{Z}_2$-graded intuitionistic L-fuzzy q-deformed quantum subspaces of A_q if it satisfies three conditions:

1) $B = (\mu_{_B}, \nu_{_B})$ is a $\mathbb{Z}_2$-graded intuitionistic L-fuzzy vector subspace of A_q

2) $\mu_{_B}(fg) \geq \mu_{_B}(x) \wedge \mu_{_B}(y), \quad \nu_{_B}(fg) \leq \nu_{_B}(f) \vee \nu_{_B}(g)$ for any $f, g \in A_q(V)$

3) $\mu_{_B}(x_j x_i) = \mu_{_B}\big((-1)^{ij} q_{ij}\, x_i x_j\big), \quad \nu_{_B}(x_j x_i) = \nu_{_B}\big((-1)^{ij} q_{ij}\, x_i x_j\big)$, for all $i < j$.

Proposition 5.3. *Let $E = A_q$ and $B = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy subspace of E defined by*

$$\mu_B(x_i) = r_i, \quad \nu_B(x_i) = s_i, \quad r_i, s_i \in L, \ \textit{s.t.} \ 0 \le r_i + s_j \le 1 \quad \forall i, j \in L,$$

and for each $f \in E$,

$$\mu_B(f) = \sup_{0 \le k'_1, \ldots, k'_n \le m} \big\{(k'_1 r_1) \wedge \ldots \wedge (k'_n r_n)\big\}$$
$$\nu_B(f) = \inf_{0 \le k'_1, \ldots, k'_n \le m} \big\{(k'_1 s_1) \vee \ldots \vee (k'_n s_n)\big\},$$

where

$$k'_i = \begin{cases} 0, & \textit{if } k_i = 0, \\ 1, & \textit{if } k_i \ge 1. \end{cases} \tag{5.4}$$

Then $B = (\mu_B, \nu_B)$ is an intuitionistic L-fuzzy vector subspace of E.

Proof. Clearly for each $f, \ g \in E,$ we have $0 \le \mu_B(f) + \nu_B(f) \le 1.$ Also:

$$\begin{aligned}
\mu_B(f) \wedge \mu_B(g) &= \sup_{0 \le k'_1, \ldots, k'_n \le m} \big\{(k'_1 r_1) \wedge \ldots \wedge (k'_n r_n)\big\} \bigwedge \sup_{0 \le l'_1, \ldots, l'_n \le l} \big\{(l'_1 r_1) \wedge \ldots \wedge (l'_n r_n)\big\} \\
&\le \sup_{0 \le k'_1, \ldots, k'_n \le m} \big\{(k'_1 r_1) \wedge \ldots \wedge (k'_k r_k)\big\} \bigvee \sup_{0 \le l'_1, \ldots, l'_n \le l} \big\{(l'_1 r_1) \wedge \ldots \wedge (l'_n r_n)\big\} \\
&= \mu_B(f + g), \\
\nu_B(f) \vee \nu_B(g) &= \inf_{0 \le k'_1, \ldots, k'_n \le m} \big\{(k'_1 s_1) \vee \ldots \vee (k'_n s_n)\big\} \bigvee \inf_{0 \le l'_1, \ldots, l'_n \le l} \big\{(l'_1 s_1) \vee \ldots \vee (l'_n s_n)\big\} \\
&\ge \inf_{0 \le k'_1, \ldots, k'_n \le m} \big\{(k'_1 s_1) \vee \ldots \vee (k'_n s_n)\big\} \bigwedge \inf_{0 \le l'_1 < \ldots < l'_n \le l} \big\{(l'_1 s_1) \vee \ldots \vee (k'_n s_n)\big\} \\
&= \nu_B(f + g).
\end{aligned}$$

Since for any $\alpha \in k$, we have:

$$\mu_B(\alpha x_i) = r_i \ge \mu_B(x_i), \quad \nu_B(\alpha x_i) = s_i \le \nu_B(x_i),$$

hence

$$\mu_B(\alpha f) = \sup_{0 \le k'_1, \ldots, k'_n \le m} \big\{(k'_1 r_1) \wedge \ldots \wedge (k'_n r_n)\big\} = \mu_B(f),$$
$$\nu_B(\alpha f) = \inf_{0 \le k'_1, \ldots, k'_n \le m} \big\{(k'_1 s_1) \vee \ldots \vee (k'_n s_n)\big\} = \nu_B(f). \qquad \square$$

Definition 5.4. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. Suppose that $A_{\bar{0}} = (\mu_{A_{\bar{0}}}, \nu_{A_{\bar{0}}})$ and $A_{\bar{1}} = (\mu_{A_{\bar{1}}}, \nu_{A_{\bar{1}}})$ are intuitionistic L-fuzzy vector subspaces of $V_{\bar{0}}, \ V_{\bar{1}}$, respectively.
Define $A'_{\bar{0}} = (\mu_{A'_{\bar{0}}}, \nu_{A'_{\bar{0}}})$ where

$$\mu_{A'_{\bar{0}}}(x) = \begin{cases} \mu_{A_{\bar{0}}}(x) & x \in V_{\bar{0}} \\ 0 & x \notin V_{\bar{0}}, \end{cases} \qquad \nu_{A'_{\bar{0}}}(x) = \begin{cases} \nu_{A_{\bar{0}}}(x) & x \in V_{\bar{0}} \\ 1 & x \notin V_{\bar{0}}, \end{cases}$$

and define $A'_{\bar{1}} = (\mu_{A'_{\bar{1}}}, \nu_{A'_{\bar{1}}})$ where

$$\mu_{A'_{\bar{1}}}(x) = \begin{cases} \mu_{A_{\bar{1}}}(x) & x \in V_{\bar{1}} \\ 0 & x \notin V_{\bar{1}}, \end{cases} \qquad \nu_{A'_{\bar{1}}}(x) = \begin{cases} \nu_{A_{\bar{1}}}(x) & x \in V_{\bar{1}} \\ 1 & x \notin V_{\bar{1}}. \end{cases}$$

Then $A'_{\bar{0}} = (\mu_{A'_{\bar{0}}}, \nu_{A'_{\bar{0}}})$ and $A'_{\bar{1}} = (\mu_{A'_{\bar{1}}}, \nu_{A'_{\bar{1}}})$ are the intuitionistic L-fuzzy vector subspaces of V. Moreover, we have $A'_{\bar{0}} \cap A'_{\bar{1}} = \tilde{0}$. So $A'_{\bar{0}} + A'_{\bar{1}}$ is the direct sum and is denoted by $A'_{\bar{0}} \oplus A'_{\bar{1}}$. If $A = (\mu_A, \nu_A)$ is an intuitionistic L-fuzzy vector subspace of V and $A = A_{\bar{0}} \oplus A_{\bar{1}}$, then $A = (\mu_A, \nu_A)$ is called a $\mathbb{Z}_2$-graded intuitionistic L-fuzzy vector subspace of V and $A_{\bar{0}}$, $A_{\bar{1}}$ are called the even part and odd part of A respectively.

Proposition 5.5. *Let $B = (\mu_B, \nu_B)$ be the intuitionistic L-fuzzy vector subset of $E = A_q$ which we defined in Example 5.3. Setting $E_{\bar{0}}$, the k-algebra generated by*

$$\{x_1^{k_1} \dots x_n^{k_n} : \sum_{i=1} k_i \equiv 0 \pmod 2\}$$

and $E_{\bar{1}}$ the k-algebra generated by

$$\{x_1^{k_1} \dots x_n^{k_n} : \sum_{i=1} k_i \equiv 1 \pmod 2\},$$

it is clear that $E = E_{\bar{0}} \oplus E_{\bar{1}}$ has the structure of a k-superalgebra. Then $B = (\mu_B, \nu_B)$ is an intuitionistic L-fuzzy q-deformed quantum subsuperalgebra of E.

Proof. 1) We define two intuitionistic L-fuzzy vector subsets $B_{\bar{0}}$, $B_{\bar{1}}$ of E by

$$\mu_{B_{\bar{0}}}(x) = \begin{cases} \mu_B(x) & \text{if } x \in E_{\bar{0}} \\ 0 & \text{if } x \in E_{\bar{1}} \end{cases} \qquad \nu_{B_{\bar{0}}}(x) = \begin{cases} \nu_B(x) & \text{if } x \in E_{\bar{0}} \\ 1 & \text{if } x \in E_{\bar{1}} \end{cases}$$

$$\mu_{B_{\bar{1}}}(x) = \begin{cases} \mu_B(x) & \text{if } x \in E_{\bar{1}} \\ 0 & \text{if } x \in E_{\bar{0}} \end{cases} \qquad \nu_{B_{\bar{1}}}(x) = \begin{cases} \nu_B(x) & \text{if } x \in E_{\bar{1}} \\ 1 & \text{if } x \in E_{\bar{0}}. \end{cases}$$

Then $B_{\bar{0}} \cap B_{\bar{1}} = \tilde{0}$. We show that $B = B_{\bar{0}} + B_{\bar{1}}$. Since each $f \in E$ can be wrtien as $f = f_1 + f_2$ which $f_1 \in E_{\bar{0}}$ and $f_2 \in E_{\bar{1}}$. Hence we have

$$\mu_{B_{\bar{0}}+B_{\bar{1}}}(f) = \sup_{f=f_1+f_2} \{\mu_{B_{\bar{0}}}(f_1) \wedge \mu_{B_{\bar{1}}}(f_2)\} = \mu_B(f_1) \wedge \mu_B(f_2) = \mu_B(f)$$

$$\nu_{B_{\bar{0}}+B_{\bar{1}}}(f) = \inf_{f=f_1+f_2} \{\nu_{B_{\bar{0}}}(f_1) \wedge \nu_{B_{\bar{1}}}(f_2)\} = \nu_B(f_1) \vee \mu_B(f_2) = \nu_B(f)$$

Thus $B = B_{\bar{0}} \oplus B_{\bar{1}}$ is a $\mathbb{Z}_2$-graded intuitionistic L-fuzzy vector subspace of E.

2) For each f, $g \in E$, we have:

$$\begin{aligned} \mu_B(f) \wedge \mu_B(g) &= \sup_{0 \le k'_1, \dots, k'_n \le m} \{(k'_1 r_1) \wedge \dots \wedge (k'_n r_n)\} \bigwedge \sup_{0 \le l'_1, \dots, l'_n \le l} \{(l'_1 r_1) \wedge \dots \wedge (l'_n r_n)\} \\ &\le \sup_{0 \le k'_1, \dots, k'_n \le m,\ 0 \le l'_1, \dots, l'_n \le l} \left\{(k'_1 r_1) \wedge \dots \wedge (k'_n r_n) \bigwedge (l'_1 r_1) \wedge \dots \wedge (l'_n r_n)\right\} \\ &\le \sup_{0 \le (k+l)'_1, \dots, (k+l)'_n \le m+l} \left\{((k+l)'_1 r_1) \wedge \dots \wedge ((k+l)'_k r_k)\right\} \\ &= \mu_B(f\ g) \end{aligned}$$

$$\begin{aligned}
\nu_B(f) \vee \nu_B(g) &= \inf_{0 \le k'_1,\ldots,k'_n \le m} \big\{(k'_1 s_1) \vee \ldots \vee (k'_n s_n)\big\} \bigvee \inf_{0 \le l'_1,\ldots,l'_n \le l} \big\{(l'_1 s_1) \vee \ldots \vee (l'_n s_n)\big\} \\
&\ge \inf_{0 \le k'_1,\ldots,k'_n \le 1,\ 0 \le l'_1,\ldots,l'_n \le m} \Big\{(k'_1 s_1) \vee \ldots \vee (k'_n s_n) \bigvee (l'_1 s_1) \vee \ldots \vee (l'_n s_n)\Big\} \\
&\ge \inf_{0 \le (k+l)'_1,\ldots,(k+l)'_n \le 1} \Big\{((k+l)'_1 s_1) \vee \ldots \vee ((k+l)'_n s_n)\Big\} \\
&= \nu_B(f\, g).
\end{aligned}$$

In the case $\hat{i} = 1$ and $k_i = l_i = 1$, then by (5.2), $x_i^{k_i + l_i} = x_i^2 = 0$. So

$$\mu_B\big((x_1^{k_1} \ldots x_n^{k_n}).(x_1^{l_1} \ldots x_n^{l_n})\big) = \mu_B(0) = 1 \ge \mu_B(x_1^{k_1} \ldots x_n^{k_n}) \bigwedge \mu_B(x_1^{l_1} \ldots x_n^{l_n})$$

$$\nu_B\big((x_1^{k_1} \ldots x_n^{k_n}).(x_1^{l_1} \ldots x_n^{l_n})\big) = \nu_B(0) = 0 \le \nu_B(x_1^{k_1} \ldots x_n^{k_n}) \bigvee \nu_B(x_1^{l_1} \ldots x_n^{l_n})$$

Therefore condition 2) of Definition 5.2 holds.

Also, for all $0 \le i < j \le n$, we have

$$\mu_B(x_j x_i) = r_j \wedge r_i = r_i \wedge r_j = \mu_B\big((-1)^{ij} q_{ij}\, x_i x_j\big),$$

$$\nu_B(x_j x_i) = s_j \vee s_i = s_i \vee s_j = \nu_B\big((-1)^{ij} q_{ij}\, x_i x_j\big)$$

and this completes the proof. □

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