Notes on Intuitionistic Fuzzy Sets
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Some new results on intuitionistic fuzzy operators

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Abstract: The concept of intuitionistic fuzzy sets is becoming very popular day by day among mathematicians and researchers from other areas of science. It has been noticed that some operators have very interesting properties in Intuitionistic fuzzy sets. That is why these properties are investigated in this study. New findings on equality are obtained and proved.
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1 Introduction

The notion of fuzzy set was introduced and developed by L.A.Zadeh [15] in 1965. In fuzzy set concept, a membership function is defined to assign each element of the reference system, a real value in the interval [0,1]. The membership value of an element is 1 indicates that the element belong to that class whereas the membership value of an element is 0 indicates that the element does not belong to the class. In fuzzy set theory, the concept of non-membership function and the hesitation margin are overlooked. In 1983, Atanassov [1] generalized this concept and presented a new conception namely intuitionistic fuzzy sets as an extension of fuzzy sets accommodating both membership and non-membership functions along with a hesitation margin. It is to be noted that in intuitionistic fuzzy set theory, the sum of the membership function and non-membership function is a value between 0 and 1. Moreover the value of hesitation margin also lies between

0 and 1. The notion of modal operators was first introduced by Atanassov [2] in 1986. Modal operators ($\Box, \Diamond$) defined over the set of all intuitionistic fuzzy sets convert every intuitionistic fuzzy set into a fuzzy set. Atanassov [4] also introduced the operators ($\boxplus, \boxtimes$) over intuitionistic fuzzy sets, as well as the operators D_α, $F_{\alpha,\beta}$, $G_{\alpha,\beta}$, $H_{\alpha,\beta}$ and $H^*_{\alpha,\beta}$. Many mathematicians and researchers [5, 6, 7, 9, 10, 11, 12, 13, 14] are working hard to develop and enrich these concepts. In this paper, we investigate various properties of these intuitionistic fuzzy operators.

2 Preliminaries

Definition 2.1 [15] *Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{\langle x, \mu_A(x) \rangle : x \in X\}$, where $\mu_A : x \to [0,1]$ is the membership function of the fuzzy set A. Fuzzy set is a collection of objects with graded membership, i.e., having degrees of membership.*

Definition 2.2 [2] *Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$, where the functions $\mu_A, \nu_A : x \to [0,1]$ define, respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$,$0 \le \mu_A(x) + \nu_A(x) \le 1$.*

Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. Then, $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A, and $\pi_A(x) \in [0,1]$ that is $\pi_A : x \to [0,1]$ and $0 \le \pi_A(x) \le 1$ for every $x \in X$. Thus, $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to the IFS A or not.

Definition 2.3 [2] *Let A, B be two IFSs in X. The basic operations are defined as follows:*

1. $A \subseteq B \iff \mu_A(x) \le \mu_B(x)$ *and* $\nu_A(x) \ge \nu_B(x), \forall x \in X$.
2. $A = B \iff \mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x), \forall x \in X$.
3. $A^C = \{\langle x, \nu_A(x), \mu_A(x) \rangle : x \in X\}$.
4. $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$.
5. $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$.
6. $A \oplus B = \{\langle x, (\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)), \nu_A(x)\nu_B(x)\rangle : x \in X\}$.
7. $A \otimes B = \{\langle x, \mu_A(x)\mu_B(x), (\nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x))\rangle : x \in X\}$.
8. $A - B = \{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle : x \in X\}$.
9. $A \triangle B = \{\langle x, \max[\min(\mu_A(x), \nu_B(x)), \min(\mu_B(x), \nu_A(x))],$ $\min[\max(\nu_A(x), \mu_B(x)), \max(\nu_B(x), \mu_A(x))]\rangle : x \in X\}$.
10. $A \times B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x) \rangle : x \in X\}$.

Definition 2.4 [3] *Let A, B and C be IFSs in X. The algebraic laws are as follows:*

1. $(A^C)^C = A.$ *[Complementary Law]*
2. $(i)A \cup A = A,$ $(ii)A \cap A = A$ *[Idempotent Law]*
3. $(i)A \cup B = B \cup A,$ $(ii)A \cap B = B \cap A.$ *[Commutative Law]*
4. $(i)(A \cup B) \cup C = A \cup (B \cup C),$ $(ii)(A \cap B) \cap C = A \cap (B \cap C).$ *[Associative Law]*
5. $(i)A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$
   $(ii)A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$ *[Distributive Laws]*
6. $(i)(A \cup B)^C = A^C \cap B^C,$ $(ii)(A \cap B)^C = A^C \cup B^C.$ *[De Morgan's Laws]*
7. $(i)A \cap (A \cup B) = A,$ $(ii)A \cup (A \cap B) = A.$ *[Absorption Laws]*
8. $(i)A \oplus B = B \oplus A,$ $(ii)A \otimes B = B \otimes A.$
9. $(i)A \oplus (B \oplus C) = (A \oplus B) \oplus C,$ $(ii)A \otimes (B \otimes C) = (A \otimes B) \otimes C.$
10. $(i)(A \oplus B)^C = A^C \otimes B^C,$ $(ii)(A \otimes B)^C = A^C \oplus B^C.$
11. $(i)A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C),$ $(ii)A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C).$
12. $(i)A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C),$ $(ii)A \otimes (B \cap C) = (A \otimes B) \cap (A \otimes C).$
    Distributive laws hold for both right and left distributions.

Definition 2.5 [8] *Let A and B be two IFSs in a nonempty set X. Then*

1. $A \ominus B = \{\langle x, \frac{1}{2}[\mu_A(x) + \mu_B(x)], \frac{1}{2}[\nu_A(x) + \nu_B(x)] \rangle : x \in X\},$
2. $A\$B = \{\langle x, (\mu_A(x).\mu_B(x))^{\frac{1}{2}}, (\nu_A(x).\nu_B(x))^{\frac{1}{2}} \rangle : x \in X\}.$

Definition 2.6 [4] *Let X be a nonempty set. If A is an IFS drawn from X, then,*

1. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X\},$
2. $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle : x \in X\}.$

For a proper IFS, $\Box A \subset A \subset \Diamond A$ and $\Box A \neq A \neq \Diamond A$.

Definition 2.7 [4] *Let X be a nonempty set. If A is an IFS drawn from X, then,*

1. $\boxplus A = \{\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2} \rangle : x \in X\},$
2. $\boxtimes A = \{\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2} \rangle : x \in X\}.$

For a proper IFS, $\boxplus A \subset A \subset \boxtimes A$ and $\boxplus A \neq A \neq \boxtimes A$.

Definition 2.8 [4] *Let* $\alpha, \beta \in [0,1] and A \in$ *IFS* X. *Then the following operators can be defined as*

1. $D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1-\alpha)\pi_A(x)\rangle : x \in X\}$.
2. $F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$, *where* $\alpha + \beta \leq 1$.
3. $G_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle : x \in X\}$, *where* $\alpha + \beta \leq 1$.
4. $H_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta\pi_A(x)\ \rangle : x \in X\}$, *where* $\alpha + \beta \leq 1$.
5. $H^*_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x))\rangle : x \in X\}$, *where* $\alpha + \beta \leq 1$.

3 Main results

Throughout this paper, intuitionistic fuzzy set and fuzzy set are denoted by IFS and FS respectively. For brevity, instead of $\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ we will use the simpler notation $\langle \mu_A(x), \nu_A(x)\rangle$.

Theorem 3.1 *Let* X *be a nonempty set. If* A *and* B *be any two IFSs drawn from* X *and* $\alpha, \beta \in [0,1]$, *where* $\alpha + \beta \leq 1$, *then*

1. $[\boxplus \boxtimes (D_\alpha(A \cup B))]^C = \boxtimes \boxplus [D_\alpha(A \cup B)]^C$
2. $[\boxtimes \boxplus (D_\alpha(A \cup B))]^C = \boxplus \boxtimes [D_\alpha(A \cup B)]^C$
3. $[\boxplus \boxtimes (D_\alpha(A \cap B))]^C = \boxtimes \boxplus [D_\alpha(A \cap B)]^C$
4. $[\boxtimes \boxplus (D_\alpha(A \cap B))]^C = \boxplus \boxtimes [D_\alpha(A \cap B)]^C$
5. $[\boxplus \boxtimes (F_{\alpha,\beta}(A \cup B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A \cup B)]^C$
6. $[\boxtimes \boxplus (F_{\alpha,\beta}(A \cup B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A \cup B)]^C$
7. $[\boxplus \boxtimes (F_{\alpha,\beta}(A \cap B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A \cap B)]^C$
8. $[\boxtimes \boxplus (F_{\alpha,\beta}(A \cap B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A \cap B)]^C$
9. $[\boxplus \boxtimes (G_{\alpha,\beta}(A \cup B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A \cup B)]^C$
10. $[\boxtimes \boxplus (G_{\alpha,\beta}(A \cup B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A \cup B)]^C$
11. $[\boxplus \boxtimes (G_{\alpha,\beta}(A \cap B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A \cap B)]^C$
12. $[\boxtimes \boxplus (G_{\alpha,\beta}(A \cap B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A \cap B)]^C$
13. $[\boxplus \boxtimes (H_{\alpha,\beta}(A \cup B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A \cup B)]^C$
14. $[\boxtimes \boxplus (H_{\alpha,\beta}(A \cup B))]^C = \boxplus \boxtimes [H_{\alpha,\beta}(A \cup B)]^C$
15. $[\boxplus \boxtimes (H_{\alpha,\beta}(A \cap B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A \cap B)]^C$

16. $[\boxtimes\boxplus(H_{\alpha,\beta}(A\cap B))]^C = \boxplus\boxtimes[H_{\alpha,\beta}(A\cap B)]^C$

17. $[\boxplus\boxtimes(H^*_{\alpha,\beta}(A\cup B))]^C = \boxtimes\boxplus[H^*_{\alpha,\beta}(A\cup B)]^C$

18. $[\boxtimes\boxplus(H^*_{\alpha,\beta}(A\cup B))]^C = \boxplus\boxtimes[H^*_{\alpha,\beta}(A\cup B)]^C$

19. $[\boxplus\boxtimes(H^*_{\alpha,\beta}(A\cap B))]^C = \boxtimes\boxplus[H^*_{\alpha,\beta}(A\cap B)]^C$

20. $[\boxtimes\boxplus(H^*_{\alpha,\beta}(A\cap B))]^C = \boxplus\boxtimes[H^*_{\alpha,\beta}(A\cap B)]^C$.

Proof. (1) Now

$$\begin{aligned}
D_\alpha(A\cup B) &= \langle \mu_{A\cup B}(x)+\alpha\pi_{A\cup B}(x), \nu_{A\cup B}(x)+(1-\alpha)\pi_{A\cup B}(x)\ \rangle\\
\boxtimes(D_\alpha(A\cup B)) &= \langle \tfrac{1}{2}(\mu_{A\cup B}(x)+\alpha\pi_{A\cup B}(x)+1), \tfrac{1}{2}(\nu_{A\cup B}(x)+(1-\alpha)\pi_{A\cup B}(x))\ \rangle\\
[\boxplus\boxtimes(D_\alpha(A\cup B))] &= \langle \tfrac{1}{4}(\mu_{A\cup B}(x)+\alpha\pi_{A\cup B}+1)(x)), \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\cup B}(x)+(1-\alpha)\pi_{A\cup B}+1](x))\ \rangle\\
[\boxplus\boxtimes(D_\alpha(A\cup B))]^C &= \langle \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\cup B}(x)+(1-\alpha)\pi_{A\cup B}+1](x)), \tfrac{1}{4}(\mu_{A\cup B}(x)+\alpha\pi_{A\cup B}+1)(x))\ \rangle
\end{aligned}$$

Again,

$$\begin{aligned}
[D_\alpha(A\cup B)]^C &= \langle \nu_{A\cup B}(x)+(1-\alpha)\pi_{A\cup B}(x), \mu_{A\cup B}(x)+\alpha\pi_{A\cup B}(x)\ \rangle\\
\boxplus[D_\alpha(A\cup B)]^C &= \langle \tfrac{1}{2}(\nu_{A\cup B}(x)+(1-\alpha)\pi_{A\cup B}(x)))), \tfrac{1}{2}(\mu_{A\cup B}(x)+\alpha\pi_{A\cup B}(x)+1)\ \rangle\\
\boxtimes\boxplus[D_\alpha(A\cup B)]^C &= \langle \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\cup B}(x)+(1-\alpha)\pi_{A\cup B}+1](x)), \tfrac{1}{4}(\mu_{A\cup B}(x)+\alpha\pi_{A\cup B}+1)(x))\ \rangle.
\end{aligned}$$

Hence

$$[\boxplus\boxtimes(D_\alpha(A\cup B))]^C = \boxtimes\boxplus[D_\alpha(A\cup B)]^C.$$

Similarly other parts can be proved. □

Theorem 3.2 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha,\beta\in[0,1]$, where $\alpha+\beta\le 1$, then*

1. $[\boxplus\boxtimes(D_\alpha(A\oplus B))]^C = \boxtimes\boxplus[D_\alpha(A\oplus B)]^C$

2. $[\boxtimes\boxplus(D_\alpha(A\oplus B))]^C = \boxplus\boxtimes[D_\alpha(A\oplus B)]^C$

3. $[\boxplus\boxtimes(D_\alpha(A\otimes B))]^C = \boxtimes\boxplus[D_\alpha(A\otimes B)]^C$

4. $[\boxtimes\boxplus(D_\alpha(A\otimes B))]^C = \boxplus\boxtimes[D_\alpha(A\otimes B)]^C$

5. $[\boxplus\boxtimes(F_{\alpha,\beta}(A\oplus B))]^C = \boxtimes\boxplus[F_{\alpha,\beta}(A\oplus B)]^C$

6. $[\boxtimes\boxplus(F_{\alpha,\beta}(A\oplus B))]^C = \boxplus\boxtimes[F_{\alpha,\beta}(A\oplus B)]^C$

7. $[\boxplus\boxtimes(F_{\alpha,\beta}(A\otimes B))]^C = \boxtimes\boxplus[F_{\alpha,\beta}(A\otimes B)]^C$

8. $[\boxtimes\boxplus(F_{\alpha,\beta}(A\otimes B))]^C = \boxplus\boxtimes[F_{\alpha,\beta}(A\otimes B)]^C$

9. $[\boxplus\boxtimes(G_{\alpha,\beta}(A\oplus B))]^C = \boxtimes\boxplus[G_{\alpha,\beta}(A\oplus B)]^C$

10. $[\boxtimes\boxplus(G_{\alpha,\beta}(A\oplus B))]^C = \boxplus\boxtimes[G_{\alpha,\beta}(A\oplus B)]^C$

11. $[\boxplus\boxtimes(G_{\alpha,\beta}(A\otimes B))]^C = \boxtimes\boxplus[G_{\alpha,\beta}(A\otimes B)]^C$

12. $[\boxtimes\boxplus(G_{\alpha,\beta}(A\otimes B))]^C = \boxplus\boxtimes[G_{\alpha,\beta}(A\otimes B)]^C$

13. $[\boxplus\boxtimes(H_{\alpha,\beta}(A\oplus B))]^C = \boxtimes\boxplus[H_{\alpha,\beta}(A\oplus B)]^C$

14. $[\boxtimes\boxplus(H_{\alpha,\beta}(A\oplus B))]^C = \boxplus\boxtimes[H_{\alpha,\beta}(A\oplus B)]^C$

15. $[\boxplus\boxtimes(H_{\alpha,\beta}(A\otimes B))]^C = \boxtimes\boxplus[H_{\alpha,\beta}(A\otimes B)]^C$

16. $[\boxtimes\boxplus(H_{\alpha,\beta}(A\otimes B))]^C = \boxplus\boxtimes[H_{\alpha,\beta}(A\otimes B)]^C$

17. $[\boxplus\boxtimes(H^*_{\alpha,\beta}(A\oplus B))]^C = \boxtimes\boxplus[H^*_{\alpha,\beta}(A\oplus B)]^C$

18. $[\boxtimes\boxplus(H^*_{\alpha,\beta}(A\oplus B))]^C = \boxplus\boxtimes[H^*_{\alpha,\beta}(A\oplus B)]^C$

19. $[\boxplus\boxtimes(H^*_{\alpha,\beta}(A\otimes B))]^C = \boxtimes\boxplus[H^*_{\alpha,\beta}(A\otimes B)]^C$

20. $[\boxtimes\boxplus(H^*_{\alpha,\beta}(A\otimes B))]^C = \boxplus\boxtimes[H^*_{\alpha,\beta}(A\otimes B)]^C$.

Proof. (5) Let us suppose that $\alpha+\beta\le 1$. Now

$$\begin{aligned}
F_{\alpha,\beta}(A\oplus B) &= \langle \mu_{A\oplus B}(x)+\alpha\pi_{A\oplus B}(x), \nu_{A\oplus B}(x)+\beta\pi_{A\oplus B}(x)\,\rangle\\
\boxtimes(F_{\alpha,\beta}(A\oplus B)) &= \langle \tfrac{1}{2}(\mu_{A\oplus B}(x)+\alpha\pi_{A\oplus B}(x)+1), \tfrac{1}{2}(\nu_{A\oplus B}(x)+\beta\pi_{A\oplus B}(x))\,\rangle\\
[\boxplus\boxtimes(F_{\alpha,\beta}(A\oplus B))] &= \langle \tfrac{1}{4}(\mu_{A\oplus B}(x)+\alpha\pi_{A\oplus B}+1)(x)), \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\oplus B}(x)+\beta\pi_{A\oplus B}+1](x))\,\rangle\\
[\boxplus\boxtimes(F_{\alpha,\beta}(A\oplus B))]^C &= \langle \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\oplus B}(x)+(1-\alpha)\pi_{A\oplus B}+1](x)), \tfrac{1}{4}(\mu_{A\oplus B}(x)+\alpha\pi_{A\oplus B}+1)(x))\,\rangle.
\end{aligned}$$

Again,

$$\begin{aligned}
[F_{\alpha,\beta}(A\oplus B)]^C &= \langle \nu_{A\oplus B}(x)+\beta\pi_{A\oplus B}(x), \mu_{A\oplus B}(x)+\alpha\pi_{A\oplus B}(x)\,\rangle\\
\boxplus[F_{\alpha,\beta}(A\oplus B)]^C &= \langle \tfrac{1}{2}(\nu_{A\oplus B}(x)+\beta\pi_{A\oplus B}(x)))), \tfrac{1}{2}(\mu_{A\oplus B}(x)+\alpha\pi_{A\oplus B}(x)+1)\,\rangle\\
\boxtimes\boxplus[F_{\alpha,\beta}(A\oplus B)]^C &= \langle \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\oplus B}(x)+\beta\pi_{A\oplus B}+1](x)), \tfrac{1}{4}(\mu_{A\oplus B}(x)+\alpha\pi_{A\oplus B}+1)(x))\,\rangle.
\end{aligned}$$

Hence

$$[\boxplus\boxtimes(F_{\alpha,\beta}(A\oplus B))]^C = \boxtimes\boxplus[F_{\alpha,\beta}(A\oplus B)]^C.$$

Similarly other parts can be proved. □

Theorem 3.3 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha,\beta\in[0,1]$, where $\alpha+\beta\le 1$, then*

1. $[\boxplus\boxtimes(D_\alpha(A-B))]^C = \boxtimes\boxplus[D_\alpha(A-B)]^C$

2. $[\boxtimes\boxplus(D_\alpha(A-B))]^C = \boxplus\boxtimes[D_\alpha(A-B)]^C$

3. $[\boxplus\boxtimes(D_\alpha(A\triangle B))]^C = \boxtimes\boxplus[D_\alpha(A\triangle B)]^C$

4. $[\boxtimes\boxplus(D_\alpha(A\triangle B))]^C = \boxplus\boxtimes[D_\alpha(A\triangle B)]^C$

5. $[\boxplus\boxtimes(F_{\alpha,\beta}(A-B))]^C = \boxtimes\boxplus[F_{\alpha,\beta}(A-B)]^C$

6. $[\boxtimes \boxplus (F_{\alpha,\beta}(A-B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A-B)]^C$

7. $[\boxplus \boxtimes (F_{\alpha,\beta}(A\triangle B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A\triangle B)]^C$

8. $[\boxtimes \boxplus (F_{\alpha,\beta}(A\triangle B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A\triangle B)]^C$

9. $[\boxplus \boxtimes (G_{\alpha,\beta}(A-B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A-B)]^C$

10. $[\boxtimes \boxplus (G_{\alpha,\beta}(A-B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A-B)]^C$

11. $[\boxplus \boxtimes (G_{\alpha,\beta}(A\triangle B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A\triangle B)]^C$

12. $[\boxtimes \boxplus (G_{\alpha,\beta}(A\triangle B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A\triangle B)]^C$

13. $[\boxplus \boxtimes (H_{\alpha,\beta}(A-B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A-B)]^C$

14. $[\boxtimes \boxplus (H_{\alpha,\beta}(A-B))]^C = \boxplus \boxtimes [H_{\alpha,\beta}(A-B)]^C$

15. $[\boxplus \boxtimes (H_{\alpha,\beta}(A\triangle B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A\triangle B)]^C$

16. $[\boxtimes \boxplus (H_{\alpha,\beta}(A\triangle B))]^C = \boxplus \boxtimes [H_{\alpha,\beta}(A\triangle B)]^C$

17. $[\boxplus \boxtimes (H^*_{\alpha,\beta}(A-B))]^C = \boxtimes \boxplus [H^*_{\alpha,\beta}(A-B)]^C$

18. $[\boxtimes \boxplus (H^*_{\alpha,\beta}(A-B))]^C = \boxplus \boxtimes [H^*_{\alpha,\beta}(A-B)]^C$

19. $[\boxplus \boxtimes (H^*_{\alpha,\beta}(A\triangle B))]^C = \boxtimes \boxplus [H^*_{\alpha,\beta}(A\triangle B)]^C$

20. $[\boxtimes \boxplus (H^*_{\alpha,\beta}(A\triangle B))]^C = \boxplus \boxtimes [H^*_{\alpha,\beta}(A\triangle B)]^C$.

Proof. (9) Let us suppose that $\alpha + \beta \leq 1$. Now

$$\begin{aligned}
G_{\alpha,\beta}(A-B) &= \langle \alpha\mu_{A-B}(x), \beta\nu_{A-B}(x) \rangle \\
\boxtimes(G_{\alpha,\beta}(A-B)) &= \langle \tfrac{1}{2}(\alpha\mu_{A-B}(x)+1), \tfrac{1}{2}(\beta\nu_{A-B}(x)) \rangle \\
[\boxplus \boxtimes (G_{\alpha,\beta}(A-B))] &= \langle \tfrac{1}{4}(\alpha\mu_{A-B}(x)+1)), \tfrac{1}{2}[\tfrac{1}{2}(\beta\nu_{A-B}(x)+1] \rangle \\
[\boxplus \boxtimes (G_{\alpha,\beta}(A-B))]^C &= \langle \tfrac{1}{2}[\tfrac{1}{2}(\beta\nu_{A-B}(x)+1], \tfrac{1}{4}(\alpha\mu_{A-B}(x)+1)) \rangle
\end{aligned}$$

Again,

$$\begin{aligned}
[G_{\alpha,\beta}(A-B)]^C &= \langle \beta\nu_{A-B}(x), \alpha\mu_{A-B}(x) \rangle \\
\boxplus[F_{\alpha,\beta}(A-B)]^C &= \langle \tfrac{1}{2}(\beta\nu_{A-B}(x)), \tfrac{1}{2}(\alpha\mu_{A-B}(x)+1) \rangle \\
\boxtimes \boxplus [G_{\alpha,\beta}(A-B)]^C &= \langle \tfrac{1}{2}[\tfrac{1}{2}(\beta\nu_{A-B}(x)+1](x)), \tfrac{1}{4}(\alpha\mu_{A-B}(x)+1))(x)) \rangle
\end{aligned}$$

Hence

$$[\boxplus \boxtimes (G_{\alpha,\beta}(A-B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A-B)]^C.$$

Similarly, the other parts can be proved. $\square$

Theorem 3.4 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then*

1. $[\boxplus \boxtimes (D_\alpha(A \ominus B))]^C = \boxtimes \boxplus [D_\alpha(A \ominus B)]^C$
2. $[\boxtimes \boxplus (D_\alpha(A \ominus B))]^C = \boxplus \boxtimes [D_\alpha(A \ominus B)]^C$
3. $[\boxplus \boxtimes (D_\alpha(A\$B))]^C = \boxtimes \boxplus [D_\alpha(A\$B)]^C$
4. $[\boxtimes \boxplus (D_\alpha(A\$B))]^C = \boxplus \boxtimes [D_\alpha(A\$B)]^C$
5. $[\boxplus \boxtimes (F_{\alpha,\beta}(A \ominus B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A \ominus B)]^C$
6. $[\boxtimes \boxplus (F_{\alpha,\beta}(A \ominus B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A \ominus B)]^C$
7. $[\boxplus \boxtimes (F_{\alpha,\beta}(A\$B))]^C = \boxtimes \boxplus [F_{\alpha,\beta}(A\$B)]^C$
8. $[\boxtimes \boxplus (F_{\alpha,\beta}(A\$B))]^C = \boxplus \boxtimes [F_{\alpha,\beta}(A\$B)]^C$
9. $[\boxplus \boxtimes (G_{\alpha,\beta}(A \ominus B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A \ominus B)]^C$
10. $[\boxtimes \boxplus (G_{\alpha,\beta}(A \ominus B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A \ominus B)]^C$
11. $[\boxplus \boxtimes (G_{\alpha,\beta}(A\$B))]^C = \boxtimes \boxplus [G_{\alpha,\beta}(A\$B)]^C$
12. $[\boxtimes \boxplus (G_{\alpha,\beta}(A\$B))]^C = \boxplus \boxtimes [G_{\alpha,\beta}(A\$B)]^C$
13. $[\boxplus \boxtimes (H_{\alpha,\beta}(A \ominus B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A \ominus B)]^C$
14. $[\boxtimes \boxplus (H_{\alpha,\beta}(A \ominus B))]^C = \boxplus \boxtimes [H_{\alpha,\beta}(A \ominus B)]^C$
15. $[\boxplus \boxtimes (H_{\alpha,\beta}(A\$B))]^C = \boxtimes \boxplus [H_{\alpha,\beta}(A\$B)]^C$
16. $[\boxtimes \boxplus (H_{\alpha,\beta}(A\$B))]^C = \boxplus \boxtimes [H_{\alpha,\beta}(A\$B)]^C$
17. $[\boxplus \boxtimes (H^*_{\alpha,\beta}(A \ominus B))]^C = \boxtimes \boxplus [H^*_{\alpha,\beta}(A \ominus B)]^C$
18. $[\boxtimes \boxplus (H^*_{\alpha,\beta}(A \ominus B))]^C = \boxplus \boxtimes [H^*_{\alpha,\beta}(A \ominus B)]^C$
19. $[\boxplus \boxtimes (H^*_{\alpha,\beta}(A\$B))]^C = \boxtimes \boxplus [H^*_{\alpha,\beta}(A\$B)]^C$
20. $[\boxtimes \boxplus (H^*_{\alpha,\beta}(A\$B))]^C = \boxplus \boxtimes [H^*_{\alpha,\beta}(A\$B)]^C$.

Proof. (1) Now

$$\begin{aligned}
D_\alpha(A \ominus B) &= \langle \mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B}(x), \nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x) \rangle \\
\boxtimes(D_\alpha(A \ominus B)) &= \langle \tfrac{1}{2}(\mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B}(x) + 1), \tfrac{1}{2}(\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x)) \rangle \\
[\boxplus \boxtimes (D_\alpha(A \ominus B))] &= \langle \tfrac{1}{4}(\mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B} + 1)(x)), \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B} + 1](x)) \rangle \\
[\boxplus \boxtimes (D_\alpha(A \ominus B))]^C &= \langle \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B} + 1](x)), \tfrac{1}{4}(\mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B} + 1)(x)) \rangle.
\end{aligned}$$

Again,

$$[D_\alpha(A \ominus B)]^C = \langle \nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x), \mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B}(x) \ \rangle$$
$$\boxplus[D_\alpha(A \ominus B)]^C = \langle \tfrac{1}{2}(\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B}(x)))), \tfrac{1}{2}(\mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B}(x) + 1) \ \rangle$$
$$\boxtimes \boxplus [D_\alpha(A \ominus B)]^C = \langle \tfrac{1}{2}[\tfrac{1}{2}(\nu_{A\ominus B}(x) + (1-\alpha)\pi_{A\ominus B} + 1](x)), \tfrac{1}{4}(\mu_{A\ominus B}(x) + \alpha\pi_{A\ominus B} + 1)(x)) \ \rangle.$$

Hence

$$[\boxplus \boxtimes (D_\alpha(A \ominus B))]^C = \boxtimes \boxplus [D_\alpha(A \ominus B)]^C.$$

Similarly, the other parts can be proved. $\square$

Theorem 3.5 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then*

1. $[(\boxplus(D_\alpha(A)) \cup (\boxtimes(D_\alpha(B))]^C = (\boxtimes[D_\alpha(A)]^C) \cap (\boxplus[D_\alpha(B)]^C)$
2. $[(\boxtimes(D_\alpha(A)) \cup (\boxplus(D_\alpha(B))]^C = (\boxplus[D_\alpha(A)]^C) \cap (\boxtimes[D_\alpha(B)]^C)$
3. $[(\boxplus(D_\alpha(A)) \cap (\boxtimes(D_\alpha(B))]^C = (\boxtimes[D_\alpha(A)]^C) \cup (\boxplus[D_\alpha(B)]^C)$
4. $[(\boxtimes(D_\alpha(A)) \cap (\boxplus(D_\alpha(B))]^C = (\boxplus[D_\alpha(A)]^C) \cup (\boxtimes[D_\alpha(B)]^C)$
5. $[(\boxplus(F_{\alpha,\beta}(A)) \cup (\boxtimes(F_{\alpha,\beta}(B))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C) \cap (\boxplus[F_{\alpha,\beta}(B)]^C)$
6. $[(\boxtimes(F_{\alpha,\beta}(A)) \cup (\boxplus(F_{\alpha,\beta}(B))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C) \cap (\boxtimes[F_{\alpha,\beta}(B)]^C)$
7. $[(\boxplus(F_{\alpha,\beta}(A)) \cap (\boxtimes(F_{\alpha,\beta}(B))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C) \cup (\boxplus[F_{\alpha,\beta}(B)]^C)$
8. $[(\boxtimes(F_{\alpha,\beta}(A)) \cap (\boxplus(F_{\alpha,\beta}(B))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C) \cup (\boxtimes[F_{\alpha,\beta}(B)]^C)$
9. $[(\boxplus(G_{\alpha,\beta}(A)) \cup (\boxtimes(G_{\alpha,\beta}(B))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C) \cap (\boxplus[G_{\alpha,\beta}(B)]^C)$
10. $[(\boxtimes(G_{\alpha,\beta}(A)) \cup (\boxplus(G_{\alpha,\beta}(B))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C) \cap (\boxtimes[G_{\alpha,\beta}(B)]^C)$
11. $[(\boxplus(G_{\alpha,\beta}(A)) \cap (\boxtimes(G_{\alpha,\beta}(B))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C) \cup (\boxplus[G_{\alpha,\beta}(B)]^C)$
12. $[(\boxtimes(G_{\alpha,\beta}(A)) \cap (\boxplus(G_{\alpha,\beta}(B))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C) \cup (\boxtimes[G_{\alpha,\beta}(B)]^C)$
13. $[(\boxplus(H_{\alpha,\beta}(A)) \cup (\boxtimes(H_{\alpha,\beta}(B))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C) \cap (\boxplus[H_{\alpha,\beta}(B)]^C)$
14. $[(\boxtimes(H_{\alpha,\beta}(A)) \cup (\boxplus(H_{\alpha,\beta}(B))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C) \cap (\boxtimes[H_{\alpha,\beta}(B)]^C)$
15. $[(\boxplus(H_{\alpha,\beta}(A)) \cap (\boxtimes(H_{\alpha,\beta}(B))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C) \cup (\boxplus[H_{\alpha,\beta}(B)]^C)$
16. $[(\boxtimes(H_{\alpha,\beta}(A)) \cap (\boxplus(H_{\alpha,\beta}(B))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C) \cup (\boxtimes[H_{\alpha,\beta}(B)]^C)$
17. $[(\boxplus(H^*_{\alpha,\beta}(A)) \cup (\boxtimes(H^*_{\alpha,\beta}(B))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \cap (\boxplus[H^*_{\alpha,\beta}(B)]^C)$
18. $[(\boxtimes(H^*_{\alpha,\beta}(A)) \cup (\boxplus(H^*_{\alpha,\beta}(B))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) \cap (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$
19. $[(\boxplus(H^*_{\alpha,\beta}(A)) \cap (\boxtimes(H^*_{\alpha,\beta}(B))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \cup (\boxplus[H^*_{\alpha,\beta}(B)]^C)$
20. $[(\boxtimes(H^*_{\alpha,\beta}(A)) \cap (\boxplus(H^*_{\alpha,\beta}(B))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) \cup (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$

Proof. (1) Now,

$$\begin{aligned}[\boxplus D_\alpha(A)] \cup [\boxtimes D_\alpha(B)] &= \langle \tfrac{1}{2}[\mu_A(x) + \alpha\pi_A(x)], \tfrac{1}{2}[\nu_A(x) + (1-\alpha)\pi_A(x) + 1]\ \rangle \\ &\quad \cup \langle \tfrac{1}{2}[\mu_B(x) + \alpha\pi_B(x) + 1], \tfrac{1}{2}[\nu_B(x) + (1-\alpha)\pi_B(x)]\ \rangle \\ &= \langle \max[\tfrac{1}{2}[\mu_A(x) + \alpha\pi_A(x)], \tfrac{1}{2}[\mu_B(x) + \alpha\pi_B(x) + 1], \\ &\quad \min[\tfrac{1}{2}[\nu_A(x) + (1-\alpha)\pi_A(x) + 1], \tfrac{1}{2}[\nu_B(x) + (1-\alpha)\pi_B(x)]\ \rangle.\end{aligned}$$

So,

$$\begin{aligned}[\boxplus(D_\alpha(A)) \cup \boxtimes(D_\alpha(B))]^C = \langle &\min[\tfrac{1}{2}[\nu_A(x) + (1-\alpha)\pi_A(x) + 1], \tfrac{1}{2}[\nu_B(x) + (1-\alpha)\pi_B(x)] \\ &\max[\tfrac{1}{2}[\mu_A(x) + \alpha\pi_A(x)], \tfrac{1}{2}[\mu_B(x) + \alpha\pi_B(x) + 1]\ \rangle.\end{aligned}$$

Again,

$$\begin{aligned}\boxtimes[D_\alpha(A)]^C \cap \boxplus[D_\alpha(B)]^C &= \langle \tfrac{1}{2}[\nu_A(x) + (1-\alpha)\pi_A(x) + 1], \tfrac{1}{2}[\mu_A(x) + \alpha\pi_A(x)]\ \rangle \\ &\quad \cap \langle \tfrac{1}{2}[\nu_B(x) + (1-\alpha)\pi_B(x)], \tfrac{1}{2}[\mu_B(x) + \alpha\pi_B(x) + 1]\ \rangle \\ &= \langle \min[\tfrac{1}{2}[\nu_A(x) + (1-\alpha)\pi_A(x) + 1], \tfrac{1}{2}[\nu_B(x) + (1-\alpha)\pi_B(x)], \\ &\quad \max[\tfrac{1}{2}[\mu_A(x) + \alpha\pi_A(x)], \tfrac{1}{2}[\mu_B(x) + \alpha\pi_B(x) + 1]\ \rangle.\end{aligned}$$

Therefore,

$$[\boxplus(D_\alpha(A)) \cup \boxtimes(D_\alpha(B))]^C = (\boxtimes[D_\alpha(A)]^C) \cap (\boxplus[D_\alpha(B)]^C).$$

Hence the proof.
Similarly other parts can be proved.

□

Theorem 3.6 *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then*

1. $[(\boxplus(D_\alpha(A)) \oplus (\boxtimes(D_\alpha(B))]^C = (\boxtimes[D_\alpha(A)]^C) \otimes (\boxplus[D_\alpha(B)]^C)$
2. $[(\boxtimes(D_\alpha(A)) \oplus (\boxplus(D_\alpha(B))]^C = (\boxplus[D_\alpha(A)]^C) \otimes (\boxtimes[D_\alpha(B)]^C)$
3. $[(\boxplus(D_\alpha(A)) \otimes (\boxtimes(D_\alpha(B))]^C = (\boxtimes[D_\alpha(A)]^C) \oplus (\boxplus[D_\alpha(B)]^C)$
4. $[(\boxtimes(D_\alpha(A)) \otimes (\boxplus(D_\alpha(B))]^C = (\boxplus[D_\alpha(A)]^C) \oplus (\boxtimes[D_\alpha(B)]^C)$
5. $[(\boxplus(F_{\alpha,\beta}(A)) \oplus (\boxtimes(F_{\alpha,\beta}(B))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C) \otimes (\boxplus[F_{\alpha,\beta}(B)]^C)$
6. $[(\boxtimes(F_{\alpha,\beta}(A)) \oplus (\boxplus(F_{\alpha,\beta}(B))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C) \otimes (\boxtimes[F_{\alpha,\beta}(B)]^C)$
7. $[(\boxplus(F_{\alpha,\beta}(A)) \otimes (\boxtimes(F_{\alpha,\beta}(B))]^C = (\boxtimes[F_{\alpha,\beta}(A)]^C) \oplus (\boxplus[F_{\alpha,\beta}(B)]^C)$
8. $[(\boxtimes(F_{\alpha,\beta}(A)) \otimes (\boxplus(F_{\alpha,\beta}(B))]^C = (\boxplus[F_{\alpha,\beta}(A)]^C) \oplus (\boxtimes[F_{\alpha,\beta}(B)]^C)$
9. $[(\boxplus(G_{\alpha,\beta}(A)) \oplus (\boxtimes(G_{\alpha,\beta}(B))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C) \otimes (\boxplus[G_{\alpha,\beta}(B)]^C)$
10. $[(\boxtimes(G_{\alpha,\beta}(A)) \oplus (\boxplus(G_{\alpha,\beta}(B))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C) \otimes (\boxtimes[G_{\alpha,\beta}(B)]^C)$

11. $[(\boxplus(G_{\alpha,\beta}(A)) \otimes (\boxtimes(G_{\alpha,\beta}(B))]^C = (\boxtimes[G_{\alpha,\beta}(A)]^C) \oplus (\boxplus[G_{\alpha,\beta}(B)]^C)$

12. $[(\boxtimes(G_{\alpha,\beta}(A)) \otimes (\boxplus(G_{\alpha,\beta}(B))]^C = (\boxplus[G_{\alpha,\beta}(A)]^C) \oplus (\boxtimes[G_{\alpha,\beta}(B)]^C)$

13. $[(\boxplus(H_{\alpha,\beta}(A)) \oplus (\boxtimes(H_{\alpha,\beta}(B))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C) \otimes (\boxplus[H_{\alpha,\beta}(B)]^C)$

14. $[(\boxtimes(H_{\alpha,\beta}(A)) \oplus (\boxplus(H_{\alpha,\beta}(B))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C) \otimes (\boxtimes[H_{\alpha,\beta}(B)]^C)$

15. $[(\boxplus(H_{\alpha,\beta}(A)) \otimes (\boxtimes(H_{\alpha,\beta}(B))]^C = (\boxtimes[H_{\alpha,\beta}(A)]^C) \oplus (\boxplus[H_{\alpha,\beta}(B)]^C)$

16. $[(\boxtimes(H_{\alpha,\beta}(A)) \otimes (\boxplus(H_{\alpha,\beta}(B))]^C = (\boxplus[H_{\alpha,\beta}(A)]^C) \oplus (\boxtimes[H_{\alpha,\beta}(B)]^C)$

17. $[(\boxplus(H^*_{\alpha,\beta}(A)) \oplus (\boxtimes(H^*_{\alpha,\beta}(B))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \otimes (\boxplus[H^*_{\alpha,\beta}(B)]^C)$

18. $[(\boxtimes(H^*_{\alpha,\beta}(A)) \oplus (\boxplus(H^*_{\alpha,\beta}(B))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) \otimes (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$

19. $[(\boxplus(H^*_{\alpha,\beta}(A)) \otimes (\boxtimes(H^*_{\alpha,\beta}(B))]^C = (\boxtimes[H^*_{\alpha,\beta}(A)]^C) \oplus (\boxplus[H^*_{\alpha,\beta}(B)]^C)$

20. $[(\boxtimes(H^*_{\alpha,\beta}(A)) \otimes (\boxplus(H^*_{\alpha,\beta}(B))]^C = (\boxplus[H^*_{\alpha,\beta}(A)]^C) \oplus (\boxtimes[H^*_{\alpha,\beta}(B)]^C)$.

Proof. (1) Now,

$$\begin{aligned}
[\boxplus D_\alpha(A)] \oplus [\boxtimes D_\alpha(B) &= \langle \tfrac{1}{2}[\mu_A(x) + \alpha\pi_A(x)], \tfrac{1}{2}[\nu_A(x) + (1-\alpha)\pi_A(x)) + 1] \,\rangle \\
&\quad \oplus \langle \tfrac{1}{2}[(\mu_B(x) + \alpha\pi_B(x) + 1], \tfrac{1}{2}[\nu_B(x) + (1-\alpha)\pi_B(x)] \,\rangle \\
&= \langle \tfrac{1}{2}(\mu_A(x) + \alpha\pi_A(x)) + \tfrac{1}{2}(\mu_B(x) + \alpha\pi_B(x) + 1) \\
&\quad - \tfrac{1}{2}(\mu_A(x) + \alpha\pi_A(x))\tfrac{1}{2}(\mu_B(x) + \alpha\pi_B(x) + 1), \\
&\quad \tfrac{1}{2}(\nu_A(x) + (1-\alpha)\pi_A(x) + 1)\tfrac{1}{2}(\nu_B(x) + (1-\alpha)\pi_B(x)) \,\rangle.
\end{aligned}$$

So,

$$\begin{aligned}
[\boxplus(D_\alpha(A)) \oplus \boxtimes(D_\alpha(B))]^C &= \langle \tfrac{1}{4}(\nu_A(x) + (1-\alpha)\pi_A(x) + 1)(\nu_B(x) + (1-\alpha)\pi_B(x)), \\
&\quad \tfrac{1}{2}(\mu_A(x) + \alpha\pi_A(x)) + \tfrac{1}{2}(\mu_B(x) + \alpha\pi_B(x) + 1) \\
&\quad - \tfrac{1}{4}(\mu_A(x) + \alpha\pi_A(x))(\mu_B(x) + \alpha\pi_B(x) + 1) \,\rangle.
\end{aligned}$$

Again,

$$\begin{aligned}
\boxtimes[D_\alpha(A)]^C \otimes \boxplus[D_\alpha(B)]^C &= \langle \tfrac{1}{2}[\nu_A(x) + (1-\alpha)\pi_A(x) + 1], \tfrac{1}{2}[\mu_A(x) + \alpha\pi_A(x)] \,\rangle \\
&\quad \otimes \langle \tfrac{1}{2}[\nu_B(x) + (1-\alpha)\pi_B(x)], \tfrac{1}{2}[\mu_B(x) + \alpha\pi_B(x) + 1] \,\rangle \\
&= \langle \tfrac{1}{4}(\nu_A(x) + (1-\alpha)\pi_A(x) + 1)(\nu_B(x) + (1-\alpha)\pi_B(x)), \\
&\quad \tfrac{1}{2}(\mu_A(x) + \alpha\pi_A(x)) + \tfrac{1}{2}(\mu_B(x) + \alpha\pi_B(x) + 1) \\
&\quad - \tfrac{1}{4}(\mu_A(x) + \alpha\pi_A(x))(\mu_B(x) + \alpha\pi_B(x) + 1) \,\rangle.
\end{aligned}$$

Therefore,

$$[\boxplus(D_\alpha(A)) \oplus \boxtimes(D_\alpha(B))]^C = (\boxtimes[D_\alpha(A)]^C) \otimes (\boxplus[D_\alpha(B)]^C).$$

Hence the proof.
Similarly other parts can be proved. □

Remark 3.7 Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then

1. $(\boxplus(H^*_{\alpha,\beta}(A)) \cup (\boxplus(H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A \cup B)$
2. $(\boxtimes(H^*_{\alpha,\beta}(A)) \cup (\boxtimes(H^*_{\alpha,\beta}(B)) \neq \boxtimes H^*_{\alpha,\beta}(A \cup B)$
3. $(\boxplus(H^*_{\alpha,\beta}(A)) \cap (\boxplus(H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A \cap B)$
4. $(\boxtimes(H^*_{\alpha,\beta}(A)) \cap (\boxtimes(H^*_{\alpha,\beta}(B)) \neq \boxtimes H^*_{\alpha,\beta}(A \cap B)$
5. $(\boxplus(H^*_{\alpha,\beta}(A)) \oplus (\boxplus(H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A \oplus B)$
6. $(\boxtimes(H^*_{\alpha,\beta}(A)) \oplus (\boxtimes(H^*_{\alpha,\beta}(B)) \neq \boxtimes H^*_{\alpha,\beta}(A \oplus B)$
7. $(\boxplus(H^*_{\alpha,\beta}(A)) \otimes (\boxplus(H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A \otimes B)$
8. $(\boxtimes(H^*_{\alpha,\beta}(A)) \otimes (\boxtimes(H^*_{\alpha,\beta}(B)) \neq \boxtimes H^*_{\alpha,\beta}(A \otimes B)$
9. $(\boxplus(H^*_{\alpha,\beta}(A))\triangle(\boxplus(H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A\triangle B)$
10. $(\boxtimes(H^*_{\alpha,\beta}(A))\triangle(\boxtimes(H^*_{\alpha,\beta}(B)) \neq \boxtimes H^*_{\alpha,\beta}(A\triangle B)$
11. $(\boxplus(H^*_{\alpha,\beta}(A)) - (\boxplus(H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A - B)$
12. $(\boxtimes(H^*_{\alpha,\beta}(A)) - (\boxtimes(H^*_{\alpha,\beta}(B)) \neq \boxtimes H^*_{\alpha,\beta}(A - B)$
13. $(\boxplus(H^*_{\alpha,\beta}(A)) \times (\boxplus(H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A \times B)$
14. $(\boxtimes(H^*_{\alpha,\beta}(A)) \times (\boxtimes(H^*_{\alpha,\beta}(B)) \neq \boxtimes H^*_{\alpha,\beta}(A \times B)$
15. $(\boxplus(H^*_{\alpha,\beta}(A))\$(\boxplus(H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A\$B)$
16. $(\boxtimes(H^*_{\alpha,\beta}(A))\$(\boxtimes(H^*_{\alpha,\beta}(B)) \neq \boxtimes H^*_{\alpha,\beta}(A\$B)$

Proof. We consider a counter-example. Let $A = \langle 0.7, 0.2, 0.1 \rangle$ and $B = \langle 0.8, 0.1, 0.1 \rangle$. Here, we consider $\alpha = 0.4$ and $\beta = 0.2$.

$$(\boxplus H^*_{\alpha,\beta}(A)) \oplus (\boxplus H^*_{\alpha,\beta}(B)) = \langle 0.2776, 0.4290 \rangle$$

and

$$\boxplus H^*_{\alpha,\beta}(A \oplus B) = \langle 0.188, 0.6604 \rangle.$$

So $(\boxplus H^*_{\alpha,\beta}(A)) \oplus (\boxplus H^*_{\alpha,\beta}(B)) \neq \boxplus H^*_{\alpha,\beta}(A \oplus B)$
Similarly the other parts of the remark can be shown. □

The results of the above Remark 3.7 are also true for the operators D_α ,$F_{\alpha,\beta}$, $G_{\alpha,\beta}$ and $H_{\alpha,\beta}$ that means if $H^*_{\alpha,\beta}$ is replaced by any one operatorD_α,$F_{\alpha,\beta}$, $G_{\alpha,\beta}$ or $H_{\alpha,\beta}$ then the remark stated above is also true.

4 Conclusion

New properties of some operators in intuitionistic fuzzy sets have been investigated in this paper. Using the features of intuitionistic fuzzy operators some new equalities have been obtained.These equalities are very useful because they are shorter and more practical. These equalities could be made use in many application areas of intuitioistic fuzzy operators and will provide more practical solutions.

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