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# A measure extension theorem

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**Abstract**

In the paper continuous set functions are considered where the additional condition is substituted by max-min condition:$\mu(A \cup B) = \max(\mu(A), \mu(B))$, $\mu(A \cap B) = \min(\mu(A), \mu(B))$. For such functions the extension theorem is proved from an algebra to the generalized $\sigma$-algebra.

## 1 Introduction

The notion of maxitive measure was introduced by Shilkret $\mu(A \cup B) = max(\mu(A), \mu(B))$.
In our approach we prefer a symmetric notion.
Let $\mathcal{R}$ be an algebra of subsets of space $\Omega$.
We shall study M-measure $\mu : \mathcal{R} \longrightarrow [0, 1]$ satisfying the following properties:

1. $\mu(\Omega) = 1,\ \mu(\emptyset) = 0$;
2. $\mu(A \cup B) = \max(\mu(A), \mu(B))$,
   $\mu(A \cap B) = \min(\mu(a), \mu(B))$ for any A,B$\in \mathcal{R}$;
3. If $A_n \nearrow A$, $B_n \searrow B$, $A_n, B_n \in \mathcal{R}$, (n=1,2,...), $A, B \in \mathcal{R}$,
   then $\mu(A_n) \nearrow \mu(A)$ and $\mu(B_n) \searrow \mu(B)$.

*Remark 1.1* *Let* $A_n \in \mathcal{R},\ (n = 1, 2, ...),\ A_n \subset A_{n+1}$ *and* $\bigcup_{n=1}^{\infty} A_n = A$ *then* $A_n \nearrow A$.
*Let* $B_n \in \mathcal{R},\ (n = 1, 2, ...),\ B_n \supset B_{n+1}$ *and* $\bigcap_{n=1}^{\infty} B_n = B$ *then* $B_n \searrow B$.

The main result of the paper is the following theorem.

**Theorem:** To any M-measure $\mu$ defined on an algebra $\mathcal{R}$ there exists exactly one M-measure $\overline{\mu}$ on $\sigma(\mathcal{R})$ extending $\mu$.

# 2 Proof of theorem

Denote by $\mathcal{R}^+$ the family of all sets A that can be presented in the form

$$A = \bigcup_{n=1}^{\infty} A_n,\ A_n \subset A_{n+1},\ A_n \in \mathcal{R}\ (n = 1, 2, ...).$$

Evidently $\mu(A_n) \leq\ \mu(A_{n+1}) \leq\ \mu(\Omega) \leq\ 1\ (n = 1, 2, ...)$, hence there exists $\lim\limits_{n \longrightarrow \infty}(A_n)$.

We want to prove that the limit does not depend on the choice of the sequence $(A_n)$ but only the set A.

**Lemma 2.1** *Let $A_n, B_m \in \mathcal{R}\ (m, n = 1, 2...),\ A_n \nearrow A,\ B_m \nearrow A$. Then $\lim\limits_{n \longrightarrow \infty} \mu(A_n) = \lim\limits_{m \longrightarrow \infty} \mu(B_m)$.*

*Proof.*
At first we fix an index m.
Then

$$A_n \cap B_m \nearrow A \cap B_m = B_m,$$

hence

$$\mu(B_m) = \lim_{n \longrightarrow \infty} \mu(A_n \cap B_m) \leq \lim_{n \longrightarrow \infty} \mu(A_n).$$

Since the inequality holds for any m, we obtain

$$\lim_{m \longrightarrow \infty} \mu(B_m) \leq \lim_{n \longrightarrow \infty} \mu(A_n).$$

Now we fix an index n and analogue we obtained

$$\lim_{m \longrightarrow \infty} \mu(A_n) \leq \lim_{n \longrightarrow \infty} \mu(B_m),$$

hence $\lim\limits_{n \longrightarrow \infty} \mu(A_n) = \lim\limits_{m \longrightarrow \infty} \mu(B_m)$.

**Definition 2.2** *Denote by $\mathcal{R}^+$ the set of all $A \subset \Omega$ such that there exist $A_n \in \mathcal{R}$ $(n = 1, 2, ...),\ A_n \nearrow A$. Then we define $\mu^+(A) = \lim\limits_{n \longrightarrow \infty} \mu(A_n)$.*

**Proposition 2.3** *If $A_n \in \mathcal{R}^+,\ A_n \nearrow A$, then $A \in \mathcal{R}^+$ and $\lim\limits_{n \longrightarrow \infty} \mu^+(A_n) = \mu^+(A)$.*

*Proof.*
Since $A_n \in \mathcal{R}^+$, there are $A_{n,i} \in \mathcal{R}$ such that $A_{n,i} \nearrow A_n,\ (i \longrightarrow \infty)$.
Put

$$B_n = \bigcup_{i=1}^{n} A_{i,n}\ (n = 1, 2, ...).$$

Then $B_n \in \mathcal{R}\ (n = 1, 2, ..)$,

$$B_n = \bigcup_{i=1}^{n} A_{i,n} \subset \bigcup_{i=1}^{n} A_{i,n+1} \subset \bigcup_{i=1}^{n+1} A_{i,n+1} = B_{n+1}.$$

Moreover

$$B_n = \bigcup_{i=1}^{n} A_{i,n} \subset \bigcup_{i=1}^{n} A_i = A_n \subset A,$$

hence

$$\mu^+(A) = \lim_{n\longrightarrow\infty} \mu^+(B_n) \le \lim_{n\longrightarrow\infty} \mu^+(A_n) \le \mu^+(A).$$

Since $\mu$ is self dual, also the dual notion can be defined.

**Definition 2.4** *Denote by* $\mathcal{R}^-$ *the set of all* $A \in \Omega$ *such that there exists* $A_n \in \mathcal{R}$ $(n = 1, 2, ...),\ A_n \searrow A$. *Then we define* $\mu^-(A) = \lim_{n\longrightarrow\infty} \mu(A_n)$.

**Proposition 2.5** *If* $A_n \in \mathcal{R}^-,\ A_n \searrow A$, *then* $A \in \mathcal{R}^-$, *and* $\lim_{n\longrightarrow\infty} \mu^-(A_n) = \mu^-(A)$.

**Proposition 2.6** *If* $A, B \in \mathcal{R}^+,\ C, D \in \mathcal{R}^-$ *then*

$$A \cup B \in \mathcal{R}^+, A \cap B \in \mathcal{R}^+, C \cup D \in \mathcal{R}^-, C \cap D \in \mathcal{R}^-$$

*and*

$$\mu^+(A \cup B) = \max(\mu^+(A), \mu^+(B)),\quad \mu^+(A \cap B) = \min(\mu^+(A), \mu^+(B))$$
$$\mu^-(C \cup D) = \max(\mu^-(C), \mu^-(D)),\quad \mu^-(C \cap D) = \min(\mu^-(C), \mu^-(D)).$$

*Proof.*
Let $A_n, B_n \in \mathcal{R},\ A_n \nearrow A,\ B_n \nearrow B$.
Then

$$A_n \cup B_n \nearrow A \cup B,\ A_n \cap B_n \nearrow A \cap B$$

and

$$\mu^+(A \cup B) = \lim_{n\longrightarrow\infty} \mu(A_n \cup B_n) = \lim_{n\longrightarrow\infty} \max(\mu(A_n), \mu(B_n)) =$$
$$\max(\lim_{n\longrightarrow\infty} \mu(A_n), \lim_{n\longrightarrow\infty} \mu(B_n)) = \max(\mu^+(A), \mu^+(B)).$$

$$\mu^+(A \cap B) = \lim_{n\longrightarrow\infty} \mu(A_n \cap B_n) = \lim_{n\longrightarrow\infty} \min(\mu(A_n), \mu(B_n)) =$$
$$\min(\lim_{n\longrightarrow\infty} \mu(A_n), \lim_{n\longrightarrow\infty} \mu(B_n)) = \min(\mu^+(A), \mu^+(B)).$$

The dual assertion can be proved analogously.

**Proposition 2.7** *If $B \in \mathcal{R}^+,\ C \in \mathcal{R}^-$ and $C \subset B$, then $\mu^-(C) \leq \mu^+(B)$.*

*Proof.*
Let $B_n \in \mathcal{R},\ B_n \nearrow B$.
Then

$$C = (C \setminus B_n) \cup (C \cap B_n),$$

hence

$$\mu^-(C) = \mu^-((C \setminus B_n) \cup (C \cap B_n)) = \max(\mu^-(C \setminus B_n), \mu^-(C \cap B_n)) \leq$$

$$\leq \max(\mu^-(C \setminus B_n), \mu(B_n)) \leq \max(\mu^-(C \setminus B_n), \mu^+(B)).$$

On the other hand

$$C \setminus B_n \in \mathcal{R},\ \textit{then } C \setminus B_n \searrow C \setminus B \textit{ and } C \setminus B = \emptyset.$$

Therefore

$$\mu^-(C) \leq \max(\lim_{n \longrightarrow \infty} \mu^-(C \setminus B_n), \mu^+(B)) = \max(0, \mu^+(B)) = \mu^+(B).$$

**Definition 2.8** *For any $A \subset \Omega$ put*

$$\mu^*(A) = \inf\{\mu^+(B); B \in \mathcal{R}^+,\ B \supset A\},$$

$$\mu_*(A) = \sup\{\mu^-(C); C \in \mathcal{R}^-,\ C \subset A\}.$$

**Proposition 2.9** *For any $A \in \Omega$ $\mu_*(A) \leq \mu^*(A)$.*

*Proof.*
Let $C \in \mathcal{R}^-,\ C \subset A,\ B \in \mathcal{R}^+,\ B \supset A$.
By Prop. 2.7 $\mu^-(C) \leq \mu^+(B)$, hence

$$\mu_*(A) = \sup\{\mu^-(C); C \in \mathcal{R}^-,\ C \subset A\} \leq \mu^+(B)$$

and therefore

$$\mu_*(A) \leq \inf\{\mu^+(B); B \in \mathcal{R}^+,\ A \subset B\} = \mu^*(A).$$

**Proposition 2.10** *If $A_n \nearrow A,\ B_n \searrow B$, then $\mu^*(A_n) \nearrow \mu^*(A),\ \mu_*(B_n) \searrow \mu_*(B)$.*

*Proof.*
Since $A_n \subset A$ evidently $\mu^*(A_n) \leq \mu^*(A)$,hence $\lim_{n \longrightarrow \infty} \mu^*(A_n) \leq \mu^*(A)$. On the other hand to any $\varepsilon > 0$ there exists $B_n \in \mathcal{R}^+,\ B_n \supset A_n$ such that

$$\mu^*(A_n) + \varepsilon > \mu^+(B_n).$$

Put $C_n = \bigcup_{i=1}^{n}(B_i)$. Then $C_n \in \mathcal{R}^+,\ C_n \supset \bigcup_{i=1}^{n}(A_i) = A_n$ and

$$\mu^+(C_n) = \max_{1 \leq i \leq n} \mu^+(B_i) < \max_{1 \leq i \leq n}(\mu^*(A_i) + \varepsilon) = \max_{1 \leq i \leq n}(\mu^*(A_i)) + \varepsilon = \mu^*(A_n) + \varepsilon.$$

Therefore

$$\mu^*(A) \le \mu^+(\bigcup_{n=1}^{\infty} C_n) = \lim_{n\longrightarrow\infty} \mu^+(C_n) \le \lim_{n\longrightarrow\infty} \mu^*(A_n) + \varepsilon.$$

Since the inequality holds for $\varepsilon > 0$, we obtain

$$\mu^*(A) \le \lim_{n\longrightarrow\infty} \mu^*(A_n)$$

Then it is true,that

$$\lim_{n\longrightarrow\infty} \mu^*(A_n) = \mu^*(A).$$

The dual assertion can be proved similarly.

**Definition 2.11** *Define* $\mathcal{M} = \{A \in \Omega;\ \mu^*(A) = \mu_*(A)\}$.

**Proposition 2.12** $\mathcal{M}$ *is a monotone family, i.e.* $A_n, B_n \in \mathcal{M}$ $(n = 1, 2, ...)$, $A_n \nearrow A,\ B_n \searrow B$ *implies that* $A \in \mathcal{M},\ B \in \mathcal{M}$.

*Proof.*
We have

$$\mu_*(A) \le \mu^*(A) = \lim_{n\longrightarrow\infty} \mu^*(A_n) = \lim_{n\longrightarrow\infty} \mu_*(A_n) \le \mu_*(A)$$

hence

$$\mu_*(A) = \mu^*(A),\ A \in \mathcal{M}.$$

Similarly

$$\mu_*(B) = \lim_{n\longrightarrow\infty} \mu_*(B_n) = \lim_{n\longrightarrow\infty} \mu^*(B_n) \ge \mu^*(B)$$

Then also $B \in \mathcal{M}$.

**Proposition 2.13** $\mathcal{R} \subset \mathcal{M},\ \mu^*$ *is an extension of* $\mu$.

*Proof.*
Evidently $\mu(A) = \mu^+(A) = \mu^-(A)$. Therefore $\mu^*(A) \le \mu^+(A) = \mu^-(A) \le \mu_*(A)$, but also $\mu^*(A) \ge \mu_*(A)$.
Then $\mu(A) = \mu^*(A) = \mu_*(A)$.

**Proposition 2.14** *If* $\sigma(\mathcal{R})$ *is the* $\sigma$*-algebra generated by* $\mathcal{R}$*, then* $\sigma(\mathcal{R}) \subset \mathcal{M}$.

*Proof.*
We have proved that $\mathcal{R} \subset \mathcal{M}, \mathcal{M}$ is a monotone family. Therefore $\mathcal{M}$ contains the monotone family generated by $\mathcal{R}$, and this family coincides with $\sigma(\mathcal{R})$.

**Proposition 2.15** *Let* $\overline{\mu} = \mu^* \mid \sigma(\mathcal{R})$. *Then* $\overline{\mu}$ *is an M-measure,* $\overline{\mu} \mid \mathcal{R} = \mu$.

*Proof.*
Since $\overline{\mu} = \mu^* \mid \sigma(\mathcal{R}),\ \mu = \mu^* \mid \mathcal{R}$, we have $\mu = \overline{\mu} \mid \mathcal{R}$, hence $\overline{\mu}(\Omega) = 1,\ \overline{\mu}(\emptyset) = 0$.
By Prop.2.12 we obtain that $\overline{\mu}$ is continuous.
Finally to any $A, B \in \sigma(\mathcal{R})$ and any $\varepsilon > 0$ there are $C \in \mathcal{R}^+,\ D \in \mathcal{R}^+,\ C \supset A,\ D \supset B$ such that

$$\overline{\mu}(A) + \varepsilon = \mu^*(A) + \varepsilon > \mu^+(C),$$

$$\overline{\mu}(B) + \varepsilon = \mu^*(B) + \varepsilon > \mu^+(D).$$

Therefore

$$\max(\overline{\mu}(A) + \varepsilon,\ \overline{\mu}(B) + \varepsilon) > \max(\mu^+(C), \mu^+(D)) =$$
$$= \mu^+(C \cup D) \geq \mu^+(A \cup B) = \overline{\mu}(A \cup B).$$

Since the inequality holds for any $\varepsilon > 0$, we have $\max(\overline{\mu}(A),\ \overline{\mu}(B)) \geq \overline{\mu}(A \cup B)$. The opposite inequality can be proved similarly.
Then we obtained, that

$$\max(\overline{\mu}(A),\ \overline{\mu}(B)) = \overline{\mu}(A \cup B)$$

and

$$\min(\overline{\mu}(A),\ \overline{\mu}(B)) = \overline{\mu}(A \cap B)$$

**Proof of Theorem**

The existence of $\overline{\mu}$ was proved in *Proposition 2.15*, now we shall prove the uniqueness.
Let $\nu : \sigma(\mathcal{R}) \longrightarrow [0,1]$ be an M-measure $\nu \mid \mathcal{R} = \mu$.
Put

$$\mathcal{K} = \{A \in \sigma(\mathcal{R}); \nu(A) = \overline{\mu}(A)\}.$$

By the assumption $\mathcal{K} \supset \mathcal{R}$. Evidently $\mathcal{K}$ is a monotone family. Therefore $\mathcal{K}$ contains the monotone family generated by $\mathcal{R}$, but this family coincides with $\sigma(\mathcal{R})$.
Hence there exists exactly one measure $\overline{\mu}$ on $\sigma(\mathcal{R})$, which is extending of measure $\mu$.

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# References

[1] Krachounov, M.: Intuitionistic probability and intuitionistic fuzzy sets. In: First International Workshop on Intuitionistic Fuzzy Sets, Generalized Nets and Knowledge Engeneering (E. El-Darzi. R. Atanassov, P. Chountas eds.) Univ.of Westminister, London 2006, 18-24.

[2] Pap, E.: Pseudodditive measures and their applications. In: Handbook of Measure Theory (E. Pap ed.). Elsevier, Amsterdam 2002, 1403 - 1465.

[3] Riečanová, Z.: About $\delta$-additive and $\delta$-maxitive measures. Math. Slovaca 32, 1982, 413 -436.

[4] Shilkret, N.: Maxitive measure and integration. Indag.Math. 33 (1971), 109 - 116.