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# The weak intuitionistic fuzzy implication based on $\Delta^{*}$ operation 

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#### Abstract

In this paper, a new weak intuitionistic fuzzy implication is introduced based on the recently defined $\Delta *$ operation. Fulfillment of some axioms and properties, together with Modus Ponens and Modus Tollens inference rules, are investigated. The negation induced by the studied implication is presented.


Keywords: Intuitionistic fuzzy implication, Intuitionistic fuzzy logic.
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## 1 Introduction

In 1983, Atanassov presented in [1] a concept of some kind of vague sets, called Intuitionistic Fuzzy Sets (IFSs). The concept directly alluded to the concept of Fuzzy Sets (FSs) introduced in 1965 by L. A. Zadeh. IFS, however, differs from FS, because of the introduced independence of the membership degree and non-membership degree of an element $x$ to a set $A$. While in the FS the non-membership degree of elements $x$ to the FS $A$ is equal to $1-\mu_{A}(x)$, where $\mu_{A}(x)$ is the membership degree, Atanassov introduced separate values $\mu_{A}(x)$ and $\nu_{A}(x)$ of membership and non-membership of $x$ to the IFS $A$.

In the Intuitionistic Fuzzy Logic (IFL) the truth-value of a variable $x$ is given by ordered pair $\langle a, b\rangle$, where $a, b, a+b \in[0,1]$. We will call such a pair (see [5]) an Intuitionistic Fuzzy Pair (IFP). The numbers $a$ and $b$ are interpreted as the degrees of validity and non-validity of $x$. We denote the truth-value of $x$ by $V(x)$.

We denote the variable with truth-value true in the classical logic by $\underline{1}$ and the variable false by $\underline{0}$. For these variables it holds therefore: $V(\underline{1})=\langle 1,0\rangle$ and $V(\underline{0})=\langle 0,1\rangle$. We will also use the variable with the truth-value $\langle 0,0\rangle$, which will be called full ignorance $\underline{\mathrm{FI}}$. So: $V(\underline{\mathrm{FI}})=\langle 0,0\rangle$. Such a variable does not exist in either the classical or the fuzzy logic.

We call the variable $x$ an Intuitionistic Fuzzy Tautology (shortly: IFT), if and only if for $V(x)=\langle a, b\rangle$ it holds that $a \geq b$, and an Intuitionistic Fuzzy co-Tautology (IFcT), if it holds that $a \leq b$.

For a variable $x$ we define the value of negation of $x$ in the typical form $V(\neg x)=\langle b, a\rangle$.
For the IF pairs we can define various operations. One of them is the operation $\Delta^{*}$ introduced in [24] by Dworniczak. It is somewhat a 'dual operation' to the operation $\triangle$, introduced by Atanassova in [12] and investigated by Atanassova and Dworniczak in [13]. Therefore, the operation $\Delta^{*}$ generates the weak intuitionistic fuzzy implication analogous to the implication presented in [14]. Because of this analogy, the layout of the present paper is identical to [14]. However, it should be emphasized, that it deals with a different operation.

Definition 1 ([24]). For IFPs $\langle a, b\rangle$ and $\langle c, d\rangle$, the $\triangle *$ operation is defined as follows:

$$
\begin{equation*}
\langle a, b\rangle \triangle *\langle c, d\rangle=\left\langle\frac{2-b-d}{4-a-b-c-d}, \frac{2-a-c}{4-a-b-c-d}\right\rangle . \tag{1}
\end{equation*}
$$

For the next considerations, we introduce firstly some ordering relation for the intuitionistic truth-values. For $V(x)=\langle a, b\rangle$ and $V(y)=\langle c, d\rangle$ where $a, b, c, d, a+b, c+d \in[0,1]$, we denote $V(x) \preceq V(y)$ if and only if $a \leq c$ and $b \geq d$. The notation $V(x) \succeq V(y)$ means $V(y) \preceq V(x)$.

One of the important logical connectives in the IFL is the Intuitionistic Fuzzy Implication (IFI). In this paper, we will omit the formal difference between an implication as a logical connective and an implicator as a binary operator. For some considerations making this difference is important.

The general conditions for the IFI were given first by Cornelis and Deschrijver [16], Cornelis, Deschrijver and Kerre [18, 19], Cornelis, Deschrijver, Cock and Kerre [17], and later by Liu and Wang [26], and Zhou, Wu and Zhang [28]. The conditions are based on the conditions known for the classical fuzzy implication (see e.g. [15], Def 1.1.1., p. 2).

Definition 2 ([16], Def. 4.2, p.6). Let $V(x), V\left(x_{1}\right), V\left(x_{2}\right), V(y), V\left(y_{1}\right), V\left(y_{2}\right) \in L$ be any intuitionistic truth-values (intuitionistic fuzzy pairs). The intuitionistic fuzzy implicator is the mapping $I: L^{2} \rightarrow L$, fulfilling the properties:
(IFI 1) if $V\left(x_{1}\right) \preceq V\left(x_{2}\right)$, then $I\left(V\left(x_{1}\right), V(y)\right) \succeq I\left(V\left(x_{2}\right), V(y)\right)$,
(IFI 2) if $V\left(y_{1}\right) \preceq V\left(y_{2}\right)$, then $I\left(V(x), V\left(y_{1}\right)\right) \preceq I\left(V(x), V\left(y_{2}\right)\right)$,
(IFI 3) $\quad I(V(\underline{1}), V(\underline{1}))=V(\underline{1})$,
(IFI 4) $\quad I(V(\underline{1}), V(\underline{0}))=V(\underline{0})$,
(IFI 5) $\quad I(V(\underline{0}), V(\underline{0}))=V(\underline{1})$,
(IFI 6) $\quad I(V(\underline{0}), V(\underline{1}))=V(\underline{1})$.
We can see that the condition (IFI 6) can be omitted. The (IFI 6) condition can be obtained as a corollary from the (IFI 5) and (IFI 2) conditions.

We can find in the literature the definition of the intuitionistic fuzzy implicator (implication) without the conditions (IFI 1) and (IFI 2) (see, e.g., [27], Def. 10, p. 3). However, it is the isolated case, and, moreover, neglecting the monotonicity conditions (IFI 1) and (IFI 2) is inappropriate and allows too much freedom in defining of the 'implicator' or 'implication'.

In the literature on the subject, almost 200 different intuitionistic fuzzy implications were noticed (see, e.g., [2-4]). One of them is presented by Atanassova in [6]. Such a type of implication is called by Dworniczak in [22] a Weak Intuitionistic Fuzzy Implication (WIFI). The WIFIs are studied in [7-11, 14, 20-23].

Definition 3 ([22], Def. 2, p. 13). The Weak Intuitionistic Fuzzy Implication (WIFI) is the logical connective $\Rightarrow$, fulfilling the conditions:
(WIFI 1) if $V\left(x_{1}\right) \preceq V\left(x_{2}\right)$, then $V\left(x_{1} \Rightarrow y\right) \succeq V\left(x_{2} \Rightarrow y\right)$,
(WIFI 2) if $V\left(y_{1}\right) \preceq V\left(y_{2}\right)$, then $V\left(x \Rightarrow y_{1}\right) \preceq V\left(x \Rightarrow y_{2}\right)$,
(WIFI 3) $\underline{0} \Rightarrow y$ is an IFT,
(WIFI 4) $x \Rightarrow \underline{1}$ is an IFT,
(WIFI 5) $\underline{1} \Rightarrow \underline{0}$ is an IFcT.
We will call the operation 'weak' because the (WIFI 3)-(WIFI 5) conditions are mainly defined in the 'strong' form as (IFI 3)-(IFI 6) (see e.g. [16, 18]).

The most important kind of IFIs is called an ( $S, N$ )-implication (or an $S$-implication). It is an implication with the truth value $I(V(x), V(y))=S((V(x)), V(y))$, where $S$ is an $s$-norm and $N$ is some negation operator. In this case, the $s$-norm $S$ must be an intuitionistic counterpart of the classical $s$-norm (see e.g. [18]).

## 2 Main results

We introduce now a new weak intuitionistic fuzzy implication $\rightarrow^{*}$. The implication given below is a result of using the $\Delta^{*}$ operation in the role of the $s$-norm in the $(S, N)$ - implication. Because of this fact we will call it the implication based on $\triangle^{*}$ operation. The negation is in this case the classical negation $\neg$.

Symbolically we write: $V\left(x \rightarrow^{*} y\right)=V(\neg x) \triangle^{*} V(y)$.
We can therefore formulate the following Theorem 1.
Theorem 1. Let $V(x)=\langle a, b\rangle$ and $V(y)=\langle c, d\rangle$ be the truth-values of the variables $x$ and $y$, respectively, and $a, b, c, d, a+b, c+d \in[0,1]$. The intuitionistic logical connective $\rightarrow^{*}$, with the truth-value:

$$
V\left(x \rightarrow^{*} y\right)=\left\langle\frac{2-a-d}{4-a-b-c-d}, \frac{2-b-c}{4-a-b-c-d}\right\rangle
$$

is a weak intuitionistic fuzzy implication (WIFI).
Proof. As a preliminary note, the pair $\left\langle\frac{2-a-d}{4-a-b-c-d}, \frac{2-b-c}{4-a-b-c-d}\right\rangle$ is an IFP because:

$$
0 \leq \frac{2-a-d}{4-a-b-c-d} \leq 1
$$

and

$$
0 \leq \frac{2-b-c}{4-a-b-c-d} \leq 1,
$$

and

$$
0 \leq \frac{2-a-d}{4-a-b-c-d}+\frac{2-b-c}{4-a-b-c-d}=1 \leq 1 .
$$

The connective $\rightarrow^{*}$ fulfills conditions (WIFI 1)-(WIFI 5) because of the following reasoning:

- (WIFI 1) Let $V(y)=\langle c, d\rangle$. If $\left\langle a_{1}, b_{1}\right\rangle=V\left(x_{1}\right) \preceq V\left(x_{2}\right)=\left\langle a_{2}, b_{2}\right\rangle$, then $a_{1} \leq a_{2}$ and $b_{1} \geq b_{2}$, and, consequently, also $\left(1-a_{1}\right)\left(1-b_{2}\right) \geq\left(1-a_{2}\right)\left(1-b_{1}\right)$. Therefore,

$$
(1-d)\left(1-b_{2}-1+b_{1}\right)+(1-c)\left(1-a_{1}-1+a_{2}\right)+\left(\left(1-a_{1}\right)\left(1-b_{2}\right)-\left(1-a_{2}\right)\left(1-b_{1}\right)\right) \geq 0
$$

so

$$
\begin{aligned}
& (1-d)\left(1-b_{2}\right)+(1-c)\left(1-a_{1}\right)+\left(1-a_{1}\right)\left(1-b_{2}\right) \geq(1-d)\left(1-b_{1}\right)+(1-c)\left(1-a_{2}\right)+\left(1-a_{2}\right)\left(1-b_{1}\right), \\
& \left(1-a_{1}\right)\left(1-a_{2}\right)+\left(1-a_{1}\right)(1-d)+\left(1-a_{2}\right)(1-d)+(1-c)(1-d)+(1-d)(1-d)+ \\
& +(1-d)\left(1-b_{2}\right)+(1-c)\left(1-a_{1}\right)+\left(1-a_{1}\right)\left(1-b_{2}\right) \geq\left(1-a_{1}\right)\left(1-a_{2}\right)+\left(1-a_{1}\right)(1-d)+ \\
& +\left(1-a_{2}\right)(1-d)+(1-c)(1-d)+(1-d)(1-d)+(1-d)\left(1-b_{1}\right)+(1-c)\left(1-a_{2}\right)+\left(1-a_{2}\right)\left(1-b_{1}\right), \\
& \left(1-a_{1}+1-d\right)\left(1-a_{2}+1-b_{2}+1-c+1-d\right) \geq\left(1-a_{2}+1-d\right)\left(1-a_{1}+1-b_{1}+1-c+1-d\right),
\end{aligned}
$$

and finally

$$
\frac{2-a_{1}-d}{4-a_{1}-b_{1}-c-d} \geq \frac{2-a_{2}-d}{4-a_{2}-b_{2}-c-d} .
$$

In the same manner, we can check the inequality

$$
\frac{2-b_{1}-c}{4-a_{1}-b_{1}-c-d} \leq \frac{2-b_{2}-c}{4-a_{2}-b_{2}-c-d} .
$$

Therefore,

$$
\left\langle\frac{2-a_{1}-d}{4-a_{1}-b_{1}-c-d}, \frac{2-b_{1}-c}{4-a_{1}-b_{1}-c-d}\right\rangle \succeq\left\langle\frac{2-a_{2}-d}{4-a_{2}-b_{2}-c-d}, \frac{2-b_{2}-c}{4-a_{2}-b_{2}-c-d}\right\rangle
$$

Hence, $V\left(x_{1} \rightarrow^{*} y\right) \succeq V\left(x_{2} \rightarrow^{*} y\right)$, and the proof of (WIFI 1) is completed.

- (WIFI 2) Let $V(x)=\langle a, b\rangle$. If $\left\langle c_{1}, d_{1}\right\rangle=V\left(y_{1}\right) \preceq V\left(y_{2}\right)=\left\langle c_{2}, d_{2}\right\rangle$, therefore, $c_{1} \leq c_{2}$ and $d_{1} \geq d_{2}$, and, consequently, $\left(1-c_{1}\right)\left(1-d_{2}\right) \geq\left(1-c_{2}\right)\left(1-d_{1}\right)$, then:

$$
(1-a)\left(c_{1}-c_{2}\right)+(1-b)\left(d_{2}-d_{1}\right)+\left(\left(1-c_{2}\right)\left(1-d_{1}\right)-\left(1-c_{1}\right)\left(1-d_{2}\right)\right) \leq 0,
$$

so

$$
\begin{aligned}
& (1-a)\left(1-c_{2}-1+c_{1}\right)+(1-b)\left(1-d_{1}-1+d_{2}\right)+\left(1-c_{2}\right)\left(1-d_{1}\right)-\left(1-c_{1}\right)\left(1-d_{2}\right) \leq 0 \\
& \left(1-a+1-d_{1}\right)\left(1-a+1-b+1-c_{2}+1-d_{2}\right) \leq\left(1-a+1-d_{2}\right)\left(1-a+1-b+1-c_{1}+1-d_{1}\right)
\end{aligned}
$$

and finally

$$
\frac{2-a-d_{1}}{4-a-b-c_{1}-d_{1}} \leq \frac{2-a-d_{2}}{4-a-b-c_{2}-d_{2}} .
$$

In the same manner, we can check the inequality

$$
\frac{2-b-c_{1}}{4-a-b-c_{1}-d_{1}} \geq \frac{2-b-c_{2}}{4-a-b-c_{2}-d_{2}} .
$$

Therefore,

$$
\left\langle\frac{2-a-d_{1}}{4-a-b-c_{1}-d_{1}}, \frac{2-b-c_{1}}{4-a-b-c_{1}-d_{1}}\right\rangle \preceq\left\langle\frac{2-a-d_{2}}{4-a-b-c_{2}-d_{2}}, \frac{2-b-c_{2}}{4-a-b-c_{2}-d_{2}}\right\rangle .
$$

Hence, $V\left(x \rightarrow^{*} y_{1}\right) \preceq V\left(x \rightarrow^{*} y_{2}\right)$, and the proof of (WIFI 2) is completed.

- (WIFI 3) Let $V(y)=\langle c, d\rangle$. It is, by definition, $V\left(\underline{0} \rightarrow^{*} y\right)=\left\langle\frac{2-d}{3-c-d}, \frac{1-c}{3-c-d}\right\rangle$. Since the inequality $\frac{2-d}{3-c-d} \geq \frac{1-c}{3-c-d}$ is equivalent to $c+1 \geq d$, which holds for $c, d \in[0,1]$, therefore $\underline{0} \rightarrow^{*} y$ is an IFT.
- (WIFI 4) Let $V(x)=\langle a, b\rangle$. It is, by definition, $V\left(x \rightarrow^{*} \underline{1}\right)=\left\langle\frac{2-a}{3-a-b}, \frac{1-b}{3-a-b}\right\rangle$. Since the inequality $\frac{2-a}{3-a-b} \geq \frac{1-b}{3-a-b}$ is equivalent to $b+1 \geq a$, and this holds for $a, b \in[0,1]$. Therefore, $x \rightarrow^{*} \underline{1}$ is an IFT.
- (WIFI 5) We have $V\left(\underline{1} \rightarrow^{*} \underline{0}\right)=\left\langle\frac{0}{2}, \frac{2}{2}\right\rangle=V(\underline{0})$ and therefore $\underline{1}^{*} \underline{0}^{*}$ is an IFcT.

The implication $\rightarrow^{*}$ fulfills conditions (IFI 4) and (IFI 6) of Definition 2 but does not fulfill (IFI 3) and (IFI 5). It satisfies only (IFI 3') and (IFI 5') in the form:
(IFI 3') $V\left(\underline{1} \rightarrow^{*} \underline{1}\right)=\left\langle\frac{1}{2}, \frac{1}{2}\right\rangle$,
(IFI 5') $\quad V\left(\underline{0} \rightarrow{ }^{*} \underline{0}\right)=\left\langle\frac{1}{2}, \frac{1}{2}\right\rangle$.
Both values are not equal to $V(\underline{1})$, however, both are IFTs. Moreover, for any variable $x$ the following is true

$$
V\left(x \rightarrow^{*} x\right)=\left\langle\frac{1}{2}, \frac{1}{2}\right\rangle
$$

In the literature ${ }^{1}$ on fuzzy implications (not necessarily intuitionistic fuzzy implications), besides (WIFI 1)-(WIFI 5) or (IFI 1)-(IFI 6), the following axioms are also postulated:
(IFI 7) $\quad V(\underline{1} \Rightarrow y)=V(y)$,
(IFI 8) $\quad V(x \Rightarrow x)=V(\underline{1})$,
(IFI 9) $\quad V(x \Rightarrow(y \Rightarrow z))=V(y \Rightarrow(x \Rightarrow z))$
(IFI 10) $V(x \Rightarrow y)=V(\underline{1})$ iff $V(x) \preceq V(y)$,
(IFI 11) $V(x \Rightarrow y)=V(N(y) \Rightarrow N(x))$, where $N$ is a negation,
where $x, y, z$ are variables with the truth-value $V(x)=\langle a, b\rangle, V(y)=\langle c, d\rangle, V(z)=\langle e, f\rangle$, where $a$, $b, c, d, e, f, a+b, c+d, e+f \in[0,1]$, and $\Rightarrow$ is an implication.

Theorem 2. The implication $\rightarrow$ *
a) does not satisfy (IFI 7), (IFI 8), and (IFI 9),
b) does not satisfy ( IFI 10), but $V\left(x \rightarrow^{*} y\right)=V(\underline{1})$ iff $V(x)=V(\underline{0})$ and $V(y)=V(\underline{1})$, therefore, if $V\left(x \rightarrow^{*} y\right)=V(1)$, then $V(x) \preceq V(y)$,
c) satisfies (IFI 11) with $N=\neg$.

Proof. a)
(IFI 7) $V\left(1 \rightarrow^{*} y\right)=\left\langle\frac{1-d}{3-c-d}, \frac{2-c}{3-c-d}\right\rangle \neq\langle c, d\rangle$,
(IFI 8) $V\left(x \rightarrow^{*} x\right)=\left\langle\frac{1}{2}, \frac{1}{2}\right\rangle \neq\langle 1,0\rangle$,
(IFI 9) It is easy to show that in general the equality does not hold. Counterexample: $a=d=f=1, b=c=e=0$.
b) (IFI 10) If $V\left(x \rightarrow^{*} y\right)=V(\underline{1})$, i.e., $\frac{2-a-d}{4-a-b-c-d}=1$ and $\frac{2-b-c}{4-a-b-c-d}=0$, then $2-a-d=4-a-b-c-d$ and $b+c=2$, which holds only for $b=c=1$. Therefore, $V(x)=\langle 0,1\rangle \preceq\langle 1,0\rangle=V(y)$.
In the other direction, if $V(x) \preceq V(y)$, i.e., $a \leq c$ and $b \geq d$, then it is not necessarily true that $V\left(x \rightarrow^{*} y\right)=V(\underline{1})$. Counterexample: $a=b=c=d=\frac{1}{2}$.
c) (IFI 11) Since $V(\neg x)=\langle b, a\rangle$ and $V(\neg y)=\langle d, c\rangle$, then

$$
V\left(\neg y \rightarrow^{*} \neg x\right)=\left\langle\frac{2-a-d}{4-a-b-c-d}, \frac{2-b-c}{4-a-b-c-d}\right\rangle=V\left(x \rightarrow^{*} y\right) .
$$

## Remarks:

(R1) The implication $\rightarrow^{*}$ does not satisfy (IFI 7), however, if $\underline{1} \rightarrow^{*} y$ is an IFT, then $y$ is an IFT and if $y$ is an IFcT then $\underline{1} \rightarrow^{*} y$ is an IFcT.

[^0](R2) The implication $\rightarrow^{*}$ does not satisfy (IFI 8), however, $x \rightarrow^{*} x$ is an IFT.
(R3) The implication $\rightarrow^{*}$ does not satisfy (IFI 10), however, if $V(x) \preceq V(y)$, then $x \rightarrow^{*} y$ is an IFT.

It is also easy to check that the implication $\rightarrow^{*}$ does not satisfy the equivalent of the classical (two-valued) logic axioms. Namely, it is:

$$
V\left(\underline{0} \rightarrow^{*} \underline{0}\right)=V\left(\underline{1} \rightarrow^{*} \underline{1}\right)=\left\langle\frac{1}{2}, \frac{1}{2}\right\rangle \neq V(\underline{1}),
$$

although

$$
V\left(\underline{1} \rightarrow^{*} \underline{0}\right)=\langle 0,1\rangle=V(\underline{0}) \text { and } V\left(\underline{0} \rightarrow^{*} \underline{1}\right)=\langle 1,0\rangle=V(\underline{1}) .
$$

However, we should notice that $\underline{0} \rightarrow^{*} \underline{0}$ and $\underline{1} \rightarrow^{*} \underline{1}$ are IFTs.
As we can see the implication $\rightarrow *$ is therefore not a generalization of the classical implication.
There exist two basic rules of inference: Modus Ponens and Modus Tollens. These tautologies are given in the two-valued logic in the form

$$
(p \wedge(p \Rightarrow q)) \Rightarrow q \text { and }((p \Rightarrow q) \wedge N(q)) \Rightarrow N(p)
$$

respectively.
We assume that the Modus Ponens rule in the IFL-case is as follows:

$$
\text { if } x \text { is an IFT and } x \Rightarrow y \text { is an IFT, then } y \text { is an IFT. }
$$

Similarly, we assume that the Modus Tollens rule in the IFL-case as follows:

$$
\text { if } x \Rightarrow y \text { is an IFT and } y \text { is an IFcT, then } x \text { is an IFcT. }
$$

Theorem 3. The implication $\rightarrow^{*}$ :
a) satisfies Modus Ponens in the IFL-case,
b) satisfies Modus Tollens in the IFL-case.

Proof. Let $V(x)=\langle a, b\rangle$ and $V(y)=\langle c, d\rangle$.
a) Let $x$ and $x \rightarrow^{*} y$ be IFTs. Then $a \geq b$ and $\frac{2-a-d}{4-a-b-c-d} \geq \frac{2-b-c}{4-a-b-c-d}$, and further $a-b \geq 0$ and $b+c \geq a+d$. So, $0 \leq a-b$ and $a-b \leq c-d$. It follows that $0 \leq c-d$. Hence, $c \geq d$ and $y$ is an IFT.
b) Let $x \rightarrow^{*} y$ be an IFT and $y$ be an IFcT. Then $\frac{2-a-d}{4-a-b-c-d} \geq \frac{2-b-c}{4-a-b-c-d}$, and $c \leq d$. Consequently, $b+c \geq a+d$ and $d-c \geq 0$. So, $0 \leq d-c$ and $d-c \leq b-a$. It follows that $0 \leq b-a$. Hence, $a \leq b$, therefore $x$ is an IFcT.

## Remarks:

(R4) For $V(x)=\langle 1,0\rangle$, if $x \rightarrow^{*} y$ would be an IFT, i.e., $\frac{2-a-d}{4-a-b-c-d} \geq \frac{2-b-c}{4-a-b-c-d}$, then we would have $c \geq 1+d$, therefore, it must be $c=1$ and $d=0$. Hence, $V(y)=\langle 1,0\rangle$.
(R5) If $x \rightarrow^{*} y$ would be an IFT, i.e., $\frac{2-a-d}{4-a-b-c-d} \geq \frac{2-b-c}{4-a-b-c-d}$, then for $V(y)=\langle 0,1\rangle$ there holds the inequality $b \geq a+1$, therefore, $V(x)=\langle 1,0\rangle$.

One of the fundamental tautologies of classical logic is the relationship between the implication and negation. This relationship says that the truth-value of negation of the variable $x$ is equal to the value of the logical implications of the antecedent $x$ and the consequent false.

Symbolically, this tautology is written in the classical logic in the form of $N(x) \Leftrightarrow(x \Rightarrow 0)$. Using this relationship, we can-for every intuitionistic fuzzy implication-designate a corresponding negation, called a generated (induced) negation.

Theorem 4. Let $V(x)=\langle a, b\rangle$. The negation $N^{*}$ generated by the implication $\rightarrow^{*}$ is expressed by formula:

$$
V\left(N^{*}(x)\right)=\left\langle\frac{1-a}{3-a-b}, \frac{2-b}{3-a-b}\right\rangle
$$

Proof. By definition of the $\rightarrow^{*}$ implication.

## Remarks:

(R6) $V\left(N^{*}(\underline{0})\right)=\left\langle\frac{1}{2}, \frac{1}{2}\right\rangle$,

$$
\begin{aligned}
& V\left(N^{*}(\underline{1})\right)=\langle 0,1\rangle=V(\underline{0}), \\
& V\left(N^{*}(\underline{\mathrm{FI}})\right)=\left\langle\frac{1}{3}, \frac{2}{3}\right\rangle .
\end{aligned}
$$

(R7) For any variable $x$ the negation $N^{*}(x)$ is an IFcT.
(R8) The negation $N^{*}(x)$ is not involutive because

$$
V\left(N^{*}\left(N^{*}(x)\right)\right)=\left\langle\frac{2-b}{2(3-a-b)}, \frac{4-2 a-b}{3-a-b}\right\rangle \neq\langle a, b\rangle=V(x) .
$$

The first equality in Remark (R6) shows that negation $N^{*}$ does not fulfill the basic property of negations in the form $V\left(N^{*}(\underline{0})\right)=V(\underline{1})$, however $N^{*}(\underline{0})$ is an IFT. The property presented in Remark (R6) should not be satisfied because the negation of the IFT should be an IFcT and the negation of the IFcT should be an IFT. For this reason, negation $N^{*}$ should not be used in further applications.

## 3 Conclusion

In the paper, a new fuzzy intuitionistic implication with its basic properties is presented. The implication may be the subject of further research, both in terms of its properties or comparisons with other intuitionistic fuzzy implications, and possible applications. The applications, for example, may relate to fuzzy control, reasoning with incomplete or uncertain information, or multiple criteria decision making, especially with varying degrees of criteria importance.

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[^0]:    ${ }^{1}$ Various authors give these and other axioms following [25], pp. 308, 310.

